ALL GAMES HAVE EQUILIBRIA IN FINITELY ADDITIVE STRATEGIES

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Abstract. A game is specified by a non-empty set of players, and for each player, a non-empty set of actions and a bounded von Neumann-Morgenstern utility function. We show that all games have equilibria in finitely additive mixed strategies. We also specify a finite approximation method for finding the equilibria that put unit mass on the set of iteratively weakly undominated strategies. Corollaries include the existence of equilibria in countably additive mixed strategies for compact games with special kinds of discontinuous utility functions.

Contents

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1. Introduction

Glicksberg [1952] showed that finite player games with jointly continuous payoffs and compact sets of actions have equilibria in countably additive mixed strategies. Joint continuity is crucial for the result. Sion and Wolfe [1957] gave a zero-sum, compact continuum game where the discontinuities prevent the existence of approximate equilibria in countably additive mixed strategies.

Over the intervening decades, considerable effort has been devoted to identifying sufficient conditions weaker than joint continuity for the existence of equilibria in countably additive mixed strategies. We take a different path. We drop the requirement of countable additivity on mixed strategies and establish the following result, subject to the most minimal of requirements.

Theorem. All games have finitely additive mixed-strategy equilibria.

There are no cardinality assumptions on the player set, nor on the sets of actions available to the players, nor are there any assumptions beyond boundedness on the agents' utility functions.

We make two contributions beyond this result. First, we give a method for finding a closed non-empty set of finitely additive mixed strategy equilibria with two properties: they put zero mass on the iteratively weakly dominated strategies; and they represent the limits of equilibria on finite approximations to the game. The second contribution starts by identifying probabilities as continuous linear functionals on the set of all bounded functions. By restricting these linear functionals to different vector subspaces of possible utility functions, we give simple proofs of some extant countably additive equilibrium existence results as well as some new results.

The distinction between countably additive and finitely additive probabilities hinges on whether or not the formula $p(\cup_{n\in S} B_n) = \sum_{n\in S} p(B_n)$

can be extended from $\{B_n : n \in S\}$ being a finite disjoint collection of sets to being a countably infinite disjoint collection. Belying the long and sharply contentious debate about which choice is the correct one, this has often been erroneously portrayed as a "merely technical" issue.

Our point of view is that correctness must be conditional on the uses being made of probabilities. For some uses one choice will be better, and for other uses the other will be better. We contend that the better choice for games with infinite action sets is finite additivity. To put our contention in context, we give an overview of the advantages and disadvantages of using finitely additive probabilities that have arisen in various contexts.

1.1. **Advantages and their Contexts.** Any finitely or countably additive probability on a collection of subsets of a space has many extensions to the class of all subsets of that space. All of them are finitely additive but not necessarily countably additive. Savage [1972] and de Finetti [1974, 1975] argued extensively that probabilities, understood as someone's degree of belief, should be allowed to be attached to every set. While the argument has not convinced everyone, Savage, and to a lesser extent de Finetti, wanted nonatomic subjective probabilities on the class of all subsets of a subjective state space. There are no countably additive nonatomic probabilities on the class of all subsets, at least not within the usual axiomatizations of mathematics, but there are nonatomic finitely additive probabilities.

A second part of their argument for finite additivity in models of Bayesian priors appears in de Finetti [1974, pp. 299-231]. He notes that under the usual assumptions in probability theory, a finitely additive probability can be extended, perhaps in many ways, to any class of sets, while countably additive probabilities, when they can be extended at all, extend uniquely. He argues that this extra expressive power is an important part, perhaps even a crucial part, of the description of beliefs.

Viewed through the lens of modern multiple prior subjective state space theories of choice under ambiguity, it is tempting to use the set of extensions as a set of priors. While this does not capture all of de Finetti's nuanced distinctions between imprecision, indeterminacy and unverifiability, it provides a set of priors with the same properties as those associated with the econometrics of unobservable variables.[1]

[1]See Stinchcombe [2016] for this. That paper also showed that using the set of finitely additive extensions as the set of priors for modern multi-prior choice under ambiguity models allows a small, but still interesting, class of ambiguous choice problems to be successfully modeled and analyzed.

When one assigns probability to all subsets of a space, measurability arguments are drastically simplified, and this paper will make central use of these simplifications. Measurability issues have played a central role in the theory of large population games initiated by Schmeidler [1973]. He used a nonatomic probability space to model the set of agents, which handily captures the idea that each player is a vanishingly small portion of the whole population. A difficulty arises because the theory of Nash equilibrium is built on independent randomization by the agents, and a continuum of independent choices involves some rather complicated mathematics.

If the large population is modeled as a nonatomic distribution over, say, a metric space of characteristics and that distribution is countably additive, then measurable functions are, with high probability, continuous. In order that there be a well-defined distribution of equilibrium actions, the mapping from characteristics to actions chosen must be Borel measurable. Therefore, in the presence of countable additivity, it is a high probability event that people whose characteristics are close choose close actions. This precludes independent randomization, an observation with a long history.

As far back as the work in Doob [1937, Theorem 2.2], it has been known that a continuous-time stochastic process can not be have non-degenerate independent realizations at all points in time and simultaneously have measurable time paths. This implies that when the players, modeled as points on an interval subset of the real line, or as points on any set measurably isomorphic to the real line, are independently randomizing over, say, Heads and Tails, one cannot assign probability to the set of players who choose Heads. The solution to these difficulties requires an enrichment of the σ-algebra over the product of the measurable spaces as in Sun [2006, §6] and the references cited there. Starting by modeling the population using a finitely additive distribution on the class of all subsets of the space of characteristics as in Cerreia-Vioglio et al. [2022], and as we do, drastically simplifies this process.

For us, the central advantage of the class of finitely additive probabilities is their compactness. Probabilities on a class of sets $\mathcal{E}$ can be understood as points in the product space $[0,1]^{\mathcal{E}}$, i.e., in the space of all functions from $\mathcal{E}$ to $[0,1]$. By Tychonov's theorem, this space is compact in the product topology. In the product topology, a net (generalized sequence) of probabilities p^α converges to a probability p if $p^\alpha(E)$ converges to $p(E)$ for all sets E in the class of sets $\mathcal{E}$. Since finite sums are continuous in their arguments, the set of finitely additive probabilities is a closed subset of the compact product space. For present purposes, there are two particularly useful implications of this compactness.

First, if a class of closed sets of probabilities has the finite intersection property, then it has a non-empty intersection. To put it differently, finitely satisfiable properties of probabilities can always be represented. The crucial finite satisfiability property for us shows up in the set of limits of equilibria on finite approximations to the games.

For F a finite set of actions, let $Eq(F)$ denote the closed set of all limits of equilibria along a nets (generalized sequences) of finite approximations to the game that eventually all contain F. If F' is another finite set, then $Eq(F) \cap Eq(F') = Eq(F \cup F')$. This means that the class of sets, $Eq(F)$, indexed by finite sets of actions has the finite intersection property. The necessarily non-empty intersection represent equilibria that are limits along nets that are "exhaustive," nets allow every player every action. As we will see, this class of equilibria can still be analyzed as if the game were finite if one uses the exhaustive hyperfinite sets from nonstandard analysis that correspond to exhaustive nets.

Second, since every bounded function is a uniform limit of simple functions, a net of probabilities p^α defined on the class of all subsets of a space converges to a probability p if and only if its integrals against all bounded functions, f, converge, $\int f\, dp^\alpha \to \int f\, dp$. In particular, the linear functional $p \mapsto \int f\, dp$ is always continuous, hence always has a maximum on the compact set of probabilities.

One can say a good bit more about the set of maxima, and this will provide a segue to contexts in which the disadvantages of finitely additive probabilities appear. The set of finitely additive probabilities is both compact and convex. The Krein-Milman theorem tells us that the compact, convex set of maximizers for a continuous linear objective function $p \mapsto \int f\, dp$ always has a solution in the set of extreme points of the set of probabilities. The extreme points are **Z1** probabilities, that is, probabilities with $p(B)$ equal to either **Z**ero or **1** for all sets B. These represent point masses on the solutions to the problem of maximizing a bounded function and yield a maximum in the set of Z1's for any bounded function.

There are many more Z1's than there are points in the original space. Yosida and Hewitt [1952, §4] used the set of Z1's to represent the finitely additive Z1's as countably additive point mass probabilities on that larger space. This is a set of points that do not appear in the original space. Yosida and Hewitt [1952, p. 56-8] used properties of these points to get at "various peculiar properties that finitely additive measures . . . may exhibit." The peculiar properties of the new points are at the heart of many of the disadvantages.

1.2. **Disadvantages and their Contexts.** We will show that for every game, there are finitely additive equilibria that put full mass on the set of iteratively weakly undominated strategies. We will also give an example of a two-player game with the compact strategy sets $[0, 1]$ and jointly continuous utility functions for which the set of iteratively weakly undominated strategies is empty. The way that these statements are consistent is that the finitely additive equilibria are Z1's that, for any $\epsilon > 0$, put unit mass on the interval $(0, \epsilon)$. The decidedly peculiar property of these Z1's is that that the intersection of these sets, $\cap_{\epsilon>0}(0, \epsilon)$, is empty.

The Z1's that represent these equilibria are point masses on "new points." Thus, even in the class of games analyzed in Glicksberg [1952], there are phenomena of game theoretic interest that are not captured by the original set of actions. The disadvantage is that the new points must be added to the model. One might argue that if the modeler had wanted these points in the model, then they would have included them. But from our point of view, the use of the larger set of finitely additive probabilities presumes that they are already there.

These new points also appear if one wishes to use the Bayesian subjective state space model developed in Savage [1972] to model choices between different continuously distributed random variables and simultaneously avoid money pumps. Stinchcombe [1997] collected the various money pump examples from the literature and then systematically added the new points to the subjective state spaces in a fashion that minimally expanded the model and avoided money pumps. Again, the ostensibel disadvantage is that the points were not in the original model, and one may find that objectionable.

From de Finetti [1975, p. 353], we have the following about the new points.

> The basic idea is the possibility of stretching the interpretation in such a way as to be able to attribute the "missing" probability in the partition to new fictitious entities in order that everything adds up properly. In some cases, in order to salvage countable additivity, it is even claimed that the new entities are not fictitious, but real.

From our point of view, it is perhaps better to think of the new points as having always been in the model even though they were not explicitly mentioned.

Another disadvantage of finitely additive probabilities is the failure of Fubini's theorem for finite product spaces. The set of functions on a finite product of spaces that have a well-defined integral for all products of finitely additive probabilities is very small. This can make defining or understanding equilibrium utilities problematic.

Stinchcombe [2005] develops a theory of set-valued integration from which one can take selections in a fashion that respects the Nash equilibrium best-response properties. This is an awkward undertaking, and Examples demonstrate the difficulties of systematically selecting from the set of integrals. Here we circumvent the problem by noting that products of finitely supported Nash equilibria define the probability of all subsets of the products of spaces of actions. This means that their limits specify a probability on the class of all subsets of the product space. This in turn means that every bounded utility function has a well-defined integral.

A final disadvantage of finitely additive probabilities for games arises when players in nonatomic continuum are identified with characteristics such as their utility functions. Such characteristics are points in e.g. the unit ball in an infinite dimensional space of functions. As discussed in Stinchcombe [2023, §8], the Z1's that support the finitely additive probabilities on such unit balls are quite difficult to work with. We will address these difficulties in a companion paper on infinite population games that is under preparation.

1.3. **Outline.** The next section gives the notation necessary to formally state the main existence result, Theorem A. It then gathers the necessary background for the rest of the paper. As Corollaries to the existence of finitely additive equilibria, this section uses the background on duality to deliver known and new countably additive equilibrium existence results.

The subsequent section systematically uses nets of finite approximations to give finitely additive equilibria for three different two-player games games with discontinuous payoffs. None of these games have countably additive equilibria, and only one of them has approximate countably additive equilibria. These games highlight aspects of the differences between the countably and the finitely additive mixed strategies, and also highlight differences in the approaches one can take to the class of finitely additive equilibria. In general, the limits of equilibria taken along nets of exhaustive finite approximations to a game are an easier way to analyze the finitely additive equilibria, and the equilibria of games using the exhaustive hyperfinite sets from nonstandard analysis are an easier way to analyze the nets of equilibria.

The next section contains two results. First, Theorem B shows that the set of finitely additive equilibria is a compact set and that the equilibrium correspondence is upper hemi-continuous. This means that the set of finitely additive equilibria satisfies the same basic robustness criteria as the set of countably additive equilibria that we are used to. Next, Theorem C shows that there is a non-empty, compact set of finintely additive equilibria that represent the limits of equilibria along nets of finite approximations to the game and which put unit mass on the iteratively weakly undominated strategies. While Theorems A and B concern the set of all finitely additive equilibria, Theorem C concerns the subset of equilibria that we find most reasonable.

We then turn to the payoff equivalences of countably additive and finitely additive equilibria. For these to hold, one must have very special kinds of discontinuities in the utility functions. There are some relations, unfortunately rather tenuous, between these special discontinuities and the well-behaved discontinuities that have dominated the study of countably additive equilibrium existence for compact games.

The last section contains some concluding remarks and sketches how finitely additive mixtures seem likely to make contributions to two very different areas in game theory. Finally, proofs not in the text are gathered in the appendix.

2. Finitely Additive Equilibria

We begin with a formal description of the class of games under consideration. Essentially the only assumption that we make is that the von Neumann Morgenstern utility functions are bounded, an assumption necessary to preclude the St. Petersburg paradox and similar phenomena. We then turn to the relevant properties of the set of finitely additive probabilities.

First, there is no loss in assuming that they are *total*, that is, they are defined on the class of all subsets of the space under consideration. This property drastically simplifies measurability considerations. Second, they are compact, which, in the absence of measurability issues, entails the existence of optima for problems with bounded utility functions. It also entails the existence of probabilities with 'finitely satisfiable' properties.

2.1. Equilibria.

The starting point is the set of total probabilities.

Definition 2.1. *For any non-empty set X, 2^X denotes the class of all subsets of X. A* ***total probability on*** *X is a function $p : 2^X \to [0, 1]$ that satisfies $p(X) = 1$ and $p(B_1 \cup B_2) = p(B_1) + p(B_2)$ for all disjoint*

$B_1, B_2 \in 2^X$. The set of probabilities on the class of all subsets of X is denoted by $\Delta(X)$ or Δ when X is clear from context.

By induction, if $\{B_n : n = 1, \ldots, N\}$ is a finite collection of disjoint sets, a total probability must satisfy $p(\cup_{n=1}^N B_n) = \sum_{n=1}^N p(B_n)$.

Definition 2.2. *A **game** $\Gamma = (A_i, u_i)_{i \in I}$ is specified by:*

(1) *a non-empty set of players, I;*
(2) *for each $i \in I$, a non-empty set of actions A_i; and*
(3) *for each $i \in I$, a bounded von Neumann-Morgenstern utility function $u_i : A \to [-B, +B]$ where $A = \times_{j \in I} A_j$.*

An equivalent formulation of the profile of utility functions is that utilities are specified by a function $u : A \to [-B, +B]^I$.

The set of total probabilities on A_i is denoted by $\Delta_i = \Delta(A_i)$ and the set of total probabilities on $A := \times_{i \in I} A_i$ is denoted by Δ. The linear extension of u_i from A to Δ is also denoted by u_i, that is, $u_i(\mu) = \int_A u_i(a)\, d\mu(a)$ for any $\mu \in \Delta$. In the same way, the linear of extension of u is also denoted by u, that is, $u(\mu) = (u_i(\mu))_{i \in I} \in [-B, +B]^I$.

Nash equilibria may involve independent randomization by the players.

Definition 2.3. *A probability $\widehat{\mu} \in \Delta$ is a **independent extension** of $(\mu_i)_{i \in I} \in \times_{i \in I} \Delta_i$ if for all finite $I_F \subset I$, for all $B_j \subset A_j$ for $j \in I_F$,*

$$\widehat{\mu}(\text{proj}^{-1}(\times_{j \in I_F} B_j)) = \textstyle\prod_{j \in I_F} \mu_j(B_j). \tag{1}$$

We use the following game-theoretic notation, for $a \in A$, $i \in I$, and $b_i \in A_i$, the point $a \backslash b_i \in A$ is defined by $\text{proj}_j(a \backslash b_i) = a_j$ for $j \neq i$ and $\text{proj}_i(a \backslash b_i) = b_i$. And we extend this notation to mixtures over A, for μ a probability on A, $i \in I$ and $b_i \in A_i$, $\mu \backslash b_i$ is the image measure of μ under the mapping $a \mapsto a \backslash b_i$ from A to A.

Definition 2.4. *An independent extension μ^* of $(\mu_i^*)_{i \in I}$ is a **Nash equilibrium** if for all $i \in I$ and all $b_i \in A_i$, $u_i(\mu^*) \geq u_i(\mu^* \backslash b_i)$.*

With these in place, we can state our major result.

Theorem A. *For any game Γ satisfying the assumptions* (1)-(3) *above, an equilibrium exists.*

Here is an outline of the proof: take larger and larger finite approximations, I_F, to the set of players, I; combine that with larger and large finite approximations, H_i, to the actions sets of the $i \in I_F$; use Nash's existence theorem to get an equilibrium for the finite games; use the compactness of Δ to guarantee the existence of accumulation points of the equilibria; and we verify that the accumulation points are equilibria.[2]

There is a subtlety when I is infinite. To specify payoffs in a finite game $\Gamma_F = (H_i, u_i)_{i \in I_F}$, we must specify what the players $j \notin I_F$ are choosing. That choice can determine aspects of the equilibrium behavior in the limit and we allow all possible choices. We interpret the dependence as a form of players' beliefs in what some un-named set of "others" are doing. This provides an endogenous form of variability that might be called "animal spirits" or "sun spots." In the companion paper about infinite population games, we hope to characterize the class of games in which the set of equilibria does not have this variability.

In this paper, we will actually prove a good bit more. Theorem B shows that the set of equilibria is compact and that the correspondence from utility functions to the set of equilibria is upper hemi-continuous. Theorem C shows that there is a non-empty, compact set of equilibria that put all of their mass on the iteratively undominated strategies.

2.2. **Properties of Δ.** The central property of Δ is its compactness. Corollaries of compactness include include various extension results as well as a very general existence result for solutions to optimization problems.

2.2.1. *Compactness.* With $\mathcal{A}$ denoting the class of all subsets of A, every total probability can be identified with a point p in the product space $[0,1]^{\mathcal{A}}$. Being the product of compact spaces, $[0,1]^{\mathcal{A}}$ is compact in the product topology (by Tychonov's theorem). In this topology, convergence is defined by $p^\alpha \to p$ if $p^\alpha(B) \to p(B)$ for all sets B. Since finite sums are continuous in their arguments, Δ is a closed, hence compact set.

Since every bounded function on A is the uniform limit of simple functions, the usual approximation arguments show that (a) the integral of any bounded function with respect to any total probability is well defined, and (b) that $p^\alpha \to p$ if and only if $\int f\, dp^\alpha \to \int f\, dp$ for all bounded function f defined on A. It is perhaps worth emphasizing again the simplicity of the theory that comes from all functions being measurable.

2.2.2. *Extensions.* The compactness of Δ is equivalent to the property that every collection of closed subsets having the finite intersection property has a non-empty intersection.

[2]The details are in the appendix.

Example 2.1 (Extension 1)**.** *Suppose that q is a countably additive probability on the Borel σ-field of a metric space. The set of total probabilities that agree with q on the Borel σ-field is denoted $Ext(q)$. It is a non-empty, compact, convex set: for each finite collection B_F of Borel sets, the class of finitely additive total probabilities, $S(B_F)$, that agrees with q on B_F is a non-empty, compact and convex set; since $S(B_F) \cap S(B'_F) = S(B_F \cup B'_F)$, the class has the finite intersection property; and the set of total probabilities that agree with q is $\bigcap S(B_F)$ where the intersection is taken over all finite collections of Borel sets.*

One can also go in the opposite direction: every $p \in Ext(q)$ integrates the bounded continuous functions on the metric space to the same number that q does; when the countably additive q is determined by its integral against the continuous functions, each of these total probabilities p is equivalent, for all continuous purposes, to q.

Products of finitely additive probabilities also have many extensions, and this is the reason that Fubini's theorem does not hold except for very special utility functions.

Example 2.2 (Extensions 2)**.** *Suppose that q_1 and q_2 are total probabilities on A_1 and A_2. The field of measurable rectangles, $\mathcal{R}^\circ$, is the set of finite unions of sets of the form $B_1 \times B_2$, $B_i \subset A_i$. The product probability $[q_1 \times q_2]$ is defined on $\mathcal{R}^\circ$ by $[q_1 \times q_2](B_1 \times B_2) = q_1(B_1) \cdot q_2(B_2)$. By a small variant of the arguments used in the previous Example, $Ext([q_1 \times q_2])$, the set of total probabilities on $A_1 \times A_2$ that agree with $[q_1 \times q_2]$ on $\mathcal{R}^\circ$, is a non-empty, compact and convex set.*

The set of utility functions for which the product integral is well-defined is quite small and the games with such utility functions are "simple" in an essential way. We use the notation from the previous Example in the following.

Lemma 2.1. *A bounded $u : A_1 \times A_2 \to \mathbb{R}$ is the uniform limit of simple functions that are $\mathcal{R}^\circ$ measurable if and only if for every q_1 and q_2 and every $p, p' \in Ext([q_1 \times q_2])$, $\int u\, dp = \int u\, dp'$.*

This follows from Harris et al. [2005, Theorem 1]. That paper shows that games with utility functions that are uniform limit of simple $\mathcal{R}^\circ$-measurable functions are equivalent, in all of the essential ways, to games with compact metric spaces of actions and jointly continuous utility functions.

The following example will matter for our analysis of the Sion and Wolfe [1957] game in the next section.

Example 2.3. *Suppose that for* $(x, y) \in [0, 1]$*,*

$$u(x, y) = \begin{cases} -1 & \text{if } x < y \\ 0 & \text{if } x = y \\ +1 & \text{if } x > y, \end{cases}$$

and that q_1 *and* q_2 *are both* Z1*'s that put unit mass on each interval* $(\frac{1}{2} - \epsilon, \frac{1}{2})$*. There are elements* p_{-1}, p_0*, and* p_{+1} *in* $Ext([q_1 \times q_2])$ *such that* $\int u\, dp_{-1} = -1$*,* $\int u\, dp_0 = 0$*, and* $\int u\, dp_{+1} = +1$*.*

To see why, let L_{-1}*,* L_0*, and* L_{+1} *be lines through* $(1/2, 1/2)$ *with slopes* $1/2$*,* 1*, and* 2*. For each of these lines, the set of* $p \in Ext([q_1 \times q_2])$ *that put unit mass on that line and each* $(\frac{1}{2} - \epsilon, \frac{1}{2}) \times (\frac{1}{2} - \epsilon, \frac{1}{2})$ *is a collection of compact convex sets having the finite intersection property.*

Fubini's theorem says that the order of integration does not matter for products of countably additive probabilities. The order does matter for finitely additive probabilities.

Example 2.4. *In the previous example, define* $f(y) = \int u(x, y)\, dq_1(x)$*. The function* $y \mapsto f(y)$ *is equal to* -1 *for* $y \geq \frac{1}{2}$ *and is equal to* $+1$ *for* $y < \frac{1}{2}$*. This means that* $\int f(y)\, dq_2(y) = +1$*. Thus, integrating first w.r.t.* x *and then* y *gives* $+1$*. For integrating in the opposite order, define* $g(x) = \int u(x, y)\, dq_2(y)$*. This is equal to* $+1$ *for* $x \geq \frac{1}{2}$ *and is equal to* -1 *for* $x < \frac{1}{2}$ *so that* $\int g(x)\, dq_1(x) = -1$*.*

2.2.3. *Extreme Points and Optimization.* The extreme points of Δ are the **Z1**'s, the probabilities with $p(B)$ being either equal to Zero or to 1 for all sets B. The class of Z1's is closed (since the limit of a net of 0's and 1's can only be a 0 or a 1), hence compact. Here is the existence of optima result referred to above, any bounded function on A has a maximand in the set of Z1's.

Lemma 2.2. *For any bounded* $f : A \to \mathbb{R}$*, there is a* Z1 *that solves* $\max_{p \in \Delta} \int f\, dp$*.*

Proof. Let $r = \sup_{a \in A} f(a)$, and for each $n \in \mathbb{N}$, let F_n denote the necessarily non-empty set of $a \in A$ with $r - \frac{1}{n} \leq f(a) \leq r$, and let S_n denote the closure of the set of Z1's that put mass 1 on F_n. The class $\{S_n : n \in \mathbb{N}\}$ has the finite intersection property, and any Z1 in its intersection solves the maximization problem. □

2.3. **Finite Approximations to Games.** To talk about limits of approximate games in such a fashion that we can guarantee that every action of every player is included, we require the following generalization of sequences.

Definition 2.5. *A **directed set** is a pair $(D, \succsim)$ where D is a nonempty set and $\succsim$ is a transitive binary relation on D satisfying: $\alpha \succsim \alpha$ for all $\alpha \in D$, and for all $\alpha, \beta \in D$, there exists $\gamma \in D$ with $\gamma \succsim \alpha$ and $\gamma \succsim \beta$. A **net** in a set X is a mapping $\alpha \mapsto x_\alpha \in X$ from a directed set D to X.*

A sequence is the special case of a net where the directed set, $(D, \succsim)$, is $(\mathbb{N}, \geq)$. We will need nets of finite approximations to a game as well as nets of equilibria for those approximate games.

Definition 2.6. *A **net of finite approximations to a set** X is a mapping $\alpha \mapsto F_\alpha$ from a directed set $(D, \succsim)$ to the class of finite subsets of X. We write $F_\alpha \uparrow \infty$ if the net **exhausts** X, that is, if for all finite $F \subset X$, there exists an $\alpha \in D$ such that for all $\beta \succsim \alpha$, $F \subset F_\beta$.*

Sequences of finite sets can exhaust countable sets. The right choice of indexing set shows that nets of finite sets can exhaust any set.

Example 2.5. *Let D denote the class of finite subsets of a set X. For $F, F' \in D$, define $F \succsim F'$ if $F \supset F'$. Taking the mapping from D to the finite sets to be the identity mapping, we have, for all finite $F \subset X$, there exists an $\alpha \in D$, namely $\alpha = F$, such that for all $\beta \succsim \alpha$, $F \subset F_\beta$.*

For a finite player game $\Gamma = (A_i, u_i)_{i \in I}$, we study equilibria that arise as accumulation points or as limits of equilibria on the games $\Gamma_\alpha = (F_{i,\alpha}, u_i)_{i \in I}$ where for each $i \in I$, the mapping $\alpha \mapsto F_{i,\alpha}$ exhausts A_i. Let μ_α be the product total probability induced by an equilibrium for the finite game Γ_α. It is a textbook exercise to show that the compactness of Δ implies that the net $\alpha \mapsto \mu_\alpha$ has a non-empty set of accumulation points.

Definition 2.7. *For $A = \times_{i \in I} A_i$, a total probability $\mu \in \Delta$ is a **limit point** of the net $\alpha \mapsto \mu_\alpha$ of total probabilities if for all sets $B \subset A$ and all $\epsilon > 0$, there exists an $\alpha \in A$ such that for all $\beta \succsim \alpha$, $|\mu_\beta(B) - \mu(B)| < \epsilon$, and μ is an **accumulation point** of the net $\alpha \mapsto \mu_\alpha$ if for all sets $B \subset A$, all $\epsilon > 0$, and all α, there exists $\beta \succsim \alpha$ such that $|\mu_\beta(B) - \mu(B)| < \epsilon$.*

To more easily work with the set of all accumulation points of all equilibria on all exhaustive nets of finite approximations to a given game, the next section will use the exhaustive hyperfinite sets constructed in nonstandard analysis.[3] Exhaustive hyperfinite sets allow

[3]For a textbook treatment of nonstandard analysis, see Hurd and Loeb [1985], where Chapter II.4 shows how to guarantee that exhaustive hyperfinite sets exist. Lindstrøm [1988] gives a development of nonstandard analysis that builds closely on the sequence- and net-based intuitions that permeate analysis.

us to analyze the net of finite games as if it is a single finite game, while at the same time taking into account e.g. the iterated deletion of weakly dominated strategies. We will, mostly, use the hyperfinite sets for developing intuition and provide proofs based on nets.

2.4. **Duality and Restrictions.** There is a dual space representation of Δ that provides a method of restricting finitely additive equilibria to their actions on subspaces of functions. For some games, this allows us to obtain countably additive equilibria from finitely additive equilibria. Let $\mathbb{B} = \mathbb{B}(A)$ denote the Banach space of bounded functions on A with the sup norm. The essential mathematics is contained in the following

Lemma 2.3. *If p is a total probability on A, then the mapping $f \mapsto L(f;p) := \int f\, dp$ from $\mathbb{B}(A)$ to $\mathbb{R}$ is a Lipschitz continuous, linear, monotonic functional that integrates the constant function, 1_A, to 1. If $L : \mathbb{B}(A) \to \mathbb{R}$ is a Lipschitz continuous, linear, monotonic functional that integrates the constant function to 1, then there is a unique total probability p_L such that $L(f) = \int f\, dp_L$.*

Proof. The given properties of the mapping $f \mapsto L(f;p)$ are immediate for simple functions. The simple functions are sup norm dense in $\mathbb{B}(A)$, and the integral is the unique Lipschitz continuous functional that extends the integral from the dense subset of simple functions.

If $L : \mathbb{B}(A) \to \mathbb{R}$ has the given properties, then defining $p_L(B) = L(1_B)$ gives a total probability. The linearity of $L(\cdot)$ implies that that $L(f) = \int f\, dp_L$ for all simple functions f, and the Lipschitz continuity of $L(\cdot)$ and the sup norm density of the simple functions implies that $L(f) = \int f\, dp_L$ for all $f \in \mathbb{B}(A)$. □

Every sup norm closed vector subspace of $\mathbb{V} \subset \mathbb{B}$ gives rise to a compact, but not generally Hausdorff, topology $\tau_{\mathbb{V}}$ on Δ defined by $p^\alpha \to_{\tau_{\mathbb{V}}} p$ if $\int g\, dp^\alpha \to \int g\, dp$ for all $g \in \mathbb{V}$. Thus, the convergence defined above is in the $\tau_{\mathbb{B}}$-topology, also known as the **weak*-topology for finitely additive probabilities**.

2.4.1. *Continuous Compact Games.* If the product space A is a compact Hausdorff space and $\mathbb{V} = C(A)$ is the sup norm closed set of continuous functions on A, then (the Riesz representation theorem tells us that) each restriction of $L(\cdot;p)$ to $\mathbb{V}$ is associated with a unique countably additive $q_{ca} = q_{ca}(p)$ on the Baire, and hence on the Borel σ-fields. The countably additive q_{ca} is defined on a much smaller σ-field of subsets than p is. On that smaller class of sets, q_{ca} is determined by the restriction that it integrates every (necessarily bounded) $g \in \mathbb{V}$ to the same number that p does, i.e. $\int g\, dq_{ca} = \int g\, dp$. When $\Gamma = (A_i, u_i)_{i\in I}$

is a compact and continuous game, this delivers countably additive equilibrium existence.

Corollary A.1 (Generalized Glicksberg)**.** *If each A_i is a compact Hausdorff space and each $u_i : A \to [-B, +B]$ is continuous in the product topology on A, then for any finitely additive equilibrium p^*, if the countably additive $q^* = q_{ca}(p^*)$ is an equilibrium.*

Proof. For any $i \in I$ and any $b_i \in A_i$, the functions $a \mapsto u_i(a)$ and $a \mapsto u_i(a\backslash b_i)$ are both continuous. Thus, a finitely additive p^* is an equilibrium if and only if every p on the Borel σ-field that integrates continuous functions to the same number is an equilibrium. □

Since a finitely additive equilibrium exists, this means that a countably additive equilibrium exists. When I is finite, this existence result is due to Glicksberg [1952]. Here I is arbitrary.[4]

2.4.2. *Continuous Polish Games.* When A is not compact, a finitely additive p must satisfy an additional condition in order that there be a countably additive q that integrates all of the bounded continuous functions to the same number.

A metric space is *Polish* if it is both complete and separable. A finite or countable product of Polish metric spaces with the product topology is itself Polish. It is well-known that any countably additive probability q on the Borel subsets of a Polish space is **tight**, that is, for any $\epsilon > 0$, there is a compact K such that $q(K) > 1 - \epsilon$. The following is the appropriate generalization for general finitely additive probabilities.

Definition 2.8. *A finitely additive p on the class of all subsets of a Polish space X is **near-tight** if for all $\epsilon > 0$, there is a compact $K \subset X$ such that $p(K^\delta) > 1 - \epsilon$ for all $\delta > 0$ where K^δ is the set of $b \in X$ such that $d(b, K) < \delta$.*

We will now consider the case that $\mathbb{V} = C_b(X)$ is the sup norm closed set of all continuous bounded functions on a Polish space X.

Definition 2.9. *For (X, d) a Polish space, two probabilities p_1 and p_2 on a class of subsets of X that contain the Borel σ-field are **continuously equivalent** if for all $f \in C_b(X)$, $\int f\, dp_1 = \int f\, dp_2$. If $q = q_{ca}(p)$ is a countably additive Borel probability that is continuously equivalent*

[4]For a finite set of players, the generality of compact Hausdorff spaces rather than compact metric spaces is illusory. Stinchcombe [2005, Corollary 3, p. 217] shows that, after identifying strategically equivalent strategies, the game is equivalent to one with compact metric spaces of actions.

to a near-tight finitely additive p, then q is the ***countably additive version of*** *p.*

As part of a nonstandard analysis characterization of countably additive probabilities on Polish spaces, Stinchcombe [2023, Corollary 4.1] shows that a finitely additive p on the class of all subsets of a Polish space is near-tight if and only if it has a (necessarily unique) countably additive version. These considerations deliver the following.[5]

Corollary A.2. *If I is finite or countable, each A_i is Polish and each $u_i : A \to [-B, +B]$ is continuous in the product topology on A, then a near-tight p^* is an equilibrium if and only if every continuously equivalent p is an equilibrium if and only if the countably additive $q^* = ca(p^*)$ is an equilibrium.*

3. Two-Person Games on the Line

In this section we cover three two-person games, all of them having action sets that are a subsets of the real line, none of them having countably additive equilibria. We begin with an asymmetric Hôtelling game due to Simon and Zame [1990] in which there are countably additive ϵ-equilibria for every $\epsilon > 0$, but no countably additive equilibria. After deleting the weakly dominated strategies along an exhaustive net of finite games, the limits have a particularly simple form. We then turn two games in which there are no approximate countably additive equilibria: the classic Sion and Wolfe [1957] game; and a version the "pick the largest number" game.

For the analysis of the Hôtelling game, we give a nets-of-finite-approximations argument and then give the hyperfinite sets version of the same argument. For the second and third games, we start with the hyperfinite analysis and sketch how one turns this into an argument along the nets of finite approximations.

There is a long history of taking standard parts of hyperfinite objects. The particulars of directly representing finitely additive probabilities in the way that we do goes back to Robinson [1964], and details of the general procedure are gathered in Stinchcombe [2023, §2]. These techniques allow us to sidestep the complications of keeping track of the details of the net of finite approximations while we analyze equilibria. This is particularly useful when, for each finite game in the net of finite games that exhausts the actions sets of the players, we iteratively delete weakly dominated strategies.

[5]The omitted proof directly parallels the previous argument for the generalized Glicksberg result.

3.1. **An Asymmetric Hôtelling Game.** There is a uniform distribution of consumer locations on the interval $[0, 1]$. Due to licensing restrictions, player 1 can only pick a location, x, in the interval $A_1 = [0, 0.8]$ while player 2 can only pick a location, y, in the interval $A_2 = [0.8, 1]$. Given choices $x < y$, consumers go to the location closest to their own location, and what the consumer at the midpoint $\frac{1}{2}x + \frac{1}{2}y$ does has no effect on payoffs. But if $x = y = 0.8$, the consumers view the two as perfect substitutes, with $\frac{1}{2}$ patronizing one of the players and the remainder patronizing the other. The discontinuity of the payoffs at $(x, y) = (0.8, 0.8)$ means that there is no countably additive equilibrium, although there are ϵ equilibria for every $\epsilon > 0$.

Lemma 3.1. *With μ_1^* being a finitely additive* Z1 *putting mass* 1 *on* $(0.8-\epsilon, 0.8)$ *for all* $\epsilon > 0$ *and μ_2^* being the countably additive point mass on $a_2 = 0.8$, (μ_1^*, μ_2^*) is an equilibrium for the asymmetric Hôtelling game yielding equilibrium utilities* $(u_1^*, u_2^*) = (0.8, 0.2)$. *Further, along all exhaustive nets of finite approximations, all nets of equilibria that survive deletion of weakly dominated strategies have limits of this form.*

Proof. Verifying that μ^* is an equilibrium is immediate. For the rest, in any finite approximation $(F_{1,\alpha}, F_{2,\alpha})$ containing $(0.8, 0.8)$, let $h'_{1,\alpha} \in F_{1,\alpha}$ be the largest element smaller than 0.8. Any $h_{1,\alpha} < h'_{1,\alpha}$ in $F_{1,\alpha}$ is weakly dominated for 1 and any $h_{2,\alpha} > 0.8$ in $F_{2,\alpha}$ is weakly dominated by 0.8 for player 2. After eliminating these strategies in Γ_α, the unique equilibrium in the 2×1 game is point mass on the pair $(h'_{1,\alpha}, 0.8)$. Taking limits as $F_{1,\alpha}$ becomes exhaustive guarantees that for all $\epsilon > 0$, there exists an α such that for all $\beta \succsim \alpha$, $h'_{1,\beta} \in (0.8-\epsilon, 0.8)$. The limit Z1's must have $\mu_1^*((0.8-\epsilon, 0.8)) \equiv 1$ while μ_2^* is the countably additive point mass on 0.8. □

Starting with a set X, a *hyperfinite subset* is a set constructed from nets of finite subsets of X (using an ultrapower construction) in nonstandard analysis. A hyperfinite set is *exhaustive* if contains every finite subset of X, and these correspond to sets constructed from exhaustive nets of finite subsets. A key virtue of these sets is that they satisfy many of the same logical properties as finite sets.

The hyperfinite analysis of the asymmetric Hôtelling game replaces A_1 and A_2 with exhaustive hyperfinite sets H_1 and H_2.

A special point for player 1. For player 1, let h'_1 be the largest element of H_1 strictly less than than 0.8. This point exists because, far enough along the net of finite subsets that goes into the construction of H_1, there is a largest element strictly less than 0.8. Since H_1 is exhaustive, $0.8 - \epsilon \in H_1$ for every positive $\epsilon > 0$, which in turn means that

h_1' belongs to every interval $(0.8-\epsilon, 0.8)$, that is, h_1' is **infinitesimally close to** 0.8.

A special point for player 2. Because H_2 is exhaustive, $0.8 \in H_2$, and we let $h_2 = 0.8$.

Deletion of weakly dominated strategies. For player 1, any $h_1 < h_1'$ is weakly dominated by h_1'. For player 2, any $h_2' > 0.8$ is weakly dominated for player 2 by 0.8. After removing the weakly dominated strategies from the game, player 1 has only the strategies h_1' and 0.8 which player 2 has only the strategy, 0.8. Given that player 2 is playing 0.8, player 1's choice is between h_1', which yields a payoff infinitesimally close to 0.8 and 0.8, which yields the payoff of 0.5. The only Z1s corresponding to this are of the form given above.

3.2. **A Colonel Blotto Game.** Sion and Wolfe [1957] give an asymmetric, two battlefield Colonel Blotto game where both players have one unit of force to allocate between the battlefields. The player allocating the larger/smaller force to a battlefield wins/loses, and equal force allocations lead to ties. The payoffs are additive, with +1 for each battlefield won, −1 for each one lost, and 0 for ties. There is, however, one asymmetry in the game, player 2 starts with an advantage of 0.5 immobile units of force already present in the second battlefield.

Let x and $(1-x)$ denote player 1's force allocations to the first and second battle fields, and y and $(1-y)$, denote player 2's force allocations. The payoffs are given by

$$\begin{aligned} v_1(x,y) &= \operatorname{sgn}(x-y) + \operatorname{sgn}((1-x)-(1.5-y)), \text{ and} \\ v_2(x,y) &= -\, u_1(x,y). \end{aligned} \tag{2}$$

To make the payoffs stay in the interval $[-1,+1]$, Sion and Wolfe [1957] add +1 to player 1's payoffs so that utilities are given by

$$u_1(x,y) = \begin{cases} -1 & \text{if } x < y < x + \frac{1}{2} \\ 0 & \text{if } x = y \text{ or } y = x + \frac{1}{2} \\ 1 & \text{otherwise,} \end{cases} \tag{3}$$

and $u_2(x,y) = -u_1(x,y)$. Diagramatically, we can represent $u_1(\cdot,\cdot)$ as in Figure 1.

Sion and Wolfe [1957] show that this game has no countably additive ϵ-equilibria for a range of strictly positive ϵ. We give finitely additive representations of the limits of "reasonable" equilibria for this game, providing a contrast with both the Sion and Wolfe result and the literature on discontinuous games that has studiously avoided games of this sort.

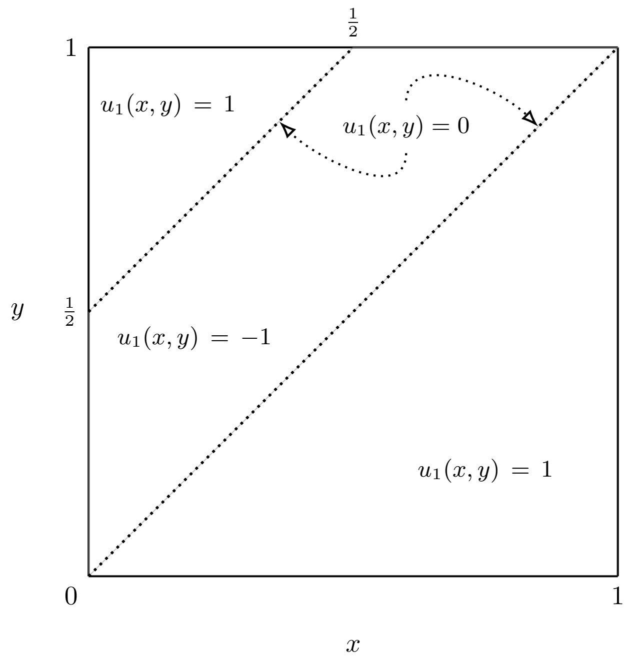

FIGURE 1. The Sion-Wolfe Game [1957]

The finitely additive equilibria depend on details of the hyperfinite action sets, equivalently on the details of the net of finite approximations. Player 2's advantage of 1/2 in the second battlefield means that fine details of the approximation around $(0.5, 0.5)$ can matter. Let H_1 and H_2 denote the two players' exhaustive hyperfinite action sets, and throughout, let h_i' denote player i's largest strategy strictly less than 0.5 in H_i.

We analyze two cases, the fully symmetric one and one of the asymmetric ones. In the fully symmetric case, we suppose that $H_1 = H_2$, equivalently, that the nets of finite approximations satisfy $F_{1,\alpha} = F_{2,\alpha}$. In the asymmetric case, we suppose that $h_1' < h_2'$, that is, that 2 can better approximate 0.5 from below than 1 can. This is equivalent to working withh nets of finite approximations $F_{i,\alpha}$ in which player 2's largest strategy less than 0.5 larger than the corresponding strategy for 1. The represent the equilibrium in the symmetric case, we must specify the product extension as the issues in Example 2.3 are crucial.

Lemma 3.2 (The symmetric version). *If the exhaustive hyperfinite sets satisfy $H_1 = H_2$ and h' is the largest element of H_i that is strictly less than* 0.5*, then after interated deletion of weakly dominated strategies, the unique equilibrium has player* 1 *playing* 0*, h' and* 1 *with probabilities*

1/5, 1/5 and 3/5 while player 2 plays h', 0.5, and 1 with probabilities 1/5, 1/5 and 3/5, and this delivers utilities $(2/5, -2/5)$.

Letting ν be a Z1 that puts mass 1 on every interval $(0.5-\epsilon, 0.5)$, we have $\mu_1^* = \frac{1}{5}\delta_0 + \frac{1}{5}\nu + \frac{3}{5}\delta_1$ and $\mu_2^* = \frac{1}{5}\nu + \frac{1}{5}\delta_{0.5} + \frac{3}{5}\delta_1$. To give the independent extension, μ^* of (μ_1^*, μ_2^*) that represents the hyperfinite equilibrium, we must represent the tie that happens in the hyperfinite game when both players play h'. To this end, let L_ϵ be the line joining the points $(0.5-\epsilon, 0.5-\epsilon)$ and $(0.5, 0.5)$ and let η be a Z1 on $[0,1]\times[0,1]$ that satisfies $\eta(L_\epsilon) = 1$ for all $\epsilon > 0$. The independent extension μ^* representing the equilibrium is given in the following table.

Mass	Z1's with the given mass
1/25	η, $\delta_0\times\nu$, $\delta_0\times\delta_{0.5}$, and $\nu\times\delta_{0.5}$
3/25	$\delta_0\times\delta_1$, $\nu\times\delta_1$, $\delta_1\times\nu$, and $\delta_1\times\delta_{0.5}$
9/25	$\delta_1\times\delta_1$.

In the next equilibrium, the product extension needs no extra specification.

Lemma 3.3 (An asymmetric version)**.** *If $h_1' < h_2'$ then there is a unique equilibrium in the hyperfinite game that survives iterated deletion of weakly dominated strategies. The finitely additive equilibrium that represents it is $\mu^* = (\mu_1^*, \mu_2^*)$ where μ_1^* is the countably additive mixture $\frac{1}{3}\delta_0 + \frac{2}{3}\delta_1$ and μ_2^* is the finitely additive $\frac{1}{3}\eta + \frac{2}{3}\delta_1$ where η is a Z1 satisfying $\eta((0.5-\epsilon, 0.5)) \equiv 1$ and the equilibrium utilities are $(u_1^*, u_2^*) = (\frac{1}{3}, -\frac{1}{3})$.*

In particular, along all exhaustive filters of finite approximations where player 2's largest element strictly less than 0.5 is larger than player 1's, all limits of equilibria that survive iterated deletion of weakly dominated strategies are of this form.

3.3. **The Largest Integer Game.** The action sets are $A_1 = A_2 = \mathbb{N}$. With $\Phi(\cdot)$ a strictly increasing cdf on $[0,\infty)$, the utilities are given by $u_i(a_i, a_j) = \text{sgn}(a_i - a_j) + \Phi(a_i)$. If one player is using a countably additive strategy, then the supremum of the achievable payoffs for the other player is 2. Since this is true for both players and the supremum of the $\Phi(\cdot)$ part of the utility function is 1, there is no countably additive ϵ-equilibrium for any $\epsilon < 1$.

Replace each A_i by an exhaustive hyperfinite H_i and let $\bar{h}_i$ be the maximal element of H_i. Since every strategy $h_i' \in H_i$ is strictly dominated by any larger element of H_i, the unique equilibrium is $(\bar{h}_1, \bar{h}_2)$. There are three cases: $\bar{h}_1 > \bar{h}_2$; $\bar{h}_1 = \bar{h}_2$; and $\bar{h}_1 < \bar{h}_2$. Corresponding to these three cases are the three purely finitely additive equilibria that arise as limits of exhaustive hyperfinite filters: the corresponding equilibrium payoffs, (u_1^*, u_2^*), are either $(2,0)$, $(1,1)$, or $(0,2)$. In each case, if μ^* is the finitely additive equilibrium, then it puts mass "to the right of" all of the actions in $\mathbb{N}$, that is to say, for any $(N_1, N_2) \in A_1 \times A_2$, $\mu^*(\{(n_1, n_2) : n_1 \geq N_1,\ n_2 \geq N_2\}) = 1$.

In the symmetric version of the Sion-Wolfe game, we needed to explicitly give the product extension, in this game, we must also do this.

(1) Because of the failure of Fubini's theorem demonstrated in Example 2.4, the marginals of the equilibrium do not determine payoffs. In this game, perhaps the easiest way to see this is to use hyperfinite sets rather than nets. [Stinchcombe, 2023, Cor. 5.2, p. 648] shows that for any Z1, p, putting mass "to the right of" $\mathbb{N}$, and any infinite integer $N \in {}^*A_i$, there is a pair of infinite integers m, M with $m < N < M$ with $p(B) = \delta_m({}^*B) = \delta_M({}^*B)$ for all $B \subset \mathbb{N}$. Thus, even if the marginals of an equilibrium μ^* are equal, the payoffs may be either $(2,0)$, $(1,1)$, or $(0,2)$ depending on the product extension that represents the hyperfinite equilibrium.
(2) When using the finitely additive strategies, the original sets of strategies for the players do *not* yield a dense set of possible payoffs. The strategy μ^* involves both players playing "to the right of" the set $\mathbb{N}$. This means that it is possible that an equilibrium μ^* delivers the payoffs $(u_1^*, u_2^*) = (2,0)$ while we simultaneously have $\sup_{n_1 \in A_1} u_1(\mu^* \backslash n_1) = 0 < 2$.

The last point demonstrates the relative ease of using hyperfinite sets, or nets of finite sets, to analyze the finitely additive equilibria.

4. Properties of the Set of Equilibria

One might worry that the use of "exotic" probabilities would deliver an equilibrium theory that is not recognizable. This section is meant to systematically allay some of those worries and give a more nuanced view of when the "exotic" does and does not arise.

For a game $\Gamma(u) = (A_i, u_i)_{i \in I}$, $Eq(\Gamma(u))$ denotes the set of finitely additive equilibria when the utility function is given by $u = (u_i)_{i \in I}$. The basic structural results from finite games carry over with much the same proofs. Theorem B shows that the set of equilibria is closed and that the equilibrium correspondence from utility functions to the corresponding set of equilibria is upper hemi-continuous.

Theorem C concerns equilibria that put zero mass on the set of iteratively weakly undominated strategies. The iterated deletion of weakly dominated strategies is one of the most powerful and widely used methods for finding a "reasonable" set of equilibrium predictions from a game theory model. The "exotic" properties of finitely additive equilibria are on full display here, but these properties are necessary to capture phenomena of game theoretic interest even in the tamest class of infinite action games.

Example 4.1 gives a two-person game on $[0,1] \times [0,1]$ having jointly continuous utility functions in which the set of iteratively undominated strategies for either player belongs to $(0, 1/2^n]$ for every $n \in \mathbb{N}$. In the original set of strategies, there is no representation for this set. In every element of an exhaustive net of finite approximations, there is an action that is not weakly dominated, and for large finite sets, that action is close to 0. In a direct parallel, the finitely additive equilibria that represent play of these actions are Z1's that puts mass 1 on $(0, \epsilon)$ for every $\epsilon > 0$.

4.1. **Basic Structural Results.** We parametrize games by their utility functions, $u \mapsto \Gamma(u)$ where $u = (u_i)_{i \in I}$ with each $u_i : A \to [-B, +B]$ for some $B > 0$. We give the set of utility functions the product supnorm topology, that is, $u^\alpha \to u$ if (and only if) for all $i \in I$, the supnorm distance, $\|u_i^\alpha(\cdot) - u_i(\cdot)\|$, converges to 0.

The proofs of the following claims are minor variants on the textbook arguments for finite games. Recall that a correspondence with a closed graph is upper hemi-continuous if the range is a compact Hausdorff space.

Theorem B. *For any game* Γ *satisfying the assumptions* (1)-(3) *above,*
(1) $Eq(\Gamma(u))$ *is a closed subset of* Δ, *and*
(2) *the equilibrium correspondence set* $\{(u,p) : p \in Eq(u)\}$ *is closed.*

Proof. Let p^α be a net in $Eq(\Gamma)$ and suppose that $p^\alpha \to p$. Because each p^α is an equilibrium, for all α, for all $i \in I$, and all $b_i \in A_i$, $\int u_i(a)\, dp^\alpha(a) \geq \int u_i(a \backslash b_i)\, dp^\alpha(a)$. Since $p^\alpha \to p$ and each $u_i(\cdot)$ is bounded,

$$\begin{aligned} &\textstyle\int u_i(a)\, dp^\alpha(a) \to \int u_i(a)\, dp(a) \text{ and} \\ &\qquad\qquad \textstyle\int u_i(a\backslash b_i)\, dp^\alpha(a) \to \int u_i(a\backslash b_i)\, dp(a), \end{aligned} \tag{4}$$

which implies that $\int u_i(a)\, dp(a) \geq \int u_i(a\backslash b_i)\, dp(a)$.

For upper hemi-continuity, take arbitrary nets $p^\alpha \to p$ and $u^\alpha \to u$ with $p^\alpha \in Eq(u^\alpha)$. We must show that $p \in Eq(u)$. Suppose, for the purpose of establishing a contradiction, that p is not an equilibrium of $\Gamma(u)$. This requires that for some $i \in I$ and some $b_i \in A_i$, there exists a strictly positive r such that

$$(\ddagger) \quad \textstyle\int u_i(a)\, dp = \int u_i(a\backslash b_i) - r.$$

From the triangle inequality,

$$\left|\int u_i^\alpha \, dp^\alpha - \int u_i \, dp\right| \leq \left|\int u_i^\alpha \, dp^\alpha - \int u_i \, dp^\alpha\right| + \left|\int u_i \, dp^\alpha - \int u_i \, dp\right|. \quad (5)$$

The first term on the right goes to 0 because $\|u_i^\alpha - u_i\|$ goes to 0 and the second term goes to 0 because $p^\alpha \to p$. By the same argument,

$$\left|\int u_i^\alpha(a\backslash b_i) \, dp^\alpha(a) - \int u_i(a\backslash b_i) \, dp(a)\right| \to 0. \quad (6)$$

Thus, there exists an α such that for all $\beta \succsim \alpha$, both differences are smaller than $r/2$, a contradiction to (‡). □

4.2. **Equilibria in Iteratively Undominated Strategies.** In §3, we replaced the action sets of the players by exhaustive hyperfinite sets or by exhaustive nets of finite approximations. In each case, we found unique equilibrium utilities after iterated deletion of weakly dominated strategies. The next result, Theorem C, shows that this process always delivers a non-empty, closed set of equilibria.

Our use of finitely supported probabilities in the definition of weakly dominated strategies is neither usual nor without loss of generality. However, it is appropriate for our exhaustive hyperfinite approach to games.[6]

Definition 4.1. *In a game $\Gamma = (A_i, u_i)_{i \in I}$, an action $c_i \in A_i$ is* ***weakly dominated for*** *i if there is a finitely supported probability q_i on A_i such that for all $a \in A$, $u_i(a\backslash q_i) \geq u_i(a\backslash c_i)$ and the inequality is strict for at least one a.*

Iterated dominance is defined as usual.

Definition 4.2. *For a game $\Gamma^0 = (A_i^0, u_i^0)_{i \in I}$, for each $i \in I$, let D_i^0 denote the set of weakly dominated strategies in Γ^0, let A_i^1 denote $A_i^0 \backslash D_i^0$, and define $\Gamma^1 = (A_i^1, u_i^1)_{i \in I}$ by restricting each u_i to $\times_{j \in I} A_j^1$. Iteratively apply this: given a game $\Gamma^n = (A_i^n, u_i^n)_{i \in I}$, let D_i^n denote i's weakly dominated strategies, let $A_i^{n+1} = A_i^n \setminus D_i^n$; and define $\Gamma^{n+1} = (A_i^{n+1}, u_i^{n+1})_{i \in I}$ by restricting each u_i to $\times_{j \in I} A_j^1$. Finally, let $A_i^\infty = \cap_{n \in \mathbb{N}} A_i^n$ and define the* ***game in iteratively undominated strategies*** *as $\Gamma^\infty = (A_i^\infty, u_i^\infty)_{i \in I}$ by restricting each u_i to $\times_{j \in I} A_j^\infty$.*

A game is finite if the set of players is finite and each player's set of actions is finite. For finite games, iterative deletion of weakly dominated strategies always leaves a non-empty set of strategies. We use this to take limits of equilibria in weakly undominated strategies: since

[6] We would like to note that, except when we have gone looking for games with such properties, we have not seen games where there is a distinction between strategies weakly dominated by finitely supported strategies and those weakly dominated by infinitely supported strategies.

the nets of finite approximations are exhaustive, any iteratively weakly dominated strategy is eventually excluded from every finite approximation to the game along the net.

Theorem C. *For any finite player game* Γ*, the set of finitely additive equilibria that represent the limits of equilibria in iteratively undominated strategies along exhaustive nets of finite approximations is a closed non-empty subset of the equilibria of* Γ *that put zero mass on the set of iteratively weakly dominated strategies.*

The following Example is a variant on one in Simon and Stinchcombe [1995]. It shows that situation is quite different for countably additive equilibria in compact games with jointly continuous utility functions.

Example 4.1. *With* $I = \{1, 2\}$ *and* $A_i = [0, 1]$*, suppose that the jointly continuous utility functions* $u_i(\cdot, \cdot)$ *take values in* $[0, 1]$*, are symmetric,* $u_1(a_1, a_2) = u_2(a_2, a_1)$*, and have the following properties: if either player plays* 0*, then both receive a utility of* 0*; for each* $a_j^\circ > 0$*,* $a_i \mapsto u_i(a_i, a_j^0)$ *is strictly increasing on* $[0, a_j^\circ/2]$*, and strictly decreasing on* $[a_j^\circ/2, 1]$*.*

We give the analysis of Example 4.1 in four steps, the last of which is a summary.

Step 1. **The unique countably additive equilibrium**. Whatever countably additive strategy j plays, i's best responses are a subset of $[0, 1/2]$ (and by symmetry, this is true for both players). Given that player j puts puts mass 1 on $[0, 1/2]$, player i's best responses are a subset of $[0, 1/2^2]$. By induction, for both players, best responses must put mass 1 on each set $[0, 1/2^n]$. The unique countably additive probability satisfying this is point mass on 0, hence the unique equilibrium is for both to play the weakly dominated strategy 0 with probability 1.

Step 2. **An empty set of iteratively undominated strategies**. For both players, playing 0 is weakly dominated, and in the original game, every a_i in the interval $(1/2, 1]$ is weakly dominated by $a_i = \frac{1}{2}$ because for each $a_j \neq 0$, $u_i(\cdot, a_j)$ is strictly decreasing on $[1/2, 1]$. The weakly undominated strategies for each player are a subset of $(0, 1/2]$. After deleting the weakly dominated strategies from the game, the weakly undominated strategies are a subset of $(0, 1/2^2]$. By induction, the set of iteratively weakly undominated strategies is $\cap_{n\in\mathbb{N}}(0, 1/2^n] = \emptyset$.

Step 3. **The exhaustive hyperfinite analysis**. With H_i an exhaustive hyperfinite representation of A_i, $i = 1, 2$, the previous step implies that the only iteratively undominated strategies are the h_i that are strictly positive and smaller than any $\epsilon > 0$, i.e. they are infinitesimals. The finitely additive strategies corresponding to this put mass 1 on each $(0, \epsilon)$.

Step 4. **Summary**. The game has an empty set of iteratively undominated strategies because they must belong to each $(0, \frac{1}{2^n}]$, a collection of sets with empty intersection. The finitely additive equilibria put mass 1 on all sets of the form $(0, \epsilon)$, a related collection of sets with empty intersection.

Note the difference between Theorems A and C, the first includes the countably additive equilibria putting full mass on the weakly dominated set of strategies, while the second excludes all equilibria putting mass on that set.

We have arrived at a point where the mismatch between our approach and the literature on countably additive equilibrium existence for infinite games with discontinuous payoffs is very apparent. The finitely additive equilibria in Theorem C represent the limits of equilibria of arbitarily large finite approximations to the game that survive iterated deletion of weakly dominated strategies. These equilibria encompass game theoretic concepts that countably additive equilibria cannot capture, at least not in much generality: only one of the two-person games in §3 has approximate countably additive equilibria, and even that one has no countably additive equilibrium; in the Example just given, there are iteratively undominated Z1's, but no countably additive equilibrium can play iteratively undominated strategies.

Still, there are connections with the study of equilibrium existence for infinite games which has assiduously avoided the discontinuities that preclude countably additive equilibrium existence. Though the structure of the discontinuities are not a matter of concern for our theoretical analysis, it seems worth investigating aspects of the differences in the approaches.

5. Special Discontinuities for Compact Games

For games compact sets of actions, arbitrary player sets, and jointly continuous utility functions, the countably additive equilibria are product probabilities defined on the product σ-field generated by the Borel σ-fields on the players' action sets. Every Borel probability has a compact and convex set of finitely additive total extensions (e.g. Example 2.1). From Corollary A.1, the set of finitely additive equilibria and the set of countably additive Borel equilibria are extensions or restrictions of each other for compact and continuous games. In particular, they yield the same equilibrium payoffs to the players, and we take this property as a starting point.

In this section, we investigate, for finite player games with compact metric spaces of actions, the class of Borel measurable utility functions with discontinuities special enough that the equilibrium utilities are the same for finitely additive total equilibria and countably additive Borel equilibria: Theorem D gives sufficient conditions for all of the finitely additive continuous equivalents of a countably additive equilibrium to be equilibria yielding the same payoffs; and Theorem E goes in the other direction, providing conditions under which all of the continuous equivalents of a finitely additive equilibrium are equilibria yielding the same payoffs.

At a conceptual level, the difficulties arise from a peculiar mismatch: this literature has used, and we will also use, the idea weak* topology for countably additive probabilities. But the utility functions do not integrate these probabilities continuously when we use this topology. As a result, the sufficient, and nearly necessary, conditions for finitely and countably additive continuously equivalent equilibria to yield the same utilties include the requirement that the equilibria stay away from the discontinuities.

5.1. **Continuous Equivalence.** The following is the central concept used in this section.

Definition 5.1. *Probabilities p and p' on a metric space are **continuously equivalent**, i.e. equivalent in the τ_{C_b}-topology, if they integrate all bounded continuous functions to the same number.*

It is a classic result that two countably additive Borel probabilities on a metric space are continuously equivalent if and only if they are equal. But this is far from true for total probabilities, and it is even false for finitely additive Borel probabilities. Examples make this point.

Example 5.1. *On the compact metric space $[0, 1]$, a finitely additive p is continuously equivalent to the countably additive point mass on 0, $q = \delta_0$ if and only if it puts mass 1 one every half-open set $[0, \epsilon)$. But, for example, the equilibria in Example 4.1 involve finitely additive p's that put mass 1 on each open set $(0, \epsilon)$. For the upper semicontinuous function $f(\cdot)$ with $f(0) = 1$ and $f(x) = 0$ for $x > 0$, we have $\int f(x)\,dq(x) = 1 > 0 = \int f(x)\,dp(x)$.*

Thinking of q in this Example as the equilibrium in a 1-person game, we see that if an equilibrium q puts mass on the discontinuities of the utility function, the continuously equivalent finitely additive p's need not be equilibria.

Continuous equivalence misses phenomena of game theoretic importance. In the following game, there are finitely additive equilibria that Pareto dominate the non-empty set of countably additive equilibria.

Example 5.2. *Player* 1 *picks an* $x \in [0,1]$ *and an action* a *in the two point set* $\{\alpha, \beta\}$. *Player* 2 *picks a* $y \in [0,1]$. *The utilities are given by*

$$u((x,\alpha),y) = ((2-x)(2-y),(2-x)(2-y)) \quad \textit{and} \tag{7}$$

$$u((x,\beta),y) = \begin{cases} (7(x+1)(y+1), 7(x+1)(y+1)) & \textit{if } (x,y) \neq (1,1) \\ (0,0) & \textit{if } (x,y) = (1,1). \end{cases}$$

Analysis. Play of $((0,\alpha),0)$ with probability 1 is the unique countably additive equilibrium. It yields equilibrium utilities $(4,4)$. There are finitely additive equilibria p^* that involve player 2 putting mass 1 on each open interval of y's of the form $(1-\epsilon,1)$ while player 1 puts mass 1 on the (x,β) with x in the same class of intervals. The associated equilibrium utilities are the Pareto dominant vector $(28,28)$. The countably additive $q = ca(p^*)$ that is continuously equivalent to p^* is a point mass that yields utilities $(0,0)$, which are the minimal utilities possible for the players in this game.

5.2. **Continuous Equivalence for Equilibria.** We are interested, first, in the behavior of *all* of the finitely additive continuous equivalents of a given countably additive equilibrium. We then turn to the question of whether the countably additive Borel version of a finitely additive equilibrium is necessarily an equilibrium giving the same payoffs.

5.2.1. *Upper- and Lower-Semicontinuity.* The arguments pass through two Lemmas of independent interest. For perspective on the central role that upper and lower semi-continuous functions will play, both in the first Lemma and in the two results, it is worth noting that for countably additive probabilities, $q^n \to_{C_b} q$ if and only if for all bounded upper semi-continuous functions f, $\limsup_n \int f\, dq^n \leq \limsup_n \int f\, dq$, with the reverse inequality for lower semi-continuous functions.

We will use the first Lemma to compare the utilities of deviations against finitely additive strategies and their countably additive continuously equivalent versions.

Lemma 5.1. *For* X *a compact metric space and* $f : X \to \mathbb{R}$ *a bounded upper semi-continuous function, if* p *is a total finitely additive probability on* X *and* $q = ca(p)$ *is its countably additive version, then* $\int f(x)\, dq(x) \geq \int f(x)\, dp(x)$, *and the inequality reverses if* f *is lower rather than upper semi-continuous.*

5.2.2. *The Continuous Mapping Theorem.* The next Lemma is directly analogous to the classical continuous mapping theorem for countably additive probabilities. That result tells us that if a countably additive q puts mass 0 on the discontinuities of a bounded function f and q_n is a sequence of countably additive probabilities with q_n converging to q in the weak* topology for countably additive probabilities, then $\int f\,dq_n \to \int f\,dq$. We will use the Lemma to analyze the set of games for which the finitely additive equilibria and their countably additive versions deliver the same utilities to the agents.

Lemma 5.2. *For X a compact metric space, $f : X \to \mathbb{R}$ a bounded Borel measurable function, and q a countably additive Borel probability, if q puts mass 0 on the closure of the discontinuities of f, then $\int f\,dq = \int f\,dp$ for all finitely additive p that are continuously equivalent to q.*

5.2.3. *The Relations.* We are now prepared for the result about the finitely additive continuous equivalents of a countably additive equilibrium.

Theorem D. *Suppose that q^* is a countably additive equilibrium for a finite player game with compact metric spaces of actions. If q^* puts mass 0 on the closure of the discontinuities of $u : A \to \mathbb{R}^I$ and for all $i \in I$ and all $b_i \in A_i$, the mapping $a \mapsto u_i(a\backslash b_i)$ is upper semi-continuous, then every finitely additive p that is continuously equivalent to q^* is an equilibrium that gives the same expected utility payoffs as q^*.*

Proof. By Lemma 5.2, any continuously equivalent p satisfies $\int u\,dq^* = \int u\,dp$. Since we also know that q^* is an equilibrium, for all $i \in I$ and all $b_i \in A_i$, we have

$$\int u_i(a)\,dp(a) = \int u_i(a)\,dq^*(a) \geq \int u_i(a\backslash b_i)\,dq^*(a). \tag{8}$$

Pick arbitrary $i \in I$ and $b_i \in A_i$. By assumption, $a \mapsto u_i(a\backslash b_i)$ is upper semi-continuous. By Lemma 5.1, $\int u_i(a\backslash b_i)\,dq^*(a) \geq \int u_i(a\backslash b_i)\,dp(a)$. Combining, for all $i \in I$ and all $b_i \in A_i$, $\int u_i(a)\,dp \geq \int u_i(a\backslash b_i)\,dp$ so that p is an equilibrium. □

We now give the result about the countably additive continuous equivalents of a finitely additive equilibrium.

Theorem E. *Suppose that p^* is a finitely additive equilibrium for a finite player game with compact metric spaces of actions and that $q^* = ca(p^*)$ is the countably additive version of p^*. If q^* puts mass 0 on the closure of the discontinuities of $u : A \to \mathbb{R}^I$ and for all $i \in I$ and all $b_i \in A_i$, the mapping $a \mapsto u_i(a\backslash b_i)$ is lower semi-continuous, then q^* is an equilibrium that gives the same expected utility payoffs as p^*.*

The omitted proof is a mirror image of the proof for Theorem D using the lower semi-continuous part of Lemma 5.2.

5.3. **A Different Set of Discontinuities.** The condition that the countably additive strategy put mass 0 on the closure of the discontinuity points is violated in all of the games in §3. Those are constant sum, two-player games, and for such games, we know that if one player's payoff jumps up at a discontinuity, the other's jumps down. The discovery of the efficacy of this kind of reciprocality at the discontinuity points utility functions in guaranteeing the existence of countably additive equilibria was one of the early breakthroughs in the program to identify well-behaved discontinuities (see Simon [1987]). Following the masterful synthesis and extension of that literature in Reny [2020], we have the following condition on the discontinuities of the payoffs for countably additive mixed strategies.

Definition 5.2. *For a finite player game $\Gamma = (A_i, u_i)_{i \in I}$ where each A_i is a compact metric space, the utility function $u(\cdot)$ has* ***well-behaved discontinuities*** *if for all countably additive mixed strategies $q = (q_i)_{i \in I}$ that are not equilibria, there is an open neighborhood G_q and a weak***-continuous $q \mapsto \varphi(q) = (\varphi_i(q_{-i})_{i \in I})$ with the property that for all $q' \in G_q$, there is at least one player j such that $u_j(q'_{-j}, \varphi_j(q'_{-j})) > u_j(q')$.*

The argument that a finite player, compact metric space game with well-behaved discontinuities has an equilibrium proceeds as follows: the sets of countably additive probabilities Δ_i^{ca} and Δ^{ca} are both compact in the weak* topology for countably additive probabilities, if there is no equilibrium, an assumption made for the purpose of estabilishing a contradiction, then we can take an open cover of Δ^{ca} using the open sets G_q; by compactness we can take a finite open subcover; using partitions of unity, we can glue together the functions φ associated with each element of the finite open cover into a single continuous function; since the Δ_i and Δ are also convex, the generalization of Brouwer's fixed point theorem to compact and convex metric vector spaces guarantees that this function must have a fixed point, q°. But this is absurd because it requires that at q°, somebody can change nothing and increase their payoff. This shows that for compact games with well-behaved discontinuities, there is a non-empty, closed set of countably additive equilibria.[7]

[7]One of the key advances in this literature replaced the continuous $\varphi(\cdot)$ by a correspondence having the fixed point property as well as the same "someone does strictly better" property.

The genius of the work that has used this or a similar strategy of proof for equilibrium existence lies in finding conditions that are easy to verify, widely applicable, and yet are still sufficiently restrictive as to preclude the sort of discontinuities on display in §3. Many but not all of the applications that have used well-behaved discontinuities have equilibria that put mass 0 on the closure of the discontinuities of the utility functions. To apply the results of the last two subsections to those games requires a bit more, either an upper or a lower semi-continuity condition. For some but not all of the applications, one or the other of these additional assumptions are satisfied. But a characterization of the compact games with well-behaved discontinuities for which one can pass back and forth between finitely and countably additive equilibria without changing the set of utilities seems likely to be quite elusive.

In any event, the contrast between our approach and the study of games with well-behaved discontinuities is striking — we make no assumptions on the utility function beyond boundedness.

6. Summary and Future Directions

There has always been a jarring cognitive gap between games with finite sets of actions and games with compact sets of actions. Finite games always have equilibria and finite sets of actions can approximate compact sets both exhaustively and uniformly. But unless we can appeal to special properties of the discontinuities in the utility functions, the equilibria of the finite approximations seem to have no counterparts in countably additive strategies for compact games.

This paper shows that the cognitive gap disappears if strategies are understood as finitely additive mixtures, that the gap is an artifact of the insistence on modeling the equilibria for games as countably additive strategies. That being said, there are many context dependent objections to the use finitely additive mixtures. From our point of view, if the context dependent objections to finitely additive mixtures are to have bite in game theory, they must provide a principled repudiation of the validity of finite approximations to games.

Jackson and Swinkels [2005] argue that equilibrium phenomena that depend on the fine structure of the approximations are "pathological." To at least partly avoid this issue, we include as equilibria all of the possible limits along exhaustive nets of finite games. Exhaustiveness dampens some of the variability by guaranteeing that all strategies are considered possible, and the inclusion of all limits leads to a theory that does not depend on the fine structure of the approximations.

Further, Theorem B shows that in game theory, the extent to which the finitely additive equilibria are "exotic," "peculiar," or "pathological" is somewhat limited. The closedness and upper hemi-continuity structural results about the set of equilibria for finitely additive equilibria directly mirror the properties that hold for finite games, and for the compact and continuous games with which Glicksberg [1952] began the systematic study of countably additive equilibrium existence in infinite games.

We believe that finitely additive mixed strategies has the potential to make many more contributions to to game theory. There are two areas where the techniques developed here can provide some immediate advances, extensive form games with infinite sets of actions, and infinite population games.

6.1. **Infinite Extensive Form Games.** There are several well-studied examples of extensive form games with infinite sets of actions where early choices upper hemi-continuously determine the set of later equilibrium payoffs in such a fashion as to preclude the existence of countably additive equilibria. In mechanism design models with competing principals, Myerson [1982] shows that: the set of incentive compatible mechanisms can explode upper hemi-continuously; that it is possible that early mover payoffs are strictly increasing as one nears the upper hemi-continuous explosion point; but any of the equilibrium actions taken in the larger sets necessarily force the early mover's payoffs to jump downwards. In a similar fashion, Manelli [1996] analyzes a signaling game in which the follower's best response correspondence explodes upper hemi-continuously. In approaching the discontinuity, the sender's payoff is strictly increasing, but at the discontinuity, the payoffs to later equilibrium play necessarily induce the payoffs to jump downwards. In such examples, a finitely additive equilibrium can capture the earlier player avoiding the downward jump while suffering the least possible utility consequences of avoiding the discontinuous penalty.

Another issue for countably additive mixtures in extensive form games is the "disappearance" of information in the limit. Myerson and Reny [2020, p. 497] write that

> . . . the difficulty is that the randomized signals upon which players coordinate their actions along the sequence can, in the limit, have distributions that degenerate to a point, leaving the players without access to the necessary coordination device.

This kind of disappearance of information at a point is an artifact of the insistence on countable additivity and the routine application of the weak* topology for countably additive Borel probabilities. That topology was developed for continuous problems and this is a discontinuous problem. To put it slightly differently, the disappearance depends on making a particular choice for what is meant by "in the limit."

One can see the preservation of information in the limit in the largest integer game analyzed in §3.3. There, the limit was an extreme point in the set of probabilities, i.e. a Z1. And yet it retains the information about the relative size of the players' choices along the approximating net. Stinchcombe [2023, Corollary 5.1] shows that the preservation of information in the limit is far more general than this indicates. For example, finitely additive probabilities that put unit mass on any interval $(0, \epsilon)$ can encode any distribution on any Polish metric space. The intuition is that there are uncountably many infinitesimals just to the right of 0, and functions of a uniform distribution on a hyperfinite subset of them provides the necessary variability.

All of that being said, there is still a large conceptual difficulty to be overcome for infinite extensive form games. Consider a game model in which players sequentially choose actions in, say, $[0, 1]$, and the later actions are chosen on the basis of signals that are continuous functions of the early play. There are two very different options for finite approximations to this game. One could, (A), exhaustively replace each space $[0, 1]$ and analyze the resulting net of finite extensive form games, or (B), one could start with the set of pure strategies as measurable functions from signals to later choices and replace the set of measurable functions with an exhaustive net of finite sets. The choice matters. Stinchcombe [2005, Example 2.5, p. 340] gives a game for which modeling strategy (A) gives a substantively different set of strategic structures than modeling strategy (B).

Despite the extensive argumentation in Myerson and Reny [2020] against (A) and in favor of a hybrid approach, it seems to us that to develop a finitistic[8] theory of extensive form games, a choice between the (A) and (B) must be made. For us, there is an internal consistency to exhaustively replacing the sets of actions by exhaustive hyperfinite sets or exhaustive nets. And that internal consistency is of a piece with the idea that models of infinite sets should reflect their finite origins. For game theory at least, the set of so-called "real" numbers developed by d'Alembert, Cauchy, Bolzano, Weierstrass and the mathematicians that have followed is the wrong tool.

[8]Many thanks to Don Brown for this evocative word.

6.2. **Infinite Population Games.** We know, from Theorem A, that equilibria exist for games that use any infinite set to model the players. We have several preliminary results but we do not have the full picture of how that result relates to the equilibria of the continuum population game models first studied by Schmeidler [1973]. An summary of the extensive literature that followed on this can be found in Khan and Sun [2002]. The analyses have (a) worked with a variety of countably additive nonatomic probability structures for the space of players, but crucially, (b), until very recently, that work has posited that the agents can and do correctly observe the true population distribution of actions.[9]

As to (a), extending the countably additive probability on a limited σ-field of sets of players to a larger one is always possible. A central question is how sensitive the set of equilibria is to the choice of extension. As background for such an investigation, we have used the Mas-Colell [1984] distributional equilibrium approach to continuum population games. In that approach, one defines an equilibrium as a joint distribution of agent characteristics and actions having the mutual best response property. Because one focuses on the induced joint distribution on player characteristic-action pairs, the measure theoretic differences in the population model play a much smaller role. In particular, this smaller role means that the entire issue of whether or not an equilibrium in pure strategies exists is thoroughly submerged.

Putting aside the existence of pure strategy equilibria, in studying distributional equilibria, the arguments behind Corollary A.2 on the continuous equivalence of games on Polish spaces play a central role. Provided that the distribution of agent characteristics is near-tight, one can show that exact countably additive distributional equilibria exist, and that a finitely additive joint distribution on characteristics and actions is an equilibrium if and only if it is continuously equivalent to the countably additive equilibrium. Going back to the issue of the many finitely additive extensions, we strongly conjecture that the finitely additive extensions of the continuuum population model give rise to the different continuously equivalent equilibria as exact finitely additive equilibria.

As to (b), the assumed correctness of how the population sees and interprets what is happening in the world seems far too limiting for

[9] Recent advances include Cerreia-Vioglio et al. [2022], who model players as correctly observing the true distribution of the actions taken by those in the player's peer or comparison group, and Frick et al. [2022], who model players as using the biased sample of the people that they interact with is representative of the entire population.

present day phenomena, even when people use non-representative samples as in Cerreia-Vioglio et al. [2022] and Frick et al. [2022]. We have begun the study of models in which each individual lives in their own version of reality. In these models, evidence and data is selectively curated for each individual by the advanced pattern recognition software currently deployed by profit maximizing social media firms that value addiction of their customers over accuracy. There are still objective consequences to population choices, and when they directly impinge on the people in the model, in equilibrium, they cannot be ignored. We have found it extremely convenient to start the analysis knowing that equilibria exist, and that they can be both interpreted and analyzed as limits of equilibria for finite approximations to the strategic situation being modeled.

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Appendix: Proofs Omitted from the Text

Throughout, our use of nonstandard analysis happens in a fully saturated superstructure that contains the game Γ. For a textbook coverage of this material, see Hurd and Loeb [1985], for a coverage that builds closely on the sequence- and net-based intuitions that permeate analysis, see Lindstrøm [1988].

For any probability $P \in {}^*\Delta$, we define $p = \text{st}(P)$ as the standard part in the compact Hausdorff $\tau_{\mathbb{B}}$-topology. This is equivalent to defining $p(B)$ as the standard part of the number $P({}^*B)$.

It is worth noting that we are not using the Loeb measure construction. We are instead going back to an older and more direct nonstandard representation of probabilities, see Stinchcombe [2023] for a detailed coverage of this approach.

Proof of Theorem A. Let I_F and J_F be finite set of agents and for $i \in I_F$, let B_i be a finite subset of A_i and for $j \in J_F$, let C_j be a finite subset of A_j. Define a partial order by

$$(I_F, (B_i)_{i \in I_F}) \succ (J_F, (C_j)_{j \in J_F}) \tag{9}$$

if $J_F \subset I_F$ and for all $j \in J_F \subset I_F$, $C_j \subset B_j$. By comprehensiveness, there exists a hyperfinite $(I_H, (H_i)_{i \in I_H})$ that is larger in the partial order than every finite $(I_F, (B_i)_{i \in I_F})$. Let $H = \times_{i \in I_H} H_i$.

Pick an arbitrary $z \in {}^*A$. For each $b \in H = \times_{i \in I_H} H_i$, define $v_i(b; z) = {}^*u_i(z \backslash b)$ where for $j \notin I_H$, $(z \backslash b)_j = z_j$ and for for $i \in I_H$, $(z \backslash b)_i = b_i$. By transfer of Nash's equilibrium existence theorem, the game $\Gamma_H(z) = (H_i, v_i(\cdot; z))_{i \in I_H}$ has an equilibrium, $(\gamma_i^*)_{i \in I_H}$. For $j \in {}^*I \setminus I_H$, set γ_j^* as point mass on z_j. Let γ^* be the product distribution induced by $(\gamma_i^*)_{i \in {}^*I}$ on ${}^*A$, define $\mu_i^* = \text{st}(\gamma_i)$ and $\mu^* = \text{st}(\gamma^*)$.

It is immediate that μ^* is a product extension of $(\mu_i^*)_{i \in I}$. And since every $i \in I$ belong to I_H, and for every $i \in I$, every $b_i \in A_i$ belongs to H_i, for all $i \in I$ and all $b_i \in A_i$, $u_i(\mu^*) \geq u_i(\mu^* \backslash b_i)$. $\square$

We now turn to the analysis of the symmetric hyperfinite version of the Sion-Wolfe game.

Proof of Lemma 3.2. Every $h_1 \in H_1 \cap (\frac{1}{2}, 1)$ is weakly dominated by $a_1 = 1$, and every $h_2 \in H_2 \cap [0, h')$ is weakly dominated by h'. After these strategies are eliminated from the game, every $h_1 \in (0, h') \cap H_1$ is weakly dominated by $a_1 = 0$, and 0.5 is weakly dominated by $a_1 = 1$. When 1 is only using the strategies $0, h'$, and 1, the only weakly undominated strategies for player 2 are the strategies $h', 0.5$, and 1. The payoffs in the resultant 3×3 game are given by

		Player 2		
		h'	0.5	1
	0	$(-1, +1)$	$(0, 0)$	$(+1, -1)$
Player 1	h'	$(0, 0)$	$(-1, +1)$	$(+1, -1)$
	1	$(+1, -1)$	$(+1, -1)$	$(0, 0)$

Direct examination shows that there is no pure strategy equilibrium, and that the unique distribution for player 2's actions that makes 1 indifferent is $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$ on $(h', 0.5, 1)$, and that the unique distribution for player 1's actions that makes 2 indifferent is $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$ on $(0, h', 1)$. $\square$

The following is the analysis of the asymmetric version of the Sion-Wolfe game described in the text.

Proof of Lemma 3.3. Verifying that μ^* is an equilibrium was done in the text. For the rest, let H_1 and H_2 be exhaustive hyperfinite subsets of A_1 and A_2 respectively with the property that $h_1' < h_2' < 0.5$.

First round of deletion of weakly dominated strategies

First, observe that any $h_1 \in H_1 \cap (0.5, 1)$ is weakly dominated by 1 for player 1 because moving to a higher strategy in $(0.5, 1]$ wins against every h_2 that a lower strategy beats, and either wins or ties against every h_2 that a lower strategy loses to.

Second, observe that any $h_2 \in [0, h_2') \cap H_2$ is weakly dominade by h_2' for player 2 because moving to a higher strategy in $[0, h_2')$ wins against every h_1 that a lower strategy beats, and either wins or ties against every h_1 that a lower strategy loses to.

After deleting the weakly dominated strategies, the action sets for the two players are $(H_1 \cap [0, 0.5]) \cup \{1\}$ for player 1 and $H_2 \cap [h_2', 1]$ for player 2.

Second round of deletion of weakly dominated strategies

Now consider the game with the weakly dominated strategies deleted. For player 1, playing $x = 0$ weakly dominates $0 < h_1 < \frac{1}{2}$ and playing $x = 1$ weakly dominates $\frac{1}{2}$. For player 2, the only weakly undominated strategies are $y = h_2'$ and $y = 1$.

With the weakly dominated strategies deleted, we have the 2×2 game given by

		Player 2	
		$y = h_2'$	$y = 1$
Player 1	$x = 0$	(-1,+1)	(+1,-1)
	$x = 1$	(+1,-1)	(0,0)

Direct verification shows that 1 playing $(\frac{2}{3}, \frac{1}{3})$ on $x = 0$ and $x = 1$ and 2 playing $(\frac{1}{3}, \frac{2}{3})$ on $y = h_2'$ and $y = 1$ is the unique equilibrium. □

The following proof is considerably simplified by the use of hyperfinite sets and nonstandard analysis.

Proof of Theorem C. Let $F = \times_{i \in I} F_i$ be a product of non-empty finite subsets of $\times_{i \in I} A_i$. The class of hyperfinite products $H = \times_{i \in I} H_i$ containing F is internal. For each such H, the set of equilibria in iteratively undominated strategies is internal. The union of these internal sets is itself an internal subste of ${}^*\Delta$, and the standard part of any internal set is closed in the weak* topology on finitely additive probabilities. Let $Un(F)$ denote that closed set in Δ. The class $\{Un(F) : F = \times_{i \in I} F_i\}$ has the finite intersection property, hence has non-empty, closed intersection, Un. By construction, any element of Un puts mass 0 on the set of weakly undominated strategies, and it is the standard part of an equilibrium for some hyperfinite version of the game. □

The following concerns the integrals of semi-continuous functions against continuously equivalent probabilities.

Proof of Lemma 5.1. It is sufficient to show that the inequality holds for $f(x) = 1_F(x)$, F a closed subset of X (because every non-negative upper semi-continuous function is a uniform limit of positive linear combinations of indicators of closed sets). With $F^{1/n}$ denoting the open set of points y with $d(y, F) < 1/n$, we have $A_n := (F^{1/n} \setminus F) \downarrow \emptyset$ so $q(A_n) \downarrow 0$, but since p is finitely additive, $\inf_n p(A_n) > 0$ is possible. If this happens, $\int 1_F \, dq > \int 1_F \, dp$. And since f is upper semi-continuous iff $-f$ is lower semi-continuous, the inequality reverses for lower semi-continuous functions. □

The following concerns the integrals finitely additive continuous equivalents of a countably additive probability that avoids discontinuities.

Proof of Lemma 5.2. Rescaling if necessary, we can, without loss, assume that $f(x) \in [-1, +1]$. Let F denote the closure of the set of discontinuities of f, suppose that $q = ca(p)$ is the countably additive version of a finitely additive total probability p, and that $q(F) = 0$. Let G denote the open complement of F, and pick arbitrary $\epsilon > 0$. We will show that $|\int f\, dp - \int f\, dq| < \epsilon$.

Pick $K \subset G$ such that $q(K) > 1-\epsilon/4$. By the continuous equivalence of p and q, for any $\delta > 0$, $p(K^\delta) > 1 - \epsilon/4$. Pick $\delta > 0$ such that $K^{2\delta} \subset G$. Let g denote the restriction of f to the closure of K^δ. By the usual results on the extension of continuous functions defined on closed sets, g has a continuous extension, h, to all of X with $\|h\| \leq \|g\|$. Since p and q are continuously equivalent, $\int h\, dp = \int h\, dq$. We have

$$\begin{aligned} \left| \int f\, dp - \int f\, dq \right| \leq & \\ & \left| \int f\, dp - \int h\, dp \right| + \left| \int h\, dp - \int h\, dq \right| + \left| \int h\, dq - \int f\, dq \right|. \end{aligned} \tag{10}$$

The middle term is equal to 0 by continuous equivalence. The first and the third terms are less than $\epsilon/2$ because the functions f and h agree on K^δ, a set that both probabilities assign at least mass $1 - \epsilon/4$, and the absolute value of the difference between f and h is bounded by 2 because both take values only in $[-1, +1]$. □

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