Optimal Influence Design in Networks*

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Abstract

We examine the optimal intervention of an influence designer in the presence of social learning in a network. Before learning begins, a designer implants opinions into the network to make agents' ultimate opinions as close as possible to the target opinions. By decomposing the influence matrix, which summarizes the learning structure, we transform the designer's problem into one with an orthogonal basis: implanted opinions on one cluster of agents influence only another cluster's opinions with a multiplier effect. This transformation allows us to characterize the optimal intervention under both complete and incomplete information on the network structure. The designer implants more opinions into the agents whose opinions spread well through the network in the sense that they efficiently entice others' opinions to drift away from the initial opinions and move closer to the target opinions. We also show that even though the designer does not have complete information on the network structure, when she knows how groups of agents form connections with another group, she can design the asymptotically optimal intervention in a large network. Finally, we provide examples and extensions of repeated social learning and competition.

JEL Classification: D83; D85.

Keywords: Davis-Kahan $\sin\Theta$ theorem; Singular value decomposition; Social learning; Social networks; Wedin $\sin\Theta$ theorem

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1 Introduction

1.1 Overview

In many economic situations, individuals' choices are influenced by other individuals nearby, such as friends, family, and neighbors. When individuals make a purchasing decision about a product or a voting decision for a referendum, they might refer to the opinions of the people in their social network.[1] The effects of social networks have grown stronger with the development of the Internet and social media. Online shoppers actively interact through their social media pages as well as e-commerce websites, such as Amazon or eBay. Voters eagerly share political opinions or related news articles on their own social media pages, such as Facebook or Twitter. These growing online social networks are redefining the market funnel and reshaping the political platform. Businesses are investing sizeable resources into the development of online marketing strategies,[2] and political parties are actively organizing online campaign strategies.[3]

In this paper, we examine the optimal intervention decision of an influence designer in the presence of social learning in a network of individuals. We consider a social learning process in which individuals form their opinions by taking a weighted average of their neighbors' opinions in a social network (e.g., Banerjee et al. 2021; DeGroot 1974; Della Lena 2019; Golub and Jackson 2010, 2012). In our model, before learning begins, an influence designer with limited resources can intervene to change individuals' opinions. In order to change the final outcome in favor of her objective, the designer implants opinions to lead agents with initial opinions to form new opinions that are as close as possible to the target opinion, subject to budget constraints.

In reality, for example, businesses, as influence designers in an online marketplace, actively engage in content marketing and influencer marketing. In the former, businesses directly post something on their own social media pages. In the latter, they hire outsiders, so-called influencers, who have the power of influence in a certain online social network, such as

[1] People tend to imitate the behavior of other people in a society. Psychologists use the term conformity to explain this behavior (Cialdini and Goldstein 2004). This concept has been adopted and explored in various fields of social science research, such as business marketing (e.g., Lascu and Zinkhan 1999) and political science (e.g., Sinclair 2012).

[2] Advertisers in the United States increased their spending on digital (or online) and TV advertisements in 2016 by about 15% (eMarketer 2021).

[3] New/social media executives have played important roles in U.S. presidential campaigns. Stephen Bannon, the chief executive officer of Donald Trump's 2016 presidential campaign, was the executive chairman of Breitbart News, which is a conservative online news website founded in 2007. In 2020, the social media team of Joe Biden's 2020 presidential campaign recruited social media influencers to overcome a disadvantage in his online campaign against Donald Trump.

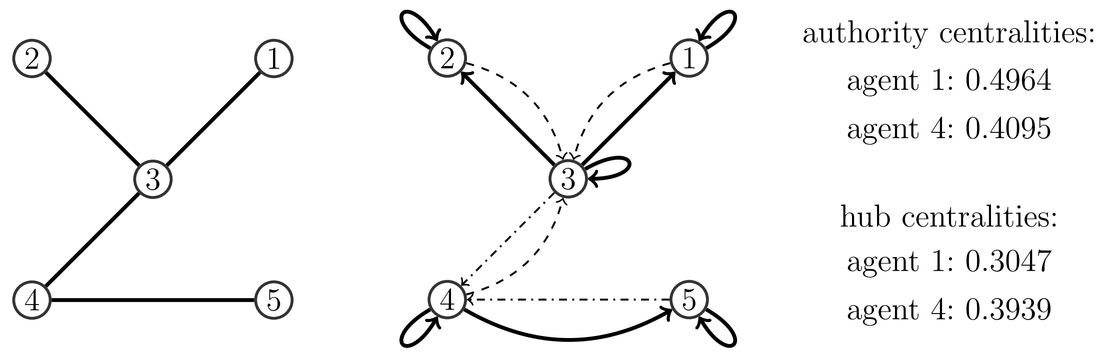
Facebook or Instagram, and ask them to share positive reviews of the firms' products.[4] In both forms of online marketing, but especially social media marketing, firms (influence designers) may control the content of the ads and/or the targeted individuals (potential customers) exposed to the ads. In this context, consumers can become messengers of firms' ads by spreading the relevant posts. They can "like" and/or share the ads, intentionally or unintentionally, to their own followers. How, then, does a designer optimally intervene in the network, and what are the consequences when she does?

By applying the singular value decomposition to the influence matrix representing the underlying learning structure among the agents in a network, we transform the designer's problem into one with an orthogonal basis: implanted opinions on one cluster of agents influence only another cluster's opinions with a multiplier effect determined by the singular value. This transformation allows us to characterize the optimal intervention fully in terms of hub and authority centrality, or singular vectors, of the influence matrix (Kleinberg 1999): the right singular vectors are associated with the hub centrality, and the left singular vectors are associated with the authority centrality. In our framework, a good hub is an individual who spreads opinions to many good authorities, and a good authority is an individual who embraces many opinions from good hubs.

See the example in Figure 1. The underlying social network is undirected as shown in Figure 1-(a), but the influence structure is represented by a directed and weighted network as in Figure 1-(b). Thus, the eigendecomposition approach by Galeotti et al. (2020) cannot be applied directly. As shown in Figure 1-(b), in terms of hub centrality associated with the largest singular value, agent 4 is more influential than agent 1 because both agents equally influence agent 3, but agent 5 is additionally influenced by agent 4. In terms of authority centrality, however, agent 1 is a better authority than agent 4 because agent 1 listens to agent 3, the best hub in the network, with a higher weight than agent 4.

Our main result, Theorem 1, characterizes how hub and authority centralities determine the optimal intervention. We show that if an agent has higher hub centrality, so that her opinion is well-spread throughout the network, the designer should inject more opinions into that agent.Let's call the agents with higher hub centralities the *influencers*. Note that, in some sense, the initial opinion of an individual is the obstacle for the designer to overcome, whereas the target opinion is the ultimate goal for her to pursue. Thus, if the target opinion is more similar to the authority centralities of the network, so that the target opinion is well-embraced in the network, the designer should more actively intervene in the network through *influencers*. However, if the initial opinion is more similar to the hub centralities of

[4]In 2015, Marriott, the international hotel chain, worked with five YouTube influencers and released content videos to celebrate reaching one million check-ins on the Marriott mobile application.



(a) undirected social network (b) directed influence structure (c) centralities

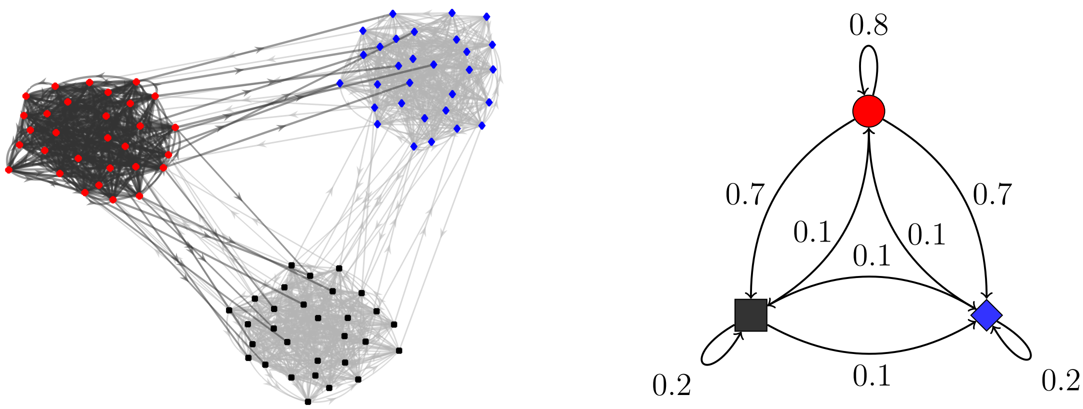
Figure 1: Example of five agents. In (a), two agents are connected by a link if they are mutual friends. Thus, the network is undirected. In (b), the influence structure is a weighted directed network. An arrow from agent j to i represents that agent i takes agent j's opinion in her opinion update. The solid arrows represent a weight of $\frac{1}{2}$, the dash-dotted arrows represent a weight of $\frac{1}{4}$, and the dashed arrows represent a weight of $\frac{1}{6}$. In (c), we highlight the authority and hub centralities of agents 1 and 4.

the network, so that the initial opinion is well-spread throughout the network, the designer should less actively intervene in the network through *influencers*.

We extend our model to a situation in which there is incomplete information about the underlying network structure. In reality, the information on an underlying network structure might not be completely known to the designer, so the analysis with Theorem 1, which requires complete information about the underlying network structure, might not be directly applicable to real-world situations. In Theorem 2, we show that when the designer knows how groups of agents form connections with one another, she can design the asymptotically optimal intervention by using a low-rank matrix approximation of the actual influence matrix. The low-rank approximation is based on the *homophily* of a social network; under a regular condition, a few factors of nodes are sufficient to explain the link formation structure of the network (Golub and Jackson 2012).

Theorem 3, which is derived from Theorem 2, is an extension of the "representative-agent" theorem by Golub and Jackson (2012). While their theorem only requires the convergence of singular values, our theorem additionally requires the convergence of singular vectors, which is proven by employing the matrix perturbation theory (Davis and Kahan 1970; Stewart and Sun 1990; Wedin 1972, 1983). The theorem enables us to construct an optimal intervention by studying a much smaller network in which there is only one (representative) agent for each type. Our result can be used to justify the development of a type-based target marketing strategy that utilizes network data based on group-level statistics, which are much easier to obtain than individual-level information.

For instance, in Figure 2-(a), there are three types of agents: red circles, blue diamonds, and black squares. We consider the situation in which, in expectation, agents of a common type are influenced by and influence other agents of the same type and different types. Our theorem shows that even when this network is not directly observable to the influence designer, she can still find an optimal strategy with a representative-agent representation of the network, as in Figure 2-(b), which neglects the specifics of the full network.



(a) homophilous influence in a large network (b) representative-agent representation

Figure 2: Example of three types. In (a), three types of agents are represented by red circles, blue diamonds, and black squares. Agents of the same type are more likely to influence one another due to homophily. However, the red circle agents influence all of the other agents more substantially. In the figure, the links from the red agents are darker. The figure in (b) is a representative-agent representation of the network in (a). There are only three nodes, each representing agents of the same type. The high weight of the links denotes the strong influence of the red circle agents.

1.2 Related Literature

The current paper is related to three strands of literature: social learning in networks, intervention in networks, and clustering techniques in machine learning.

Social learning in networks. There are two different approaches to social learning in networks: fully rational Bayesian updating models (Bala and Goyal 1998; Choi et al. 2005; Corazzini et al. 2012; Dasaratha and He 2020; Gale and Kariv 2003) and boundedly rational updating models, where agents average their neighbors' beliefs or opinions (e.g., Banerjee et al. 2021; DeGroot 1974; Della Lena 2019; DeMarzo et al. 2003; Golub and Jackson 2010, 2012; Molavi et al. 2018). The latter models are often referred to as DeGroot learning models in the literature (DeGroot 1974). The DeGroot approach is a simple heuristic learning rule in a social network, and it is widely used for its remarkable tractability. Jackson (2010) and Bramoullé et al. (2016) effectively summarize the literature on both social learning models in networks. We refer to their books for Bayesian social learning models in networks.

In this paper, we assume that agents update their beliefs according to the DeGroot model, which employs a simple heuristic learning rule, delivering a fairly complete characterization of learning dynamics as a function of network structure. This choice of social learning model has been supported by recent lab experimental works. Pogorelskiy and Shum (2021) find that subjects' information updating in networks and resulting behavior are unsophisticated and myopic. Chandrasekhar et al. (2020) and Choi et al. (2021) show that the DeGroot model explains subjects' learning behavior with considerable accuracy.

Intervention in networks. Interventions in networks can affect individuals' neighbors through network and/or spillover effects as shown in the literature on network goods, in which network externalities are generated in a social network (e.g., Fainmesser and Galeotti 2016; Galeotti et al. 2020; Radner et al. 2014; Rohlfs 1974; Shin 2017). In our framework, network externalities are heterogeneous among the agents because we allow them to have different connectivity in a network.

In terms of analysis technique, we employ the singular value decomposition approach to maximize a quadratic objective function under a budget constraint. The influence designer's problem is mathematically similar to the intervention problem investigated by Galeotti et al. (2020) in that we exploit the orthogonality of the singular vectors in the decision-maker's optimization problem. However, the current paper differs from theirs in three ways: context, network structure assumptions, and information structure assumptions. The problem studied by Galeotti et al. (2020) is motivated by public goods games, such that a decision maker maximizes agents' actions (i.e., provision of public goods). In contrast, the current paper's decision maker aims to make each agent's opinion closer to a target opinion that may differ from others in the context of (repeated) social learning. Regarding the assumption on network structure, Galeotti et al. (2020) only consider undirected networks, which is natural in the context of public games, but we cover directed networks as well as undirected network to study (repeated) social learning. Lastly, in their model, the decision maker has complete knowledge of the underlying network structure, while we provide a new mathematical tool to deal with incomplete information.

Clustering techniques in machine learning. Our paper is related to the modern clustering techniques in machine learning literature. The most closely related study is that of Kleinberg (1999), which introduces the notion of hub and authority centralities in directed networks. These concepts, along with the singular vectors of the influence matrix, provide clear explanation for our main results. We also employ the approximation techniques for stochastic block matrices introduced in the previous literature (e.g., Davis and Kahan 1970; Stewart and Sun 1990; Wedin 1972, 1983). In particular, we use two theorems in the matrix perturbation theory (Stewart and Sun 1990), the Davis-Kahan $\sin\theta$ theorem and the Wedin $\sin\theta$ theorem, which are associated with convergence of singular vectors (Davis and Kahan 1970; Wedin 1972, 1983). Benefiting from these theorems, we extend our analyses to to an incomplete information setting. The Wedin $\sin\theta$ theorem enables us to utilize the low dimensionality of the approximated matrices (Dasaratha 2020), which provides a series of comparative static results with transparent economic insights.

2 Setup

2.1 Network and Spread of Information

Network. A *network* of n agents is represented by an $n \times n$ *adjacency matrix* $\mathbf{A}$.[5] Each entry $\mathbf{A}_{ij}$ of $\mathbf{A}$ takes values in $\{0,1\}$ where $\mathbf{A}_{ij} = 1$ represents that agent i and agent j are connected by a *link* from j to i, representing that agent i listens to agent j's opinion. The *indegree* of agent i is defined by $d_i(\mathbf{A}) = \mathbf{A}_i\mathbf{1}$, where $\mathbf{A}_i$ is the ith row of the adjacency matrix, and $\mathbf{1}$ is the $n \times 1$ column vector of ones.[6] Thus, the indegree counts the number of links toward agent i. The *indegree matrix* $\mathbf{D}(\mathbf{A})$ is defined as $\mathbf{D}(\mathbf{A}) = diag(d_1(\mathbf{A}), \ldots, d_n(\mathbf{A}))$. We assume that the network is *strongly connected*; for any two agents $i, j \in N$, there is a sequence of neighbors who connect agent j to agent i.[7] This assumption implies that $d_i(\mathbf{A}) > 0$ for all i; that is, each agent listens to at least one other agent. Thus, the indegree matrix $\mathbf{D}(\mathbf{A})$ is invertible.

Learning and influence in the network. We consider a social influence model in which agents form their opinions by taking a weighted average of their neighbors' opinions: agents respect the opinions of their neighbors in the network and/or intend to conform with them (DeGroot 1974; Golub and Jackson 2010).[8] Let agents' initial private opinion vector be $\mathbf{b}^0 = (\mathbf{b}_1^0, \ldots, \mathbf{b}_n^0)^\intercal \in \mathbb{R}_+^n$, where $\mathbf{b}_i^0$ represents agent i's initial opinion, and $\mathbb{R}_+$ is the set of non-negative real numbers. We assume that agent i is boudedly rational in the sense that

[5] In this paper, we make no assumption about the adjacency matrix's symmetry, and so the adjacency matrix can be directed. Moreover, the adjacency matrix even can be assumed to be weighted, as discussed in Section 6.2.

[6] Throughout the paper, for a matrix $\mathbf{X}$, $\mathbf{X}_i$ represents the ith row vector of $\mathbf{X}$, and $\mathbf{X}^j$ the jth column vector of $\mathbf{X}$.

[7] Formally, a network is said to be strongly connected if for any pair of agents (i,j), there is a sequence of agents say, $k_0 = i, k_1, \ldots, k_l = j$ such that $A_{k_s k_{s+1}} = 1$ for all $s = 0, \ldots, l-1$.

[8] We discuss this assumption further in Section 2.2.

she updates her opinion according to the following rule:

$$\mathbf{b}_i = \frac{\alpha_i \mathbf{b}_i^0 + (1-\alpha_i)\sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{b}_j^0}{\alpha_i + (1-\alpha_i) d_i(\mathbf{A})}, \tag{2.1}$$

where $\alpha_i \in [0, \overline{\alpha}]$ with $\overline{\alpha} < 1$ represents the relative importance of agent i's private opinion. The factor of $d_i(\mathbf{A})$ in the denominator implies that the agent takes an unweighted average of her neighbors' opinions.[9]

After an exchange of opinion, the resulting opinion $\mathbf{b}$ turns out to be

$$\mathbf{b} = \underbrace{\left(\boldsymbol{\alpha} + (\mathbf{I} - \boldsymbol{\alpha})\mathbf{D}(\mathbf{A})\right)^{-1}\left(\boldsymbol{\alpha} + (\mathbf{I} - \boldsymbol{\alpha})\mathbf{A}\right)}_{\equiv \mathbf{T}} \mathbf{b}^0 = \mathbf{T}\mathbf{b}^0,$$

where $\boldsymbol{\alpha} = diag(\alpha_1, \ldots, \alpha_n)$, and $\mathbf{I}$ is the $n \times n$ identity matrix. We call $\mathbf{T}$ the *influence matrix*, which is well-defined because $\mathbf{D}(\mathbf{A})$ is invertible.[10]

Example 1 illustrates the social influence structure of two benchmark networks.

(a) complete network

(b) star network

Figure 3: Illustration of the two undirected social networks

Example 1 Consider the undirected complete network in Figure 3-(a) consisting of five agents, where all agents are linked with each other. The network is undirected as agents i and j mutually listen to each other's opinion. Assuming $\alpha_i = 0$ for all i, the corresponding adjacency, degree, and influence matrices are calculated as

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \mathbf{D}(\mathbf{A}) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

Suppose the initial opinion is given as $\mathbf{b}^0 = (1, 1, 1, 0, 0)^{\mathsf{T}}$. Then, the resulting opinion is $\mathbf{b} = \mathbf{T}\mathbf{b}^0 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right)^{\mathsf{T}}$.

[9] As we assume $\alpha_i \in [0, 1)$ for all i, our results hold for the standard DeGroot learning model where $\alpha_i = 0$ for all i as in Banerjee et al. (2021): $\mathbf{b}_i = \frac{\sum_{j=1}^{n} \mathbf{A}_{ij}\mathbf{b}_j^0}{d_i(\mathbf{A})}$. We here remark that it is important to have the property that $\frac{\alpha_i}{\alpha_i + (1-\alpha_i)d_i(\mathbf{A})}$ converges to zero as $d_i(\mathbf{A})$ diverges to infinity for the incomplete information case. See Section 6 for more details.

[10] $\mathbf{T}$ depends on $\boldsymbol{\alpha}$ and $\mathbf{A}$. To simplify the notion, we drop the dependency on them in our notation.

Now, consider the undirected star network depicted in Figure 3-(b) consisting of five peripheral agents (agent 2 to 6) and one central (agent 1). Again, assuming $\alpha_i = 0$ for all i, the corresponding adjacency, degree, and influence matrices are

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{D}(\mathbf{A}) = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose the initial opinion is $\mathbf{b}^0 = (1, 0, 0, 0, 0, 0)^\intercal$. Then, the resulting opinion is $\mathbf{b} = \mathbf{T}\mathbf{b}^0 = (0, 1, 1, 1, 1, 1)^\intercal$. Since everyone except agent 1 has the initial opinion of 0, the resulting opinion of agent 1 becomes 0. On the contrary, all the other agents are influenced only by agent 1. So, their resulting opinions become 1.

Influence design problem. We introduce an *influence designer*, who can implant opinions $\mathbf{b}'$ into the agents in a network. We assume that after the designer's intervention, the implanted opinion $\mathbf{b}'$ basically replaces the initial opinion of agents $\mathbf{b}^0$. The designer's objective is to make agents' resulting opinions as close as possible to target opinions $\mathbf{b}^*$: In particular, the designer's objective is to minimize the average of the squared differences, $\frac{1}{n}\sum_{i=1}^n (\mathbf{b}_i^* - \mathbf{b}_i)^2$, where $\mathbf{b}_i$ is the agent i's opinion after an exchange of opinions given the implanted opinion $\mathbf{b}'$, $\mathbf{b} = \mathbf{T}\mathbf{b}'$.[11]

The designer's intervention incurs a certain cost. For each agent i, the intervention cost, which can be interpreted as the cost of persuasion, is assumed to be quadratic in the difference between the implanted opinion $\mathbf{b}_i'$ and the initial opinion $\mathbf{b}_i^0$, $(\mathbf{b}_i' - \mathbf{b}_i^0)^2$. We call the net implanted opinion $\mathbf{b}_i' - \mathbf{b}_i^0$ the *injected opinion*. The designer possesses a budget of $C > 0$: The designer faces a budget constraint of $\frac{1}{n}\sum_{i=1}^n (\mathbf{b}_i' - \mathbf{b}_i^0)^2 \le C$.[12]

The influence designer's problem can be written as follows:

$$\min_{\mathbf{b}'} \quad \frac{1}{n}\sum_{i=1}^n (\mathbf{b}_i^* - \mathbf{b}_i)^2 \tag{DP 1}$$

$$\text{subject to} \quad \mathbf{b} = \mathbf{T}\mathbf{b}' \quad \text{(the exchange of opinions)}$$

$$\frac{1}{n}\sum_{i=1}^n (\mathbf{b}_i' - \mathbf{b}_i^0)^2 \le C \quad \text{(the budget constraint).}$$

Note that the exchange of opinions in the network is not based on the initial opinion $\mathbf{b}^0$, but on the implanted opinion $\mathbf{b}'$ chosen by the designer. The initial opinion $\mathbf{b}^0$ appears only in

[11] In line with the decision maker's intervention on private opinions, one might interpret it as a reduced form expression of the persuasion effect of personalized ads (e.g., Bagwell 2007; Dixit and Norman 1978).

[12] Technically, the left hand side of this budget constraint does not have to be the average persuasion cost. We can also have the total persuasion cost on the left hand side as in $\sum_{i=1}^n (\mathbf{b}_i' - \mathbf{b}_i^0)^2 \le C'$. Our analysis and main result in Theorem 1 does not depend on this setting.

the budget constraint as it works as the benchmark for calculating the intervention cost.

In order to sharpen predictions, we make two assumptions on the problem (DP 1). First, we assume that $\mathbf{b}_i^* > \mathbf{b}_i^0$ for all i, which means that every agent's initial opinion is lower than the target opinion. Second, we assume that C is so small that the designer cannot implant her ideal opinion $\mathbf{b}^*$ directly into the network: $\frac{1}{n}\sum_{i=1}^{n}(\mathbf{b}_i^* - \mathbf{b}_i^0)^2 > C$.

2.2 Discussion of the Model

The designer's problem (DP 1) is similar to a network intervention problem in Galeotti et al. (2020). While in their model agents strategically choose their actions generating externalities in a network, agents in this paper influence each other's opinion in the context of social learning. In particular, the agents update their opinions according to DeGroot's learning model (DeGroot 1974). One may require that the opinion of each individual is in $[0, 1]$. In such case, it is necessary to assume that C is sufficiently small as in Galeotti et al. (2020); otherwise, there might exist an agent whose implanted opinion chosen by the designer is strictly greater than 1.

The assumption on target opinions, $\mathbf{b}_i^* > \mathbf{b}_i^0$, implies that the designer has her own bias or directional motivation. However, this assumption does not mean that she wants to unify the agents' opinions. Note that we still allow target opinions to be fairly polarized; for example, $\mathbf{b}_i^* \approx 1$ and $\mathbf{b}_j^* \approx 0$ for some $i \neq j$.

We can think of many real-world applications of our theoretical approach. In the context of politics, a political party (or an organization) wants to intervene ballot casting decisions, so often runs political campaigns to change voters' opinions, and chooses to spread certain (dis)information. The party's effort is costly, and there is a budget constraint determined by the campaign finances. Voters may be connected with one another and exchange their opinions in a network. Each voter is less likely to vote for the party when her final opinion differs from the party's target opinion.

In the context of marketing, a company as an influence designer may offer gift cards to consumers, and the amount in the gift card is heterogeneous among the consumers. In line with the word of mouth literature in management science, the current model also represents the mechanism of influencer marketing (e.g., Kanuri et al. 2018; Kempe et al. 2003; Lambrecht et al. 2018; Mallipeddi et al. 2021). In this context, influencers are internet celebrities who are actively engaged in social media platforms such as Facebook, Twitter, Instagram, etc. They are often identified as users who have an enormous number of followers. By hiring (or paying off) some influencers, firms try to spread the word about their product in influencer reviews.

3 Analysis

3.1 Singular Value Decomposition

Facts. We here gather mathematical facts on the singular value decomposition of the influence matrix $\mathbf{T}$ that plays the key role in our analysis (Meyer 2010; Strang 2019).

Note that the influence matrix $\mathbf{T} = (\boldsymbol{\alpha} + (\mathbf{I} - \boldsymbol{\alpha})\mathbf{D}(\mathbf{A}))^{-1} (\boldsymbol{\alpha} + (\mathbf{I} - \boldsymbol{\alpha})\mathbf{A})$ is not necessarily symmetric, even when the adjacency matrix $\mathbf{A}$ is assumed to be symmetric. Therefore, the spectral decomposition technique used by Galeotti et al. (2020), which consider symmetric networks, is not applicable to our problem.[13]

The singular value decomposition of $\mathbf{T}$ provides that $\mathbf{T} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$, where:[14]

(i) $\mathbf{S} = diag(s_1, \ldots, s_n)$, where $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$ and $s_k \geq s_{k+1}$ for all k;[15]

(ii) $\mathbf{U}$ and $\mathbf{V}$ are normal matrices: $\mathbf{U}^\top\mathbf{U} = \mathbf{V}^\top\mathbf{V} = \mathbf{I}$;

(iii) the kth column vector of $\mathbf{U}$, $\mathbf{u}^k$, is a left singular vector of $\mathbf{T}$: $\mathbf{T}\mathbf{v}^k = s_k\mathbf{u}^k$;

(iv) the kth column vector of $\mathbf{V}$, $\mathbf{v}^k$ is a right singular vector of $\mathbf{T}$: $(\mathbf{u}^k)^\top\mathbf{T} = s_k(\mathbf{v}^k)^\top$.

Note that $\{\mathbf{u}^1, \ldots, \mathbf{u}^n\}$ forms an orthonormal basis of the column space of $\mathbf{T}$, and $\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ forms an orthonormal basis of the row space of $\mathbf{T}$.

Interpretation and examples. How do we interpret the $\mathbf{U}$, $\mathbf{S}$, and $\mathbf{V}$? We interpret the vectors in these matrices with a mutually reinforcing relationship of agents discussed by Kleinberg (1999): an authority and a hub.[16] In our framework, we can conceptually define the relationship of an authority and a hub in a circular arguments: a good hub is an agent that spreads opinion to many good authorities; a good authority is an agent that embraces opinions from many good hubs.

The singular value decomposition allows us to identify the hub and authority measure by breaking this circularity. Kleinberg (1999) calls the singular vectors $\mathbf{v}^1$ and $\mathbf{u}^1$ the hub and authority centrality measures, respectively. In our model, the hub and authority centrality measures are extended to n dimensions. That is, the right singular vector $\mathbf{v}^k$ represents how well the opinion spreads in the network, and its ith element, $\mathbf{v}_i^k$, represents the agent

[13] For the relationships between the singular value decomposition and the spectral decomposition, see Strang (2019).

[14] $\mathbf{U}$ and $\mathbf{V}$ need not be unique for several reasons. First, singular values and their corresponding singular vectors can be permuted. Second, if $\mathbf{u}^k$ and $\mathbf{v}^k$ are singular vectors, then $-\mathbf{u}^k$ and $-\mathbf{v}^k$ also can be singular vectors for the same singular value. Third, repeated singular values may exist, and their singular vectors do not have to be unique.

[15] Since $d_i(\mathbf{A}) > 0$ for all i, at least one singular value is strictly greater than zero.

[16] In his paper, Kleinberg (1999) motivates this measure in the problem of searching on the world wide web and tries to measure webpages' centrality.

i's outward centrality in kth dimension. The left singular vector $\mathbf{u}^k$ represents how well the opinion is embraced in the network, and its jth element, $\mathbf{u}^k_j$, represents the agent j's inward centrality in kth dimension. The diagonal matrix $\mathbf{S}$ captures the mutually reinforcing relationship between the hub and authority measures for different vectors. As the singular values are in descending order, the multiplier effect of the singular vectors is decreasing in the index.

Example 2 below discusses the centrality measures of the two networks in Figure 3.

Example 2 Consider the complete network in Figure 3. We here focus on the first singular value and the corresponding singular vectors. Since all agents are symmetric in the network, the hub and authority measures are all equal, $\mathbf{U} = \mathbf{V}$ and $\mathbf{u}^1_i = \mathbf{u}^1_j = \frac{1}{\sqrt{5}}\mathbf{1}$ for all i and j. That is, everyone's opinion spreads to the others in the same way, and everyone embraces each other's opinion in the same way.

For the star network, on the other hand, the two centrality measures are different. $\mathbf{v}^1 = (1, 0, 0, 0, 0)^\intercal$ is the hub centrality. The first agent's opinion is dominantly spread to other agents. However, the authority centrality in the first dimension is measured by $\mathbf{u}^1 = \left(0, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^\intercal$. This means that since $\alpha_i = 0$ for all i, agents 2–6 embrace the opinion of agent 1, and their authority centrality is non-zero. Finally, since $\mathbf{u}^1 = \mathbf{T}\mathbf{v}^1$, it follows that $\mathbf{u}^1_1 = 0$; in the first dimension, the implanted opinion on agent 1 will be uniformly embraced by all the other agents.

3.2 Transformation

We now transform the influence designer's problem with the concepts of centralities. From here on, for a vector $\mathbf{b} \in \mathbb{R}^n_+$, we denote the projection of $\mathbf{b}$ onto the column space of $\mathbf{T}$ by $\overline{\mathbf{b}} := \mathbf{U}^\intercal\mathbf{b}$, and the projection of $\mathbf{b}$ onto the row space by $\underline{\mathbf{b}} := \mathbf{V}^\intercal\mathbf{b}$. When the designer chooses $\mathbf{b}'$, the resulting projected opinion of the agents after the exchange of opinion can be written as $\overline{\mathbf{b}} = \mathbf{S}\underline{\mathbf{b}}'$, where $\overline{\mathbf{b}} := \mathbf{U}^\intercal\mathbf{b}$ and $\underline{\mathbf{b}}' = \mathbf{V}^\intercal\mathbf{b}'$. Thus, $\overline{\mathbf{b}}$ is the resulting opinion projected in the space of authority centrality, and $\underline{\mathbf{b}}'$ is the implanted opinion projected in the space of hub centrality.

By plugging $\mathbf{b} = \mathbf{T}\mathbf{b}'$ into the original objective function, we obtain

$$\sum_{i=1}^{n} (\mathbf{b}^*_i - \mathbf{T}_i\mathbf{b}_i)^2 = (\overline{\mathbf{b}}^*)^\intercal\overline{\mathbf{b}}^* - 2(\overline{\mathbf{b}}^*)^\intercal\mathbf{S}\underline{\mathbf{b}}' + (\underline{\mathbf{b}}')^\intercal\mathbf{S}^2\underline{\mathbf{b}}'.$$

The budget constraint is now

$$\sum_{i=1}^{n} \left(\mathbf{b}'_i - \mathbf{b}^0_i\right)^2 = \left(\underline{\mathbf{b}}' - \underline{\mathbf{b}}^0\right)^\intercal \left(\underline{\mathbf{b}}' - \underline{\mathbf{b}}^0\right) \leq C.$$

Therefore, the original problem (DP 1) is equivalent to

$$\begin{aligned} \min_{\underline{\mathbf{b}}'} \quad & \left((\overline{\mathbf{b}}^*)^\intercal \overline{\mathbf{b}}^* - 2(\overline{\mathbf{b}}^*)^\intercal \mathbf{S}\underline{\mathbf{b}}' + (\underline{\mathbf{b}}')^\intercal \mathbf{S}^2 \underline{\mathbf{b}}'\right) \\ \text{subject to} \quad & \left(\underline{\mathbf{b}}' - \underline{\mathbf{b}}^0\right)^\intercal \left(\underline{\mathbf{b}}' - \underline{\mathbf{b}}^0\right) \leq C. \end{aligned} \tag{DP 2}$$

In the original problem, the implanted opinion of an agent i influences the opinions of the other agents, and vice versa. This interdependent influence summarized by the influence matrix $\mathbf{T}$ makes the analysis and interpretation more complicated. However, interestingly, this dependency problem disappears in the transformed problem (DP 2). Since singular vectors are orthogonal to each other, the change of one projected opinion does not affect any of the other projected opinions. This convenience is presented by the following expression:

$$\overline{\mathbf{b}} = \mathbf{U}^\intercal \mathbf{b} = \begin{pmatrix} \overline{\mathbf{b}}_1 \\ \vdots \\ \overline{\mathbf{b}}_n \end{pmatrix} = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \begin{pmatrix} \underline{\mathbf{b}}'_1 \\ \vdots \\ \underline{\mathbf{b}}'_n \end{pmatrix} = \mathbf{S}\underline{\mathbf{b}}' = \mathbf{S}\mathbf{V}^\intercal \mathbf{b}'$$

A change in $\underline{\mathbf{b}}'_i$ results in a change in $\overline{\mathbf{b}}_i$ only. As a result, the singular value decomposition enables us to treat $\underline{\mathbf{b}}_i$ and $\underline{\mathbf{b}}_j$ independently. Finally, if the optimal implanted opinion is characterized by $\underline{\mathbf{b}}'$ in the projected space, then the inverse transformation $\mathbf{b}' = \mathbf{V}\underline{\mathbf{b}}'$ becomes the corresponding optimal solution.

4 Optimal Influence Design

In this section, we characterize the optimal solution to the influence design problem in terms of the hub and authority centrality measures. Benefiting from the singular value decomposition, for (DP 2), we can treat $\underline{\mathbf{b}}_i$ and $\underline{\mathbf{b}}_j$ independently for all i and j. As the optimization problem is convex, the first-order condition fully characterizes the optimal solution. Specifically, the first-order equation for $\underline{\mathbf{b}}'_k$ is

$$(s_k \overline{\mathbf{b}}^*_k - s_k^2 \underline{\mathbf{b}}'_k) = \mu(\underline{\mathbf{b}}'_k - \underline{\mathbf{b}}^0_k),$$

where $\mu > 0$ is the Lagrangian multiplier for the budget constraint of (DP 2). The left-hand side represents the marginal benefit of $\underline{\mathbf{b}}'_k$, and the right-hand side is the marginal cost. By rearranging the equation, we find the optimal solution $\underline{\mathbf{b}}'_k = \frac{s_k}{s_k^2+\mu}\overline{\mathbf{b}}^*_k + \frac{\mu}{s_k^2+\mu}\underline{\mathbf{b}}^0_k$ which is nonnegative. In a matrix form, we can write $\underline{\mathbf{b}}' = \underline{\mathbf{b}}^0 + (\mathbf{S}^2 + \mu\mathbf{I})^{-1}\mathbf{S}(\overline{\mathbf{b}}^* - \mathbf{S}\underline{\mathbf{b}}^0)$. Therefore, the solution to the original problem (DP 1) turns out to be

$$\begin{aligned} \mathbf{b}' &= \mathbf{b}^0 + \mathbf{V}(\mathbf{S} + \mu\mathbf{I})^{-1}\mathbf{S}\left(\mathbf{U}^\intercal \mathbf{b}^* - \mathbf{S}\mathbf{V}^\intercal \mathbf{b}^0\right) \\ &= \mathbf{b}^0 + \sum_{k=1}^{n} \frac{s_k}{s_k^2 + \mu}\left(||\mathbf{b}^*|| \cos(\mathbf{u}^k, \mathbf{b}^*) - s_k||\mathbf{b}^0|| \cos(\mathbf{v}^k, \mathbf{b}^0)\right)\mathbf{v}^k, \end{aligned}$$

where $||\cdot||$ is the canonical Euclidean norm of vectors.[17]

Furthermore, we can pin down the exact value of μ. As the budget constraint is binding at the optimal choice, μ solves the equation

$$\sum_{i=1}^{n}(\underline{\mathbf{b}}_i' - \underline{\mathbf{b}}_i^0)^2 = \sum_{k=1}^{n}\left(\frac{s_k}{s_k^2+\mu}\right)^2\left(\overline{\mathbf{b}}_k^* - s_k\underline{\mathbf{b}}_k^0\right)^2 = C. \tag{4.1}$$

There is a unique μ that solves Equation (4.1), since the summation is strictly decreasing in μ. This feature also implies that μ is decreasing in the budget C. As such, we can interpret μ as the shadow price of the budget.

The following theorem summarizes our analysis so far:

Theorem 1 *The solution to the influence design problem (DP 1) is*

$$\mathbf{b}' = \mathbf{b}^0 + \sum_{k=1}^{n}\frac{s_k}{s_k^2+\mu}\left(||\mathbf{b}^*||\cos(\mathbf{u}^k,\mathbf{b}^*) - s_k||\mathbf{b}^0||\cos(\mathbf{v}^k,\mathbf{b}^0)\right)\mathbf{v}^k,$$

where μ is a unique solution to the equation

$$\sum_{k=1}^{n}\left(\frac{s_k}{s_k^2+\mu}\right)^2\left(\overline{\mathbf{b}}_k^* - s_k\underline{\mathbf{b}}_k^0\right)^2 = C.$$

Here, we provide the intuition behind Theorem 1. The influence designer implants the optimal opinion $\mathbf{b}'$ in order to lead agents with the initial opinion $\mathbf{b}^0$ to have resulting opinion $\mathbf{b} = \mathbf{T}\mathbf{b}'$ as close as possible to the target opinion $\mathbf{b}^*$, subject to the budget constraint. In other words, the initial opinion is the obstacle for the designer to overcome, while the target opinion is the ultimate goal for her to pursue.

The optimal injected opinion to agent i is

$$\mathbf{b}_i' - \mathbf{b}_i^0 = \sum_{k=1}^{n}\frac{s_k}{s_k^2+\mu}\left(||\mathbf{b}^*||\cos(\mathbf{u}^k,\mathbf{b}^*) - s_k||\mathbf{b}^0||\cos(\mathbf{v}^k,\mathbf{b}^0)\right)\mathbf{v}_i^k.$$

The above expression is a weighted sum of $\mathbf{v}_i^k$, which is the i'th entry of k'th singular vector $\mathbf{v}^k$. This entry $\mathbf{v}_i^k$ measures the importance of agent i in the k'th dimension and is the only individual specific term in the summation. So, if other things are equal, the injected opinion for agent i is proportional to the level of $\mathbf{v}_i^k$. That is, the designer injects more opinions to a good hub agent whose opinion spreads well through the network.

The term $\mathbf{v}_i^k$ is weighted by two different terms: one with the singular value and the shadow price of the budget, $\frac{s_k}{s_k^2+\mu}$, and another with the centrality and the similarity measures, $\left(||\mathbf{b}^*||\cos(\mathbf{u}^k,\mathbf{b}^*) - s_k||\mathbf{b}^0||\cos(\mathbf{v}^k,\mathbf{b}^0)\right)$. Note that both of those terms are dimension specific, so they have the sub/superscript k. The first term, $\frac{s_k}{s_k^2+\mu} > 0$, captures the multiplier effect of the injected opinion in the k's dimension. If the budget C is sufficiently tight, then

[17]Throughout the paper, $||\cdot||$ represents the Euclidean norm of vectors. For matrices, we also use the notation of $||\cdot||$, and it represents the matrix norm induced by the Euclidean norm for vectors.

this term is increasing in s_k.[18] The second term includes two similarities that depend on the network structure: $\cos(\mathbf{u}^k, \mathbf{b}^*)$ measures the similarity of the k'th singular vector, $\mathbf{u}^k$, and the target opinion, $\mathbf{b}^*$: As the target becomes more similar to the k'th authority centrality, the weight increases. This similarity captures the benefit of the intervention in terms of the k'th authority centrality: If the target opinion could have been embraced well in terms of the k'th authority centrality, $\mathbf{v}_i^k$ should be considered more importantly.[19] The other similarity, $\cos(\mathbf{v}^k, \mathbf{b}^0)$, is the similarity of k'th singular vector, $\mathbf{v}^k$, and the initial opinion, $\mathbf{b}^0$: As the initial opinion becomes more similar to the k'th hub centrality, the weight decreases. This similarity captures the cost of the intervention in terms of the k's hub centrality: If the initial opinion is spread well in terms of the k's hub centrality, $\mathbf{v}_i^k$ should be considered less importantly.[20] That is, the designer should focus more on the dimension that the injected opinion on the good hub agents can more efficiently entice the others' opinions to drift away from the initial opinions and move closer to the target opinions.

Therefore, the solution in Theorem 1 implies that the designer should implant more opinions into the good hub agents (or the *influencers*) whose opinions spread well through the network in the sense that they efficiently entice the others' opinions to drift away from the initial opinions and move closer to the target opinions.[21]

5 Incomplete Information

In this section, we extend our model to the situation where there is incomplete information about the underlying network structure. To solve the influence design problem under this situation, we introduce a technique from the matrix perturbation theory that enables us to approximate the optimal intervention for large networks.

5.1 Network Formation Model

Multi-type network formation. We consider a simple network formation model that reflects many observed characteristics of the networks in reality.[22] In particular, we consider

[18]When $s_k < \sqrt{\mu}$, the first term is increasing in s_k.

[19]The higher $\cos(\mathbf{u}^k, \mathbf{b}^*)$ implies that in k's dimension the agents whose target opinions are higher embrace the influencers' opinions better. So, in this dimension, the injected opinions will be embraced well as the designer intended.

[20]The higher $\cos(\mathbf{v}^k, \mathbf{b}^0)$ implies that in k's dimension the agents whose initial opinions are higher are the influencers. So, in this dimension, less injected opinions would be required since the influencer already have higher initial opinions.

[21]Note that for some singular vectors, its similarity to the target opinion or the initial opinion can be negative. Hence, some terms of the summation can be negative, but the final value of the injected opinion will be positive. See Section 6 for examples.

[22]See Jackson (2010) and Bramoullé et al. (2016) for empirical regularities of network characteristics and network formation models generating such properties. See Jackson and Rogers (2007), Shin (2021), and references therein for other network formation models explaining empirical regularities.

the multi-type random network model proposed by Golub and Jackson (2012).[23] An agent's type represents certain distinguishing features, such as gender, age, race, education level, or any combinations of these that affect her propensity to be linked with other agents. We assume that there are $m \geq 2$ different types denoted by $\tau_1, \ldots, \tau_m$, and we let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m)$ be the vector of types. Let N_k denote the set of type-τ_k agents, and $n_k = |N_k|$ the number of type-τ_k agents. The corresponding vector of sets is denoted by $\mathbf{N} = (N_1, \ldots, N_m)$, and the vector of cardinalities by $\mathbf{n} = (n_1, \ldots, n_m)$. Note that $n = \sum_k n_k$. $\mathbf{P}$ is a symmetric $m \times m$ matrix, of which an entry $p_{kl} \in (0, 1)$ describes the probability that a type-τ_k agent has a link with a type-τ_l agent. A multi-type random network is represented by a tuple $(\boldsymbol{\tau}, \mathbf{N}, \mathbf{P})$.

The adjacency matrix $\mathbf{A}$ is a realization of the multi-type random network in which an entry $\mathbf{A}_{ij}$ with $i > j$ are independent Bernoulli random variable that takes a value of 1 with probability $p_{kl} \in (0, 1)$ when agent i's type is τ_k and agent j's type is τ_l. An entry $\mathbf{A}_{ij}$ with $i < j$ are automatically filled by $\mathbf{A}_{ij} = \mathbf{A}_{ji}$, and we let $\mathbf{A}_{ii} = 0$ for all i. This network formation model has two notable features: in their link formation, (i) agents are distinguished only by the types, and (ii) every agent of the same type forms links with the other agents in the same stochastic way.

We have three regularity conditions as below.[24]

Assumption 1 *We assume that the network $(\boldsymbol{\tau}, \mathbf{N}, \mathbf{P})$ satisfies the followings:*

(i) No vanishing types: $\frac{n_k}{n_l} \sim \mathcal{O}(1)$ *for all* $1 \leq k, l \leq m$*;*

(ii) A dense network: $p_{kl} \succeq \mathcal{O}\left(\frac{\log^2 n}{n}\right)$ *for all* $1 \leq k, l \leq m$*;*

(iii) Comparable densities: $\frac{p_{kk}}{p_{ll}} \sim \mathcal{O}(1)$ *for all* $1 \leq k, l \leq m$*.*

Assumption 1-(i) means that in terms of the number of agents of a certain type, no type is negligible compared to any other types, as the size of network becomes infinitely large. Assumption 1-(ii) implies that a network is dense enough to ensure a certain connectivity.[25] Finally, in line with Assumption 1-(i), Assumption 1-(iii) guarantees that no agent of a particular type has significantly more links than the others; in other words, every agent has

[23] In the statistics literature, this model is also called the *stochastic block model*. See Fan et al. (2020) and references therein for backgrounds and applications in the recent literature on big data analysis and machine learning.

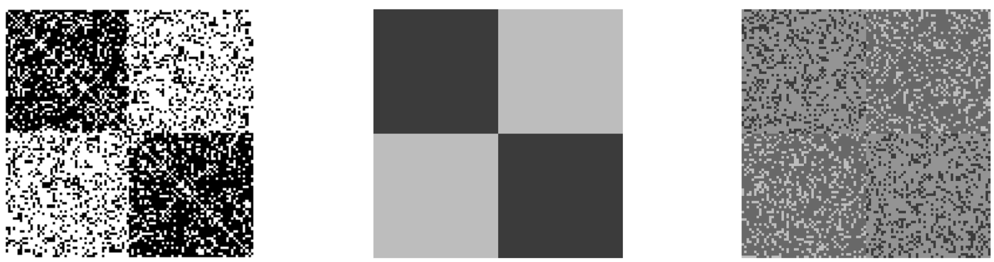
[24] Our conditions correspond to those in Definition 3 in Golub and Jackson (2012). There are certain technical differences between the two sets of assumptions, but they have similar implications, in the sense that they both guarantee the same (or corresponding) results.

[25] This assumption implies that the given parameter values of $(\boldsymbol{\tau}, \mathbf{n}, \mathbf{P})$ contain enough information to approximate the 'expected value' by the law of large number. It also implies that the probability that the realized network is connected converges to one as the network size increases to infinity.

comparably many links.[26]

Matrix perturbation theory. We assume that the influence designer does not directly observe the realized network $\mathbf{A}$, but knows the information of the network formation process represented by the tuple $(\boldsymbol{\tau}, \mathbf{N}, \mathbf{P})$: the number of agents and their types, and the probability of link formation between the agents.

Let $\overline{\mathbf{A}} = \mathbb{E}[\widetilde{\mathbf{A}}]$, where $\widetilde{\mathbf{A}}_{ij} = \mathbf{A}_{ij}$ for all $i \in N_k$ and $j \in N_l$ with $i \neq j$, and $\widetilde{\mathbf{A}}_{ii} = p_{kk}$ for all $i \in N_k$. The auxiliary expectation matrix $\overline{\mathbf{A}}$ represents the expectation matrix of $\mathbf{A}$.[27] With this notation, we can express the unobserved adjacency matrix $\mathbf{A}$ by $\mathbf{A} = \overline{\mathbf{A}} + (\mathbf{A} - \overline{\mathbf{A}})$, where $(\mathbf{A} - \overline{\mathbf{A}})$ captures the noise (or deviation) from the expectation of the adjacency matrix.



(a) realization (b) expectation (c) noise

Figure 4: Illustration of matrix perturbation approach

Figure 4-(a) is an illustration of a realized multi-type network of size 100, where there are two types of agents with the equal size of 50. The link formation probabilities are assumed to be identical across types as $p_{11} = p_{22} = \frac{2}{3} > p_{12} = p_{21} = \frac{1}{3}$. A black dot implies that there is a link between two agents. The matrix $\mathbf{A}$ can be decomposed into the expectation $\overline{\mathbf{A}}$ and the noise $\mathbf{A} - \overline{\mathbf{A}}$. Figure 4-(b) is the expectation $\overline{\mathbf{A}}$, which has rank 2, and Figure 4-(c) is the noise. One notable feature of this decomposition is that the expectation has rank 2 as

$$\begin{aligned} \overline{\mathbf{A}} &= \begin{pmatrix} p_{11}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\intercal} & p_{12}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\intercal} \\ p_{12}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\intercal} & p_{11}\mathbf{1}_{n/2}\mathbf{1}_{n/2}^{\intercal} \end{pmatrix} \\ &= \tfrac{p_{11}+p_{12}}{2} \begin{pmatrix} \mathbf{1}_{n/2} \\ \mathbf{1}_{n/2} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n/2} \\ \mathbf{1}_{n/2} \end{pmatrix}^{\intercal} + \tfrac{p_{11}-p_{12}}{2} \begin{pmatrix} \mathbf{1}_{n/2} \\ -\mathbf{1}_{n/2} \end{pmatrix} \begin{pmatrix} \mathbf{1}_{n/2} \\ -\mathbf{1}_{n/2} \end{pmatrix}^{\intercal} \end{aligned}$$

where $\mathbf{1}_{n/2}$ is the vector of ones with length $n/2$. Note that $\frac{p_{11}+p_{12}}{2}$ represents the first singular value, and $\frac{p_{11}-p_{12}}{2}$ represents the second singular value.[28] All the other singular values are

[26] One may assume that $p_{kk} > p_{kl}$ for all $1 \leq k, l \leq m$, which captures the idea of *homophily* that refers to the phenomenon that similar people tend to attach to each other more often than to dissimilar people (e.g., Bramoullé et al. 2012; McPherson et al. 2001). See Echenique and Fryer (2007) and references therein for homophily measurements. We study an example with the homophily assumption in Section 5.3.

[27] All the off-diagonal entries of $\overline{\mathbf{A}}$ are the same as the corresponding entries of $\mathbb{E}[\mathbf{A}]$.

[28] More precisely, the first singular value is $n(p_{11} + p_{12})/2$, and the second singular value is $n(p_{11} - p_{12})/2$.

zero. This observation implies that for a general multi-type network with m distinctive types, the rank of expectation of the adjacency matrix is at most m^2 as every agent of the same type has the same expected link formations. As the expectation matrix has a known form, we wish for the influence designer to achieve optimality with an intervention strategy constructed by this expectation matrix.

5.2 Characterization of the Optimal Intervention

We present a way to construct an approximated intervention $\overline{\mathbf{b}}'(n)$ for the actual optimal intervention $\mathbf{b}'(n)$, as functions of network size n. We construct the expected influence matrix as $\overline{\mathbf{T}} = \left(\boldsymbol{\alpha} + (\mathbf{I}-\boldsymbol{\alpha})\mathbf{D}(\overline{\mathbf{A}})\right)^{-1}\left(\boldsymbol{\alpha} + (\mathbf{I}-\boldsymbol{\alpha})\,\overline{\mathbf{A}}\right)$. For notational simplicity, we consider the case in which there are only two types, and $p_{11} = p_{22} > p_{12} = p_{21}$ as in the island model (Golub and Jackson 2012).[29] As such, the rank of $\overline{\mathbf{A}}$ is two, independent of network size n. We define the approximated intervention $\overline{\mathbf{b}}'(n)$ by the singular value decomposition of $\overline{\mathbf{T}}$ as in previous sections:

$$\overline{\mathbf{b}}'(n) - \mathbf{b}_0 = \sum_{k=1}^{2} \frac{\overline{s}_k}{\overline{s}_k^2 + \overline{\mu}} \left(||\mathbf{b}^*||\cos(\overline{\mathbf{u}}^k, \mathbf{b}^*) - \overline{s}_k||\mathbf{b}^0||\cos(\overline{\mathbf{v}}^k, \mathbf{b}^0)\right)\overline{\mathbf{v}}^k,$$

where $\mathbf{b}_0$ is the initial opinion, and $\overline{\mu}$ is the Lagrangian multiplier of the optimization problem with the expectation of the adjacency matrix.[30] Our goal is to show that as the network size n increases to infinity, $\mathbf{b}'(n)$ converges to $\overline{\mathbf{b}}'(n)$ up to some normalization of population size under a proper convergence notion. In this regard, we define the following:

Definition 1 $\overline{\mathbf{b}}'(n)$ is said to be *asymptotically optimal* if $\mathbf{b}'(n)$ converges in probability to $\overline{\mathbf{b}}'(n)$: for any given $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{\sqrt{n}}||\mathbf{b}'(n) - \overline{\mathbf{b}}'(n)|| > \varepsilon\right] = 0.$$

Note that the asymptotic optimality of $\overline{\mathbf{b}}'(n)$ implies that the expected value of the objective function in (DP 1), $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(\mathbf{b}_i^* - \mathbf{b}_i)^2\right]$, where $\mathbf{b} = \mathbf{T}\mathbf{b}'(n)$ converges to the expected value where $\mathbf{b} = \mathbf{T}\overline{\mathbf{b}}'(n)$.

Now, we sketch the proof of the asymptotic optimality of $\overline{\mathbf{b}}'(n)$.[31] For expositional simplicity, we here assume that $\alpha_i = 0$ for all i, and there are only two types with equal size. Let $\overline{s}_k$, $\overline{\mathbf{u}}^k$, and $\overline{\mathbf{v}}^k$ be the kth singular value and the corresponding (left and right) singular

However, as the singular vectors are normalized by the size of network n, the terms n's are cancelled out with each other in the expression.

[29] The proof in Appendix A does not rely on this simplicity.

[30] That is, $\overline{\mu}$ is a unique solution to the equation $\sum_{k=1}^{2}\left(\frac{\overline{s}_k}{\overline{s}_k^2+\overline{\mu}}\right)^2\left(\overline{\mathbf{b}}_k^* - \overline{s}_k\underline{\mathbf{b}}_k^0\right)^2 = C$.

[31] The complete proof is in Appendix A.

vectors of $\overline{\mathbf{T}}$. Recall that the approximated intervention $\overline{\mathbf{b}}'(n)$ is

$$\overline{\mathbf{b}}'(n) = \sum_{k=1}^{2} \frac{\overline{s}_k}{\overline{s}_k^2 + \mu} \left(||\mathbf{b}^*|| \cos(\overline{\mathbf{u}}^k, \mathbf{b}^*) - \overline{s}_k ||\mathbf{b}^0|| \cos(\overline{\mathbf{v}}^k, \mathbf{b}^0) \right) \overline{\mathbf{v}}^k.$$

We also denote $\overline{\mathbf{b}}'(n)$ as a low-rank approximation of $\mathbf{b}'(n)$: $\overline{\mathbf{b}}'(n)$ is constructed by $\overline{\mathbf{T}} = \mathbf{U}_2 \mathbf{S}_2 \mathbf{V}_2^\mathsf{T}$ with $\mathbf{U}_2 = [\mathbf{u}^1, \mathbf{u}^2, \mathbf{0}_n, \ldots, \mathbf{0}_n]$, $\mathbf{S}_2 = diag(s_1, s_2, 0, \ldots, 0)$, $\mathbf{V}_2 = [\mathbf{v}^1, \mathbf{v}^2, \mathbf{0}_n, \ldots, \mathbf{0}_n]$, and $\mathbf{0}_n$ is the vector of zeros of size n. The sketch of the proof has two parts: (i) By showing $||\mathbf{H}|| = ||\mathbf{T} - \overline{\mathbf{T}}||$ goes to zero as n goes to infinity, we prove that s_k converges to $\overline{s}_k$. (ii) By using the Wedin sin θ theorem, we show that the two leading left and right singular vectors of $\mathbf{T}$ converge to those of $\overline{\mathbf{T}}$.

Part (i) We find that

$$||\mathbf{H}|| = \left|\left| \left[\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})} - \frac{\overline{\mathbf{A}}_{ij}}{d_i(\overline{\mathbf{A}})} \right] \right|\right| \leq \left|\left| \left[\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})} \left(1 - \frac{d_i(\mathbf{A})}{d_i(\overline{\mathbf{A}})} \right) \right] \right|\right| + \left|\left| \left[\frac{(\mathbf{A}_{ij} - \overline{\mathbf{A}}_{ij})}{d_i(\overline{\mathbf{A}})} \right] \right|\right|,$$

where $|| \cdot ||$ represents the operator norm of matrices. Then, by the standard techniques in the literature on social and economic networks (e.g., Golub and Jackson 2012), both terms on the right-hand side of the above inequality converge in probability to zero, which relies on Assumption 1-(ii). Consequently, $||\mathbf{H}||$ converges to zero in probability, which implies that, for all k, s_k converges to $\overline{s}_k$ in probability as n increases to infinity.[32]

Part (ii) Note that the two leading singular values are strictly positive $\overline{s}_1 \geq \overline{s}_2 = \triangle > 0$, and for any given $\varepsilon > 0$, $||\mathbf{H}|| < \triangle\varepsilon$ with a high probability for large n (i.e., convergence in probability). The Wedin $\sin\theta$ theorem states that with a probability of at least $1 - \varepsilon$, the first two leading singular vectors of the expected influence matrix $\overline{\mathbf{T}}$ are close to the first two leading singular vectors of the realized influence matrix $\mathbf{T}$ (Wedin 1972, 1983). Specifically, let $\Theta(\mathbf{V}, \widehat{\mathbf{V}})$ denote the 2×2 diagonal matrix whose jth diagonal entry is the jth singular angle, and let $\sin\Theta(\mathbf{V}, \widehat{\mathbf{V}})$ be defined entry-wise. Then,

$$\max\{\sin\Theta(\mathbf{U}_0, \overline{\mathbf{U}}_0), \sin\Theta(\mathbf{V}_0, \overline{\mathbf{V}}_0)\} \leq \frac{||\mathbf{H}||}{\triangle} < \varepsilon,$$

where $\mathbf{U}_0$ and $\overline{\mathbf{U}}_0$ are the matrices having the two leading left singular vectors of $\mathbf{T}$ and $\overline{\mathbf{T}}$, respectively, and $\mathbf{V}_0$ and $\overline{\mathbf{V}}_0$ are the matrices with the two leading right singular vectors of $\mathbf{T}$ and $\overline{\mathbf{T}}$, respectively. Therefore, by the Wedin $\sin\theta$ theorem, we have the convergence

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[32] Unfortunately, the convergence of $||\mathbf{H}||$ to zero in probability is not sufficient to ensure the convergence in probability of the singular vectors even if all singular values, $\overline{s}_k$'s, converge. A popular example of this problem is the following. Let $\varepsilon > 0$ and consider two matrices $\mathbf{I}$ and $\mathbf{M}$:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \mathbf{I} + \varepsilon \mathbf{1}\mathbf{1}^\mathsf{T} = \begin{pmatrix} 1+\varepsilon & \varepsilon \\ \varepsilon & 1+\varepsilon \end{pmatrix}.$$

$\mathbf{M}$ is a small perturbation of $\mathbf{I}$ as $||\mathbf{I} - \mathbf{M}|| = 2\varepsilon$. They have similar eigenvalues as $\lambda(\mathbf{I}) = 1$ and $\lambda(\mathbf{M}) = (1 + 2\varepsilon), 1$. However, $(1,0)^\mathsf{T}$ and $(0,1)^\mathsf{T}$ are eigenvectors of $\mathbf{I}$, while the eigenvectors of $\mathbf{M}$ are $(1,1)^\mathsf{T}$ and $(1,-1)^\mathsf{T}$ for all ε.

of the two leading left and two leading right singular vectors as well as their corresponding singular values of the expected influence matrix.

The following theorem states the asymptotic optimality of the approximated intervention.

Theorem 2 *The approximated intervention* $\overline{\mathbf{b}}'(n)$ *is asymptotically optimal.*

Theorem 2 provides a powerful insight into economics. It implies that if the given network is sufficiently large, the influence designer does not have to know the complete structure of the network to design the optimal intervention. If one has enough information on the network formation process or the average connectivity between heterogeneous groups, she can find a properly approximated strategy that is statistically equivalent to the optimal strategy. In other words, to design the optimal intervention, the designer may not need any detailed micro-level network information on individual agents. Some type-specific information, such as the probability of link formation between agents, would be enough. This result may imply that the optimal intervention with incomplete information is not individual-specific, but type-specific. The following representative agent theorem formalizes this insight.

Theorem 3 (Representative agent theorem) *In the approximated intervention, the injected opinion for each agent is the same for agents of a common type:*

$$\overline{\mathbf{b}}'(n)_i - \mathbf{b}^0_i = \overline{\mathbf{b}}'(n)_j - \mathbf{b}^0_j \quad \textit{whenever } i, j \in N_k \textit{ for any } k.$$

Note that this theorem does not imply that the resulting opinions of agents of the same type would be identical after the asymptotically optimal intervention. The equivalence in the injected opinion in a group just implies that the influence designer puts the same effort or resources into agents with the same type.

5.3 Example: Island Model

Island model. Here, we characterize closed-form solutions of the optimal intervention under the incomplete information setting in an island model of two types (Golub and Jackson 2012). In particular, we restrict our attention to a symmetric case in which (i) there are n agents of two types, (ii) each type is identical in terms of their size, and (iii) the link formation probabilities of $\mathbf{P}$ are $p_{11} = p_{22} = p_s$ and $p_{12} = p_{21} = p_d$. We assume that $p_s > p_d$, which captures the idea of homophily in real social networks. Finally, without loss of generality, we assume that agents 1 to $\frac{n}{2}$ are type τ_1, and the other agents are type τ_2: $i \in N_1$ if $1 \leq i \leq \frac{n}{2}$, and $i \in N_2$ if $\frac{n}{2} < i \leq n$.

In this model, there are two non-zero singular values of $\overline{\mathbf{T}}$: $\overline{s}_1 = 1$ and $\overline{s}_2 = \frac{p_s - p_d}{p_s + p_d}$. The expected degree of each agent is $\frac{n}{2}(p_s + p_d)$. The difference between the expected number of same-type friends and the expected number of different-type friends is $\frac{n}{2}(p_s - p_d)$, which captures the intensity of homophily. Therefore, $\overline{s}_2 = \frac{p_s - p_d}{p_s + p_d}$ is the homophily-density ratio of the network. Since the influence matrix is symmetric, the singular vectors are $\overline{\mathbf{v}}^1 = \overline{\mathbf{u}}^1 = \frac{1}{\sqrt{n}}(\mathbf{1}_{n/2}, \mathbf{1}_{n/2})^\intercal$ and $\overline{\mathbf{v}}^2 = \overline{\mathbf{u}}^2 = \frac{1}{\sqrt{n}}(\mathbf{1}_{n/2}, -\mathbf{1}_{n/2})^\intercal$ by the Perron-Frobenius theorem.

In what follows, we investigate the optimal intervention $\overline{\mathbf{b}}'(n)$ and the shadow price $\overline{\mu}$ in two different cases–uniform and clustered interventions–based on the target and initial opinions ($\mathbf{b}^*$ and $\mathbf{b}^0$). Benefiting from the representative agent theorem (Theorem 3), we can simply focus on an agent of each type.

Case 1: Uniform intervention. We first consider the simplest example in which $\mathbf{b}^* = b^*\mathbf{1}$ and $\mathbf{b}^0 = b^0\mathbf{1}$, such that $0 \leq b^0 < b^* \leq 1$. Since influence matrix $\overline{\mathbf{T}}$ is symmetric, $\cos(\overline{\mathbf{u}}^1, \mathbf{b}^*) = \cos(\overline{\mathbf{v}}^1, \mathbf{b}^0) = 1$ and $\cos(\overline{\mathbf{u}}^2, \mathbf{b}^*) = \cos(\overline{\mathbf{v}}^2, \mathbf{b}^0) = 0$.[33] That is, although there are two different types of agents, they are all symmetric in terms of (i) their hub and authority centralities ($\mathbf{v}^1$ and $\mathbf{u}^1$, respectively), and (ii) their cosine similarities to the target belief and the initial belief. Therefore, the optimal intervention is uniform as

$$\overline{\mathbf{b}}'(n) = \mathbf{b}^0 + \frac{1}{1+\overline{\mu}}(b^* - b^0)\mathbf{1}.$$

The shadow price $\overline{\mu} = \frac{b^* - b^0}{\sqrt{C}} - 1$, is network independent as it does not depend on any value of $\mathbf{P}$. This result arises from the fact that there is only one principal axis that is aligned with the line connecting $\mathbf{b}^0$ and $\mathbf{b}^*$, and that principle axis is independent of $\mathbf{P}$. Moreover, since the singular value of this dimension is 1, there is no multiplier effect. As such, the optimal intervention and the corresponding shadow price are network independent.

Proposition 1 *Suppose* $\mathbf{b}^* = b^*\mathbf{1}$ *and* $\mathbf{b}^0 = b^0\mathbf{1}$. *Then, the injected opinion* $\overline{\mathbf{b}}'(n) - \mathbf{b}^0$ *is uniform as*

$$\overline{\mathbf{b}}'(n) - \mathbf{b}^0 = \frac{1}{1+\overline{\mu}}(b^* - b^0)\mathbf{1},$$

where $\overline{\mu}$ *is the unique solution to the equation*

$$\left(\frac{1}{1+\overline{\mu}}\right)^2 \left(b^* - b^0\right)^2 = C.$$

Case 2: Clustered intervention. We now consider the case in which agents are homogeneous in terms of their target beliefs, $\mathbf{b}^* = b^*\mathbf{1}$, but one group of agents has a relatively higher initial opinion than the others. Specifically, we assume that $\mathbf{b}_i^0 = b_1^0$ if $i \in N_1$, and $\mathbf{b}_i^0 = b_2^0$ if $i \in N_2$, where $0 \leq b_2^0 < b_1^0 < b^* \leq 1$. In the context of marketing, these are relatively loyal consumers, so they have more favorable opinions of the designer's product and/or service.[34] In the context of politics, one group has relatively closer bliss points to

[33] Since all the other singular vectors are orthogonal to $\overline{\mathbf{u}}^1$ and $\overline{\mathbf{v}}^1$, we also have $\cos(\overline{\mathbf{u}}^k, \mathbf{b}^*) = \cos(\overline{\mathbf{v}}^k, \mathbf{b}^0) = 0$ for all $k \geq 2$.

[34] For instance, Birke and PeterSwann (2010) find that cell phone users coordinate their choices of mobile network operators based on their pre-existing social relationships.

the designer's policy. The following proposition characterizes the optimal intervention of this case.

Proposition 2 *Suppose* $\mathbf{b}^* = b^*\mathbf{1}$ *and* $\mathbf{b}_i^0 = b_1^0$ *if* $i \in \tau_1$*, and* $\mathbf{b}_i^0 = b_2^0$ *if* $i \in \tau_2$*, where* $0 \leq b_2^0 < b_1^0 < b^* \leq 1$*. Then, the injected opinion* $\overline{\mathbf{b}}'(n) - \mathbf{b}^0$ *is clustered as*

$$\overline{\mathbf{b}}'(n) - \mathbf{b}^0 = \frac{1}{1+\overline{\mu}}\left(b^* - \left(\frac{b_1^0 + b_2^0}{2}\right)\right)\mathbf{1} + \frac{\overline{s}_2}{\overline{s}_2^2 + \overline{\mu}}\left(\frac{b_1^0 - b_2^0}{2}\right)(-\mathbf{1}_{n/2}, \mathbf{1}_{n/2})^\intercal,$$

where $\overline{s}_2 = \frac{p_s - p_d}{p_s + p_d}$*, and* $\overline{\mu}$ *is the unique solution to the equation*

$$\left(\frac{1}{1+\overline{\mu}}\right)^2\left(b^* - \left(\frac{b_1^0 + b_2^0}{2}\right)\right)^2 + \left(\frac{\overline{s}_2}{\overline{s}_2^2 + \overline{\mu}}\right)^2\left(\frac{b_1^0 - b_2^0}{2}\right)^2 = C.$$

The shadow price $\overline{\mu}$ *increases in the intensity of homophily but decreases in the network density.*

The designer's problem is illustrated in Figure 5. The rotated red ellipse and the shaded region inside of the ellipse represent the possible choices of $\underline{\mathbf{b}}'$ that the designer has under the budget constraint. The lengths of the major and minor axes of the ellipse are $\overline{s}_1 = 1$ and $\overline{s}_2 = \frac{p_s - p_d}{p_s + p_d}$, respectively. The thick blue arrow in the figure, $\overline{\mathbf{b}}'(n) - \mathbf{b}^0$, illustrates the optimal choice of injected opinion, which returns the shortest distance between the target belief and the vectors inside of the ellipse. The length of the blue arrow in the b_2-axis is longer than its length in the b_1-axis. Thus, the influence designer injects "more" opinion into the agents with lower initial opinions.[35]

Since the τ_1 type agents have more favorable initial opinions, for the designer, injection into the τ_1 type is less cost-effective than that to the τ_2 type. As such, in the characterization of $\overline{\mathbf{b}}'(n)$, the second term is strictly positive for τ_2 but strictly negative for τ_1. Since the first term is strictly positive for all agents, the coefficient in the second term, $\frac{\overline{s}_2}{\overline{s}_2^2+\overline{\mu}}\left(\frac{b_1^0 - b_2^0}{2}\right)$, represents the designer's adjustment for different groups in the intervention.

Everything else being equal, this adjustment $\frac{\overline{s}_2}{\overline{s}_2^2+\overline{\mu}}\left(\frac{b_1^0 - b_2^0}{2}\right)$ increases in the intensity of homophily, $p_s - p_d$, when the budget is sufficiently tight. This monotonicity originates in the fact that as the intensity of homophily increases, it becomes more valuable to distinguish agents according to their types. Geometrically, the minor axis of the ellipse is increasing in the intensity of homophily, which implies that the length of the blue arrow along the b_2-axis is also increasing in the intensity of homophily. Intuitively, this homophily information

[35] The cosine similarities are different as the cosine similarity of the initial opinion vector is not orthogonal to the second singular vectors. To see why, note that

$$\overline{\mathbf{u}}^2 = \overline{\mathbf{v}}^2 = \frac{1}{\sqrt{n}}(\mathbf{1}_{n/2}, -\mathbf{1}_{n/2})^\intercal, \quad \overline{\mathbf{u}}^2 \cdot \mathbf{b}^* = 0, \quad \text{and} \quad \overline{\mathbf{v}}^2 \cdot \mathbf{b}^0 = \frac{b_1^0 - b_2^0}{2} > 0.$$

The signs of $\overline{\mathbf{u}}^2$ indicate a spectral cluster of the agents, which has been well-studied in the literature on how to group nodes in a network according to some criterion (e.g., Chung 1996; Liu and Stewart 2011; Luxburg 2007).

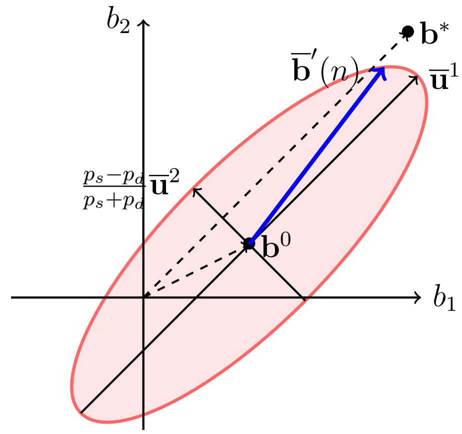


Figure 5: Illustration of the optimal intervention under incomplete information

becomes more valuable as the network density decreases; in Figure 5, the length of the minor axis decreases in the parameters representing the network density, $p_s + p_d$.

5.4 Example: Heterogeneous Group Size and Homophily

We consider two variants of the island model from the previous subsection: one example with two groups of different population sizes, and another with two groups that are homogeneous in size but differ in their homophilous network formation.

Heterogeneous group size. There are n agents of two types, where one group consists of n_1 agents of type τ_1, and the other group consists of n_2 agents of type τ_2 with $k \geq \lfloor \frac{n+1}{2} \rfloor$, without loss of generality. We assume that $n_1 = kn_2$ with $k \geq 1$; that is, the population size of type τ_1 agents is k times of the size of type τ_2 agents. We still assume that link formation probabilities of $\mathbf{P}$ are $p_{11} = p_{22} = p_s$ and $p_{12} = p_{21} = p_d$ with $p_s > p_d$.

The asymmetric group size does not change the rank of $\overline{\mathbf{T}}$, which sill has two non-zero singular values. As $\overline{\mathbf{T}}$ is a symmetric stochastic matrix, the first singular value is 1 as in the case of the symmetric island model.[36] The second singular value is calculated as $\overline{s}_2 = \frac{(p_s-p_d)(p_s+p_d)}{(p_s+kp_d)(kp_s+p_d)}$.

We know that if the agents have the same initial opinions, then a uniform intervention is optimal, as shown in Proposition 1. Thus, we consider a case in which the agents of two groups have different initial opinions, so that a clustered intervention would be optimal, and ask how the optimal clustered intervention changes when the group size heterogeneity k changes.

Intuitively, as the group size heterogeneity k increases, the first group becomes more im-

[36]When a real matrix is symmetric, each of singular values is an absolute value of an eigenvalue of the same matrix (Strang 2019).

portant; consequently, the uniform intervention focusing on the first group of agents becomes closer to the clustered intervention. This intuition is confirmed by the fact that the second singular value monotonically decreases and converges to zero as the group size heterogeneity k increases. Similarly, the shadow price $\overline{\mu}$ decreases in k.

The following proposition formalizes the discussion above:

Proposition 3 *Suppose $\mathbf{b}^* = b^*\mathbf{1}$ and $\mathbf{b}_i^0 = b_1^0 \in [0,1]$ if $i \in \tau_1$, and $\mathbf{b}_i^0 = b_2^0 \in [0,1]$ if $i \in \tau_2$, where $b_2^0 \neq b_1^0$. Then, the injected opinion $\overline{\mathbf{b}}'(n) - \mathbf{b}^0$ is clustered as*

$$\overline{\mathbf{b}}'(n) - \mathbf{b}^0 = \frac{1}{1+\overline{\mu}}\left(b^* - \left(\frac{b_1^0 + b_2^0}{2}\right)\right)\mathbf{1} + \frac{\overline{s}_2}{\overline{s}_2^2 + \overline{\mu}}\left(\frac{b_1^0 - b_2^0}{2}\right)(-\mathbf{1}_{n/2}, \mathbf{1}_{n/2})^{\mathsf{T}},$$

where $\overline{s}_2 = \frac{(p_s - p_d)(p_s + p_d)}{(p_s + kp_d)(kp_s + p_d)}$, and $\overline{\mu}$ is the unique solution to the equation

$$\left(\frac{1}{1+\overline{\mu}}\right)^2\left(b^* - \left(\frac{b_1^0 + b_2^0}{2}\right)\right)^2 + \left(\frac{\overline{s}_2}{\overline{s}_2^2 + \overline{\mu}}\right)^2\left(\frac{b_1^0 - b_2^0}{2}\right)^2 = C.$$

The shadow price $\overline{\mu}$ decreases in parameter k, the group size heterogeneity. Furthermore, the second term of the injected opinion monotonically decreases and converges to zero as k increases.

Heterogeneous homophily. Now consider the island model in which the groups are homogeneous in their size but different in their homophilous network formation. In particular, we consider an open-minded group and a close-minded group: the link formation probabilities of $\mathbf{P}$ are defined as $p_{11} = \lambda p_{12}$ with $\lambda > 1$, $p_{12} = 0$, and $p_{21} = 1$, where λ represents the degree of homophily of the τ_1 type agents; as λ becomes larger, the τ_1 type agents are less influenced by the τ_2 agent group's opinions. As such, the τ_1 type agents are open-minded: they listen to the τ_2 type agents' opinions, while the τ_2 type agents never listen to the τ_1 type agents' opinions, so that they are not influenced by any τ_1 type agents.

There are two non-zero singular values of $\overline{\mathbf{T}}$:

$$\overline{s}_1 = \frac{\sqrt{1+\lambda+\lambda^2+\sqrt{1+2\lambda(1+\lambda)}}}{1+\lambda} \quad \text{and} \quad \overline{s}_2 = \frac{\sqrt{1+\lambda+\lambda^2-\sqrt{1+2\lambda(1+\lambda)}}}{1+\lambda}.$$

Since $\overline{s}_1$ strictly decreases but $\overline{s}_2$ strictly increases in λ, the gap between the singular values monotonically decreases as the degree of homophily λ increases.[37] The influence designer's injected opinion is larger for the close-minded group because the hub centrality of the close-minded group (i.e., $\mathbf{v}_2^1$) is larger than the hub centrality of the open-minded group (i.e., $\mathbf{v}_1^1$)

[37]Note that different from the previous examples, the largest singular value is not 1 because $\overline{\mathbf{T}}$ is not symmetric.

for all $\lambda \geq 1$.[38]

We finally argue that as parameter λ increases, the difference of the injected opinions to two groups monotonically decreases and disappears eventually. To see why, note that $|\overline{s}_1 - \overline{s}_2| \to 0$ as $\lambda \to \infty$. The gap between $\mathbf{v}_1^1$ and $\mathbf{v}_2^1$ does not converge to zero as $\lambda \to \infty$; however, $\mathbf{v}_2^1 - \mathbf{v}_1^1 = \mathbf{v}_1^2 - \mathbf{v}_2^2$ for all λ. As such, the adjustment by the second term in the optimal intervention increases as λ increases, and it eventually offsets the first term. Therefore, at the limit, both groups obtain exactly the same injected opinion by the designer.

The following proposition summarizes the above discussion:

Proposition 4 *Suppose* $\mathbf{b}^* = b^*\mathbf{1}$ *and* $\mathbf{b}^0 = b^0\mathbf{1}$. *Then, the injected opinion* $\overline{\mathbf{b}}'(n) - \mathbf{b}^0$ *is larger for the close-minded group than the open-minded group. The gap between the opinions injected to the groups monotonically decreases and converges to zero as homophily heterogeneity disappears.*

6 Extensions

In this section, we present a series of further analyses of the optimal intervention under different environments.

6.1 Repeated Social Learning under Complete Information

We consider the scenario under which agents repeatedly learn from their neighbors. $\mathbf{T}^k$ is the influence matrix when agents learn from their neighbors k times. In addition to the strong connectedness of the network, we also assume that the number of agents n is odd, so that $\mathbf{T}^\infty = \lim_{k\to\infty} \mathbf{T}^k$ is well-defined.[39] When $\mathbf{b}'$ is implanted, after infinitely repeated social learning, the agents eventually reach a consensus $b \in [0,1]$ defined by $\mathbf{b} = b\mathbf{1} = \mathbf{T}^\infty \mathbf{b}'$ (Golub and Jackson 2012).[40] Since b is the consensus of the agents, it must be that $\mathbf{T}_i^\infty = \mathbf{T}_j^\infty$ for all i and j. Thus, $\mathbf{T}^\infty$ is a rank one matrix.

The rank one property of $\mathbf{T}^\infty$ is important: the optimal intervention has only one non-zero term if the agents update their social learning process sufficiently many times. Consequently, the injected opinion will be proportional to the value of hub centrality (i.e., the right sin-

[38]The right singular vector corresponding to the largest singular value is

$$\mathbf{v}^1 = \left(\frac{2}{\sqrt{4 + \frac{4(1+\lambda+\sqrt{1+2\lambda(1+\lambda)})^2}{\lambda^2}}}, \frac{2}{\sqrt{4 + \frac{-2-2\lambda+\sqrt{4+8\lambda(1+\lambda)}}{\lambda^2}}} \right)^\mathsf{T}.$$

[39]Formally, when n is odd, $\mathbf{T}$ is irreducible and aperiodic because the network is assumed to be strongly connected.

[40]It is well-known from previous studies that agents reach a consensus after repeated social learning à la DeGroot (1974), and the consensus is a weighted average of the agents' given opinions.

gular vector) that is associated with the unique non-zero singular value. Then, the question becomes what the values of the right singular vector and first singular value are.

In order to obtain a transparent interpretation of the hub centrality with a closed-form expression, we further assume that every agent takes her own opinion into account with the same importance (i.e., $\alpha_i = \alpha_j$ for all i, j). In this case, we can find that the injected opinion for agent i, $\mathbf{b}_i - \mathbf{b}_i^0$, is proportional to the value of the degree of the agent. First, we show that the hub centrality of agent i is proportional to the degree of agent i. To see why, note that $\mathbf{v}^1$ is defined as an eigenvector of $(\mathbf{T}^\infty)^\top \mathbf{T}^\infty$, which has a unique non-zero eigenvalue. Let $\mathbf{d} = (d_1(\mathbf{A}), \ldots, d_n(\mathbf{A}))^\top$ be the vector of the degrees of agents in the network. We observe that $(\mathbf{T}^\infty)^\top \mathbf{T}^\infty = \frac{n}{\mathbf{1}\mathbf{d}\mathbf{d}^\top\mathbf{1}} \mathbf{d}\mathbf{d}^\top$, and thus, $\mathbf{d}$ is an eigenvector of the unique non-zero eigenvalue. Thus, we can explicitly calculate the hub centrality, as $\mathbf{v}^1 = \frac{\mathbf{d}}{||\mathbf{d}||}$, which is proportional to the degree centrality.

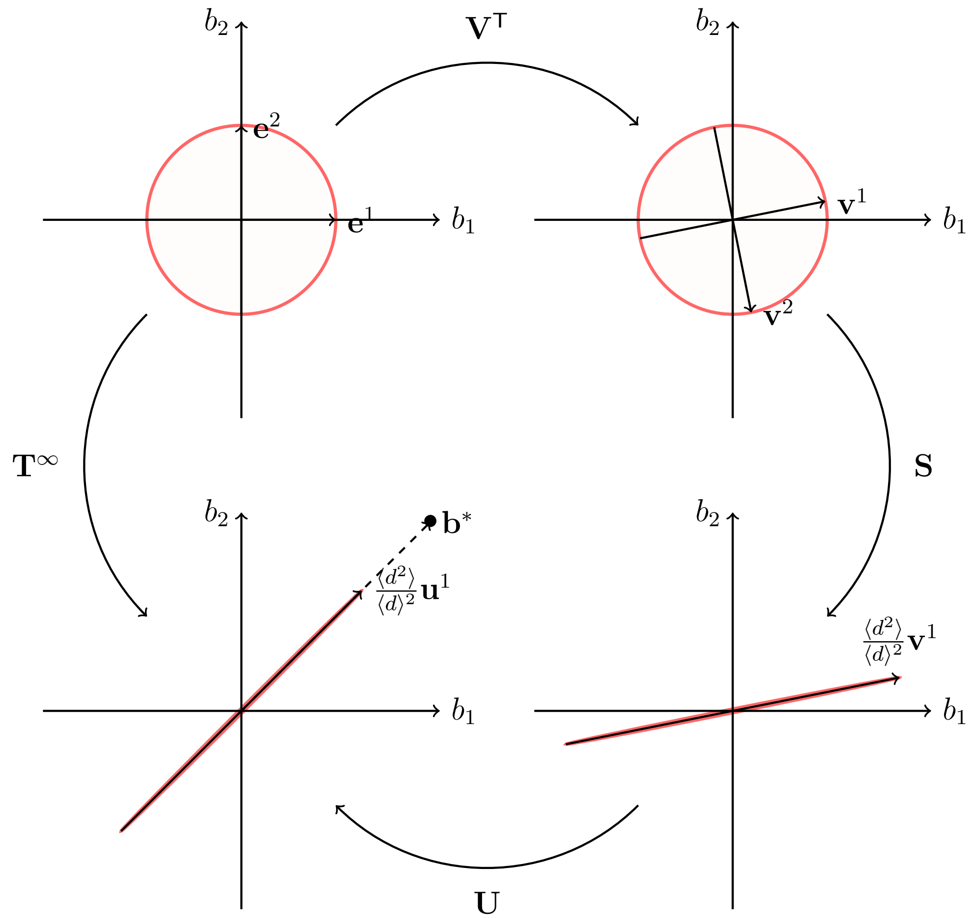


Figure 6: Illustration of the singular value decomposition of $\mathbf{T}^\infty$ for the star network of five agents. Here, we assume that $\mathbf{b}_i = \mathbf{b}_j = 0$ and $\mathbf{b}_i^* = \mathbf{b}_j^* = b^* > 0$ for all i. As the influence matrix $\mathbf{T}^\infty$ is a rank one matrix, there is only one non-zero singular value. Since $\mathbf{v}^k = 0$ for all $k \geq 2$, it is optimal to choose $\mathbf{b}_i'$ proportional to $\mathbf{u}_i^1$, the degree centrality of agent i.

Second, the authority centrality is the unit vector of ones as $\mathbf{u}^1 = \frac{\mathbf{1}}{\sqrt{n}}$, which means that every agent is identically influenced by the other agents' opinions. The corresponding eigenvalue is $s_1 = \frac{\langle d^2 \rangle}{\langle d \rangle^2}$, which is the ratio of the second moment to the square of the first moment of the degree distribution of the network.

The following proposition formally summarizes the discussions so far:

Proposition 5 *Suppose that, after the designer intervenes, agents exchange opinions repeatedly and infinitely. Then, the injected opinion* $\mathbf{b}' - \mathbf{b}^0$ *is proportional to the hub centrality associated with the unique non-zero singular value of the limit of the kth power of the influence matrix. If the adjacency matrix is symmetric, and every agent takes her own opinion into account with the same importance, then, the injected opinion* $\mathbf{b}' - \mathbf{b}^0$ *is proportional to the degree centrality:*

$$\mathbf{b}' - \mathbf{b}^0 = \frac{\langle d^2 \rangle / \langle d \rangle^2}{(\langle d^2 \rangle / \langle d \rangle^2)^2 + \mu} \left(\mathbf{b}^* \cdot \frac{\mathbf{1}}{\sqrt{n}} - \frac{\langle d^2 \rangle}{\langle d \rangle^2} \mathbf{b}^0 \mathbf{1} \cdot \frac{\mathbf{d}}{||\mathbf{d}||} \right) \frac{\mathbf{d}}{||\mathbf{d}||},$$

where $\mathbf{d} = (d_1(\mathbf{A}), \ldots, d_n(\mathbf{A}))^\intercal$*, and* μ *is calculated as in Theorem 1.*

This proposition implies that for certain types of networks, having little information about the network (i.e., the degree distribution) is sufficient to characterize the optimal intervention in the model of repeated social learning under complete information.

We close this subsection with a few remarks. First, for the intermediate case where agents interact k times, the singular value decomposition of $\mathbf{T}^k$ will provide the optimal solution in Theorem 1. Second, one might be interested in whether the optimal solution with $\mathbf{T}^k$ converges to $\mathbf{T}^\infty$. The answer is yes, and one can easily show this with the Davis-Kahan theorem (Davis and Kahan 1970). Note that the convergence arises because of the rank one property of $\mathbf{T}^\infty$: all the singular values except the largest one converge to zero, and so we only need the convergence of the first left and right singular vectors. Third, one might be interested in whether the difference between the optimal intervention and the intervention only with the degree distribution monotonically decreases to zero as k increases. This exercise is related to the question of whether the performance of the intervention with the degree distribution monotonically increases in k. The answer is no: the difference is generally governed by the behavior of the second singular value of $\mathbf{T}^k$, and for some networks, the second singular value of $\mathbf{T}^k$ may not monotonically converge to zero.[41] Finally, in Section 6.2, we discuss a general environment in which a similar intuition and results emerge.

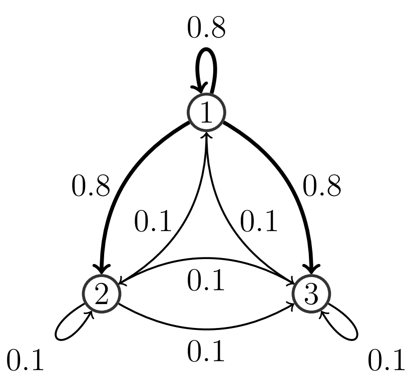
[41] We observe the monotonicity for certain networks. For instance, for the star networks, the first singular value monotonically increases to the limit, but the second singular value monotonically decreases to zero.

6.2 Alternative DeGroot Social Learning Model

Another approach to the social learning in networks starts with a $n \times n$ matrix $\mathbf{W}$ describing how n agents weight others' opinions (Bramoullé et al. 2016). In particular, agent i takes into account agent j's opinion with a weight of $\mathbf{W}_{ij} \geq 0$. The diagonal entries of $\mathbf{W}$ might include non-zero entries in order to capture the idea that some agents may take into account their own opinion. For normalization, for a given matrix $\mathbf{W}$, the influence matrix $\mathbf{T}$ is defined as $\mathbf{T}_{ij} = \frac{\mathbf{W}_{ij}}{\sum_{j=1}^{n} \mathbf{W}_{ij}}$, as the sum of influences to any agent i must be one.

Recall that in our basic setting, the optimal intervention design relies on the singular value decomposition of the influence matrix. When $\mathbf{T}$ is defined based on the given opinion weight matrix $\mathbf{W}$, the singular value decomposition of $\mathbf{W}$ exists and provides the optimal solution, as shown in Theorem 1.

Figure 7 provides an illustrative example of the influence structure. In Figure 7-(a), an arrow from agent j to agent i represents that agent i takes agent j's opinion into consideration, and agent i's updated opinion reflects agent j's opinion with the weight of the arrow. Thus, the sum of the weights of the arrows toward any agent is one. The opinion of agent 1 is given eight times as much weight as any other agent's opinion. Therefore, everyone listens to one another in a symmetric way, and so everyone has the same authority centrality associated with the first left singular vector. However, agent 1 apparently influences the agents (including himself) more than the other agents do. As such, the hub centrality of agent 1 is higher than the other agents', which is associated with the first right singular vector.



(a) influence structure

authority centralities:

$\frac{1}{\sqrt{3}}$ for agent 1,2,3

hub centralities:

agent 1: 0.9857

agent 2 and 3: 0.1231

(b) centralities

Figure 7: Three agents example

The result in Section 6.1 also extends to this situation. Suppose that learning weights are reciprocal: $\mathbf{W}_{ij} = \mathbf{W}_{ji}$ for all i and j with $i \neq j$ before normalization.[42] Then, similar to Proposition 5, it follows that the injected opinion for agent i, $\mathbf{b}_i - \mathbf{b}_i^0$, is proportional to

[42]This assumption does not mean that $\mathbf{T}_{ij} = \mathbf{T}_{ji}$.

the value of the *weighted* degree centrality of the agent, which is defined as $\frac{\omega_i}{\sum_{j=1}^{n} \omega_j}$, where the weighted degree of agent i, ω_i, is defined as $\omega_i = \sum_{k=1}^{n} \mathbf{W}_{ki}$. Thus, the weighted degree centrality measures the relative power of agent i on the opinion formation of others including herself.[43]

6.3 Influence Competition under Incomplete Information

We examine two competing influence designers attempting to develop favorable opinions of themselves among agents in a network. In the context of politics, this may represent the electoral competition between two political parties who use negative campaigns. In the context of marketing, this model may represent a duopolistic competition where two firms use "mocking advertisements."[44] In this subsection, we propose a symmetric influence design game under incomplete information and analyze a Bayes-Nash equilibrium of the game.

We consider an island model as the network formation process between agents: there are two types of agents, and each agent of the same type has the same opinion. There are two influence designers who inject opinions into the network simultaneously. For symmetry, the first designer's target opinion is assumed to be $\mathbf{0}$, and the second designer's target opinion is $\mathbf{1}$. As such, each agent's opinion represents the intensity of the agent's preference for the second designer over the first designer. Also, we consider asymmetric initial opinion vectors by letting $\mathbf{b}^0 = (b_1^0 \mathbf{1}_{n/2}, b_2^0 \mathbf{1}_{n/2})^\intercal$, where $b_1^0 + b_2^0 = 1$ and $b_1^0 > b_2^0$. Therefore, the first half of the agents have more favorable opinions toward the first designer, and the second half of the agents have more favorable opinions toward the second designer.

Let $\mathbf{b}_1'$ and $\mathbf{b}_2'$ be the implanted opinions of the first and second designers. We assume that for a given pair of implanted opinions, $(\mathbf{b}_1', \mathbf{b}_2')$, the resulting agents' opinion vector $\mathbf{b}$ is determined as the average of implanted opinions as

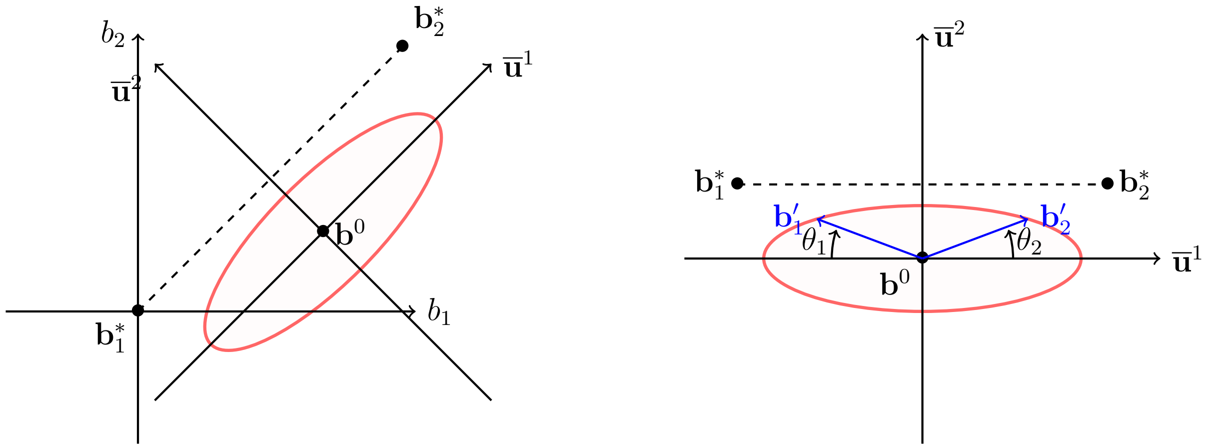
$$\mathbf{b} = \frac{1}{2}\left(\overline{\mathbf{T}}\, \mathbf{b}_1' + \overline{\mathbf{T}}\, \mathbf{b}_2'\right). \tag{6.1}$$

A designer k's budget constraint is $||\mathbf{b}_k - \mathbf{b}^0|| < C$ for any $k \in \{1, 2\}$.

Geometrically, we can transform the designer's intervention problem in this game into a problem of choosing an angle from the principal axis generated by the singular vector $\overline{\mathbf{u}}^1$.

[43] First, when the agents repeatedly learn from neighbors, the agents reach a consensus $b = \mathbf{T}^\infty = \lim_{k\to\infty} \mathbf{T}^k$ whenever $\mathbf{T}$ is strongly connected. The strong connectivity of $\mathbf{T}$ means that for any agents i and j, there is a path from i to j in the directed network generated by $\mathbf{T}$ (Jackson 2010). Second, we can show that the hub centrality of agent i is proportional to the weighted degree of agent i. Specifically, from a direct calculation, we find that ω is an eigenvector of $(\mathbf{T}^\infty)^\intercal \mathbf{T}^\infty$, where $\omega = (\omega_1(\mathbf{A}), \ldots, \omega_n(\mathbf{A}))^\intercal$ is the vector of the weighted degrees of agents. Consequently, we can calculate the hub centrality as $\mathbf{v}^1 = \frac{\omega}{||\omega||}$.

[44] Mocking advertisements that reflect an opponent's products in a bad light are a common commercial marketing strategy on social media platforms. See the recent examples of the competition between Apple and Samsung discussed in https://www.cnet.com/tech/mobile/samsung-mercilessly-mocks-iphone-x-notch-apple/.



(a) Illustration of the original problem

(b) Illustration of the transformed problem

Figure 8: Illustration of the (a) original and (b) transformed competition problems

Figure 8 illustrates this process. In the original problem, two symmetric designers would choose points on the boundary of the budget constraint in Figure 8-(a). Thus, the optimal intervention of a designer, which is a point on the boundary, can be represented as a vector rotated from the principal axis. Intuitively, as depicted in Figure 8-(b), designer 1's choice can be represented by a clockwise rotation θ_1 from the negative direction of the $\overline{\mathbf{u}}^1$-axis, and designer 2's choice is represented by a counter-clockwise rotation θ_2 from the positive direction. As neither agent has any reason to choose an angle outside of $[0, \frac{\pi}{2}]$, we can restrict our attention to the compact action profile set of $[0, \frac{\pi}{2}]^2$. We now show that a unique symmetric Nash equilibrium exists in this transformed game and thus also in the original game.

Proposition 6 *The competition game has a unique symmetric Bayes Nash equilibrium.*

To make our discussion concise, we refer to Appendix A for a detailed proof including its uniqueness.

We find two interesting features of the symmetric Nash equilibrium. First, the designers implant more extreme opinions due to the competition. Since every agent in the network is affected by the other competing influence designer, each designer has to implant more extreme opinions as a best response. Second, the resulting opinion will be moderate compared to the case without competition. As two competitors use the symmetric actions in the equilibrium, $\theta_1 - \theta_2 = 0$, the resulting opinion $\mathbf{b}$ will be on the $\overline{\mathbf{u}}^2$-axis. Without competition, however, designer 2, as the only influence designer, will implant an opinion that is located in the first quadrant from $\mathbf{b}^0$, so the resulting opinion will also be in the first quadrant.

6.4 Linear Cost

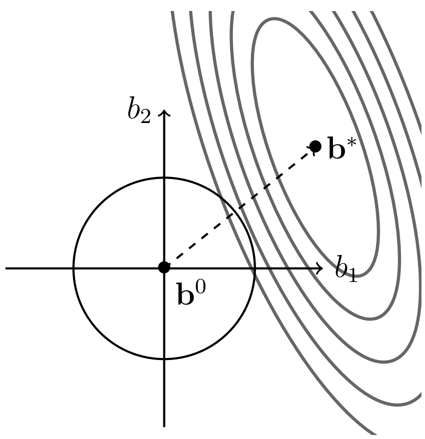
We now consider the following linear cost structure:

$$\min_{\mathbf{b}'} \quad \frac{1}{n}\sum_{i=1}^{n}(\mathbf{b}_i^* - \mathbf{b}_i)^2 \tag{DP 3}$$

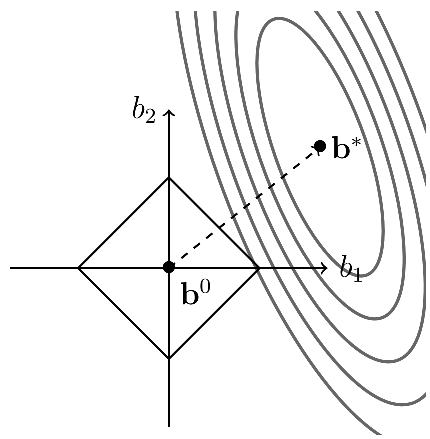
$$\text{subject to} \quad \mathbf{b} = \mathbf{T}\mathbf{b}' \quad \text{exchange of opinions}$$

$$\sum_{i=1}^{n} |\mathbf{b}_i' - \mathbf{b}_i^0| \le C \quad \text{budget constraint.}$$

The problem (DP 3) is closely related to the LASSO method in modern econometrics literature.[45] The beauty of the linear cost is that it provides node selection to design optimal influence. Take the example of the star network again, in which there are two types of nodes, central and peripheral. Figure 9-(a) is the problem under the quadratic cost, and Figure 9-(b) is the problem under the linear cost. b_1 is the implanted belief for the central agent, and b_2 is the implanted belief for the peripheral agents. The ellipse-shaped contours centered at $\mathbf{b}^*$ represent the indifference curves. As depicted in the figures, when a budget is tight (i.e., C is small), the quadratic cost induces an interior solution as $b_1, b_2 > 0$, while the linear cost induces a corner solution as $b_1 > 0$ but $b_2 = 0$. Intuitively, this result means that, in the problem under linear cost, the central agent is more cost-efficient to target for the influence designer. We can find the three remarks as follows.



(a) quadratic cost



(b) linear cost

Figure 9: Illustration of linear cost problem

[45]LASSO is an acronym of the "Least Absolute Shrinkage and Selection Operator." This method is for regression analysis that aims for variable selection through regularization. Hastie et al. (2015) provides an excellent overview of the literature on the LASSO method in statistics.

Remark 1 (Linear cost) 1. Although the linear cost provides a method to identify intuitive targets, there is no closed form solution although the standard Karush–Kuhn–Tucker method applies;

2. There could be multiple solutions solving problem (DP 3);

3. It is known that if $\mathbf{T}$ has full rank, then the solution is unique, but no closed-form solution exists (e.g., Hastie et al. 2015; Tibshirani 2013).[46]

7 Conclusion

We examine the influence designer's intervention problem. The designer would like to lead agents with an initial opinion to have new opinions as close as possible to the target opinion by injecting a new set of private opinions, subject to the budget constraint. We characterize the optimal intervention of the designer in terms of the hub and authority centrality. We decompose the influence matrix into orthogonal singular vectors: The left singular vectors are associated with the authority centrality, and the right singular vectors are associated with the hub centrality. In Theorem 1, we characterize how these factors are considered in the optimal intervention.

We also examine the situation with incomplete information about the underlying network structure. As for network formation process, we consider multi-type networks. We show that, when the influence designer has incomplete information, an asymptotically optimal intervention can be constructed by a low-rank approximation of the actual influence matrix (Theorem 2).

Two important factors should be taken into account in future research. The first factor is that firms dynamically intervene in consumers' opinions. For instance, at an early stage, the degree of intervention is severe (e.g., providing expensive gifts to specific consumers). The second factor is that firms may try to intervene in the network structure directly. For instance, a firm recommends a product reviewer's personal broadcasting channel (e.g., on YouTube or Instagram) to consumers, and once they subscribe to the broadcaster, they will be influenced by them from then on. Of course, a mix of these two factors is possible. Therefore, it is worth investigating these factors in a dynamic influence optimization model in future work.

[46] For instance, the start network with $\boldsymbol{\alpha} = 0$ has rank 2.

A Proofs

Proof of Theorem 1

Proof. We first repeat the optimization problem as

$$\min_{\underline{\mathbf{b}}'} \quad (\overline{\mathbf{b}}^*)^\intercal \overline{\mathbf{b}}^* - 2(\overline{\mathbf{b}}^*)^\intercal \mathbf{S}\underline{\mathbf{b}}' + (\underline{\mathbf{b}}')^\intercal \mathbf{S}^2 \underline{\mathbf{b}}' \tag{DP 2}$$

$$\text{subject to} \quad \left(\underline{\mathbf{b}}' - \underline{\mathbf{b}}^0\right)^\intercal \left(\underline{\mathbf{b}}' - \underline{\mathbf{b}}^0\right) \leq C.$$

As the above problem is convex, it suffices to solve the first order condition of the optimization: for all k, $(s_k \overline{\mathbf{b}}^*_k - s_k^2 \underline{\mathbf{b}}'_k) = \mu(\underline{\mathbf{b}}'_k - \underline{\mathbf{b}}^0_k)$, where $\mu > 0$ is the Lagrangian multiplier. Through rearrangements, we obtain $\underline{\mathbf{b}}'_k = \frac{s_k}{s_k^2+\mu}\overline{\mathbf{b}}^*_k + \frac{\mu}{s_k^2+\mu}\underline{\mathbf{b}}^0_k \geq 0$. Alternatively, in a matrix form, we obtain

$$\underline{\mathbf{b}}' = \underline{\mathbf{b}}^0 + (\mathbf{S}^2 + \mu\mathbf{I})^{-1}\mathbf{S}(\overline{\mathbf{b}}^* - \mathbf{S}\underline{\mathbf{b}}^0).$$

We multiply $\mathbf{V}$ on both sides of the expression, and it follows that

$$\begin{aligned}
\mathbf{b}' &= \mathbf{b}^0 + \mathbf{V}(\mathbf{S}^2 + \mu\mathbf{I})^{-1}\mathbf{S}\left(\mathbf{U}^\intercal \mathbf{b}^* - \mathbf{S}\mathbf{V}^\intercal \mathbf{b}^0\right) \\
&= \mathbf{b}^0 + \left(\mathbf{v}^1 | \cdots | \mathbf{v}^n\right) \begin{pmatrix} \frac{s_1}{s_1^2+\mu}\left((\mathbf{u}^1 \cdot \mathbf{b}^*) - s_1(\mathbf{v}^1 \cdot \mathbf{b}^0)\right) \\ \vdots \\ \frac{s_n}{s_n^2+\mu}\left((\mathbf{u}^n \cdot \mathbf{b}^*) - s_n(\mathbf{v}^n \cdot \mathbf{b}^0)\right) \end{pmatrix} \\
&= \mathbf{b}^0 + \sum_{k=1}^{n} \frac{s_k}{s_k^2 + \mu}\left((\mathbf{u}^k \cdot \mathbf{b}^*) - s_k(\mathbf{v}^k \cdot \mathbf{b}^0)\right)\mathbf{v}^k \\
&= \mathbf{b}^0 + \sum_{k=1}^{n} \frac{s_k}{s_k^2 + \mu}\left(||\mathbf{b}^*|| \cos(\mathbf{u}^k, \mathbf{b}^*) - s_k||\mathbf{b}^0|| \cos(\mathbf{v}^k, \mathbf{b}^0)\right)\mathbf{v}^k.
\end{aligned}$$

Therefore, we obtain the following expression of $\mathbf{b}'$ in the theorem.

Note that the budget constraint is binding at $\mathbf{b}'$. Hence, μ solves the equation

$$\sum_{i=1}^{n}(\underline{\mathbf{b}}'_i - \underline{\mathbf{b}}^0_i)^2 = \sum_{k=1}^{n}\left(\frac{s_k}{s_k^2 + \mu}\right)^2 \left(\overline{\mathbf{b}}^*_k - s_k\underline{\mathbf{b}}^0_k\right)^2 = C.$$

For existence, first, observe that the left-hand side of the above equation converges to zero as μ diverges to infinity. Also, when μ is zero, the left-hand side has a positive value as every term in the summation is strictly positive. As such, whenever C is small, there is a μ such that the above equation is satisfied. Finally, there is a unique μ that satisfies the above equation because each term in the summation is strictly decreasing in μ. Therefore, the theorem is proven. ■

Proof of Theorem 2

Proof. To begin with, we state the Wedin $\sin\theta$ theorem (Wedin 1972, 1983), which is an extension of the Davis-Kahan theorem for singular vectors (Davis and Kahan 1970).[47] For matrices $\mathbf{V}$ and $\widehat{\mathbf{V}}$, let $\Theta(\mathbf{V}, \widehat{\mathbf{V}})$ denote the $d \times d$ diagonal matrix whose jth diagonal entry is the jth singular angle, and let $\sin\Theta(\mathbf{V}, \widehat{\mathbf{V}})$ be defined entry-wise. Then, the following holds:

Claim 1 (Wedin $\sin\theta$ theorem) *Let $\widehat{\mathbf{M}}$ be a perturbation of $\mathbf{M}$ as $\widehat{\mathbf{M}} = \mathbf{M} + \mathbf{H}$. Suppose*

[47] See Yu et al. (2015) for general discussion of the theorems.

that $s_r(\mathbf{M}) \geq a$ *and* $s_{r+1}(\widehat{\mathbf{M}}) \leq a - \triangle$ *for some* $\triangle > 0$. *Then,*

$$\max\{\sin\Theta(\widehat{\mathbf{U}}_0, \mathbf{U}_0), \sin\Theta(\widehat{\mathbf{V}}_0, \mathbf{V}_0)\} \leq \frac{||\mathbf{H}||}{\triangle},$$

where $\mathbf{U}_0$ *and* $\mathbf{V}_0$ *represent the top-r singular subspaces of* $\mathbf{M}$, *and* $\widehat{\mathbf{U}}_0$ *and* $\widehat{\mathbf{V}}_0$ *represent the top-r singular subspaces of* $\widehat{\mathbf{M}}$.

Note that the induced matrix norm (or the operator norm) is induced by the Euclidean vector norm. The matrix norm can be replaced by other orthogonally invariant norms like the Frobenius norm $||\cdot||_{\mathrm{F}}$ (Yu et al. 2015). Also, the theorem provides that a similar inequality for $||\mathbf{v}_j - \widehat{\mathbf{v}}_j||$ holds, where $||\cdot||$ denotes the Euclidean norm.

We first prove the result for $\alpha_i = 0$ for all i. Throughout the proof, $\varepsilon > 0$ is given, and it can be arbitrarily small. Let $\mathbf{T} = \mathbf{D}(\mathbf{A})^{-1}\mathbf{A}$, $\overline{\mathbf{T}} = \mathbf{D}(\overline{\mathbf{A}})^{-1}\,\overline{\mathbf{A}}$, and $\mathbf{H} = \mathbf{T} - \overline{\mathbf{T}}$, where $\mathbf{D}(\overline{\mathbf{A}})$ is a diagonal matrix of $\mathbf{D}(\overline{\mathbf{A}})_{ii} = d_i(\overline{\mathbf{A}}) = \sum_{k=1}^{m} n_k p_{ik} > 0$. By construction, $\overline{\mathbf{T}}$ is row-stochastic, and it has a finite rank m for any n. Thus, we have

$$||\mathbf{H}|| = \Big|\Big| \left[\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})} - \frac{\overline{\mathbf{A}}_{ij}}{d_i(\overline{\mathbf{A}})}\right] \Big|\Big| \leq \Big|\Big| \left[\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})}\left(1 - \frac{d_i(\mathbf{A})}{d_i(\overline{\mathbf{A}})}\right)\right] \Big|\Big| + \Big|\Big| \left[\frac{(\mathbf{A}_{ij} - \overline{\mathbf{A}}_{ij})}{d_i(\overline{\mathbf{A}})}\right] \Big|\Big|. \tag{A.1}$$

We bound each term on the right-hand side of inequality (A.1). First, we have

$$\Big|\Big| \left[\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})}\left(1 - \frac{d_i(\mathbf{A})}{d_i(\overline{\mathbf{A}})}\right)\right] \Big|\Big| \leq \sup_{1\leq k\leq n}\left(\Big|\left(1 - \frac{d_k(\mathbf{A})}{d_k(\overline{\mathbf{A}})}\right)\Big|\right) \cdot \Big|\Big|\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})}\Big|\Big|.$$

The above inequality follows from the fact that for any matrices $\mathbf{M}$ and $\mathbf{M}'$, $||\mathbf{M}|| \leq c||\mathbf{M}||$ whenever $\mathbf{M}_{ij} < c\mathbf{M}'_{ij}$ for some $c > 0$ for all i, j. Since $p_{ij} \succeq \mathcal{O}\left(\frac{\log^2 n}{n}\right)$ for all i, j, it follows that $|d_i(\mathbf{A}) - d_i(\overline{\mathbf{A}})| \leq \frac{\varepsilon}{3} d_i(\overline{\mathbf{A}})$ for all i regardless of i's type, with probability at least $1 - \varepsilon$. This inequality is provided by Lemma A.4 in Golub and Jackson (2012), which is from Theorem 3.6 in Chung et al. (2004) and the Chernoff inequality. Thus, with probability at least $1 - \varepsilon$, it follows that

$$\sup_{1\leq k\leq n}\left(\Big|\left(1 - \frac{d_k(\mathbf{A})}{d_k(\overline{\mathbf{A}})}\right)\Big|\right) \leq \frac{\varepsilon}{3}.$$

Since $\mathbf{T}$ is row-stochastic, $\Big|\Big|\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})}\Big|\Big| \leq 1$. Thus, with probability at least $1 - \varepsilon$, we have

$$\Big|\Big| \left[\frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})}\left(1 - \frac{d_i(\mathbf{A})}{d_i(\overline{\mathbf{A}})}\right)\right] \Big|\Big| \leq \frac{\varepsilon}{3}.$$

We find an upper bound for the second term on the right-hand side of (A.1). Observe that

$$\Big|\Big| \left[\frac{(\mathbf{A}_{ij} - \overline{\mathbf{A}}_{ij})}{d_i(\overline{\mathbf{A}})}\right] \Big|\Big| \leq \frac{1}{\inf_{1\leq k\leq n} d_k(\overline{\mathbf{A}})}\Big|\Big|\mathbf{A}_{ij} - \overline{\mathbf{A}}_{ij}\Big|\Big| \leq \frac{1}{\inf_{1\leq k\leq n} d_k(\overline{\mathbf{A}})}\sqrt{\sup_{1\leq l\leq n} d_l(\overline{\mathbf{A}})}\sqrt{\log n}.$$

The first inequality holds absolutely, and the second inequality holds with probability at least $1 - \varepsilon$. Let $p_{\min} = \min_{1\leq k\leq m}\{p_{kk}\}$, $p_{\max} = \max_{1\leq k\leq m}\{p_{kk}\}$, and $n_{\min} = \min\{n_1, \ldots, n_m\}$. By the assumptions in the main text, we have

$$\frac{\sup_{1\leq l\leq n} d_l(\overline{\mathbf{A}})}{\inf_{1\leq k\leq n} d_k(\overline{\mathbf{A}})} \leq \frac{n p_{\max}}{n_{\min} p_{\min}} \sim \mathcal{O}(1),$$

which results in probability at least $1-\varepsilon$,

$$\left|\left|\left[\frac{(\mathbf{A}_{ij}-\overline{\mathbf{A}}_{ij})}{d_i(\overline{\mathbf{A}})}\right]\right|\right| \leq \sqrt{\frac{\log n}{n}\frac{1}{p_{\max}}}$$

Since $p_{\max} \succeq \mathcal{O}\left(\frac{\log^2 n}{n}\right)$, the right-hand side of the above inequality becomes smaller than $\frac{\varepsilon}{3}$. Therefore, we conclude that with probability at least $1-\varepsilon$, $||\mathbf{H}|| < \varepsilon$ for sufficiently large n.

Given that $||\mathbf{H}||$ converges to zero in probability, we find a useful fact from the Wedin $\sin\theta$ theorem. First, let $\triangle = s_m(\overline{\mathbf{T}}) - s_{m+1}(\overline{\mathbf{T}}) = s_m(\overline{\mathbf{T}}) > 0$. Second, for any $k \in \{1,\ldots,n\}$, the difference between kth singular value of $\mathbf{T}$ and kth singular value of $\overline{\mathbf{T}}$ converge to zero in probability by Weyl's theorem. Third, for any given $\varepsilon > 0$, $||\mathbf{H}|| < \triangle\varepsilon$ with probability at least $1-\varepsilon$ for sufficiently large n. Let $\mathbf{V}_r$ and $\mathbf{U}_r$ be

$$\mathbf{V}_r = \sum_{k=1}^{r} s_k \mathbf{V}\mathbf{V}^\intercal \quad \text{and} \quad \mathbf{U}_r = \sum_{k=1}^{r} s_k \mathbf{U}\mathbf{U}^\intercal.$$

Then, the following holds with probability at least $1-\varepsilon$ by the Wedin $\sin\theta$ theorem:

$$\max\{\sin\Theta(\mathbf{U}_r,\overline{\mathbf{U}}), \sin\Theta(\mathbf{V}_r,\overline{\mathbf{V}})\} \leq \frac{||\mathbf{H}||}{\triangle} < \varepsilon.$$

We show that $\frac{1}{\sqrt{n}}||\mathbf{b}'(n) - \overline{\mathbf{b}}'(n)|| < \varepsilon$ with probability at least $1-\varepsilon$. We observe

$$\begin{aligned}
&\frac{1}{\sqrt{n}}||\mathbf{b}'(n) - \overline{\mathbf{b}}'(n)|| \\
&= \frac{1}{\sqrt{n}}\left|\left|\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}\left(\mathbf{U}^\intercal\mathbf{b}^* - \mathbf{S}\mathbf{V}^\intercal\mathbf{b}^0\right) - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\left(\overline{\mathbf{U}}^\intercal\mathbf{b}^* - \overline{\mathbf{S}}\,\overline{\mathbf{V}}^\intercal\mathbf{b}^0\right)\right|\right| \\
&\leq \left|\left|\left(\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right)\frac{\mathbf{b}^*}{\sqrt{n}}\right|\right| \\
&\quad + \left|\left|\left(\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}^2\,\mathbf{V}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}^2\,\overline{\mathbf{V}}^\intercal\right)\frac{\mathbf{b}^0}{\sqrt{n}}\right|\right|.
\end{aligned} \tag{A.2}$$

We now demonstrate that the first term on the right-hand side of the above inequality becomes less than ε with probability at least $1-\varepsilon$ when n is sufficiently large. Observe

$$\begin{aligned}
&\left|\left|\left(\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right)\frac{\mathbf{b}^*}{\sqrt{n}}\right|\right| \\
&\leq \left|\left|\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| \cdot \underbrace{\left|\left|\frac{\mathbf{b}^*}{\sqrt{n}}\right|\right|}_{\leq 1} \\
&\leq \left|\left|\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| \\
&\leq \left|\left|\mathbf{V}(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S}\,\mathbf{U}^\intercal - \mathbf{V}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\mathbf{U}^\intercal\right|\right| + \left|\left|\mathbf{V}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| \\
&\leq \left|\left|\mathbf{V}\right|\right| \cdot \left|\left|(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S} - (\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\right|\right| \cdot \left|\left|\mathbf{U}^\intercal\right|\right| + \left|\left|\mathbf{V}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| \\
&= \underbrace{\left|\left|(\mathbf{S}^2+\mu\mathbf{I})^{-1}\mathbf{S} - (\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\right|\right|}_{< \frac{\varepsilon}{8} \text{ as } \mathbf{S}\to\overline{\mathbf{S}} \text{ and } \mu\to\overline{\mu} \text{ in probability}} + \left|\left|\mathbf{V}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2+\overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\varepsilon}{8} + \left|\left|\mathbf{V}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| \\
&\leq \frac{\varepsilon}{8} + \left|\left|\mathbf{V}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\mathbf{U}^\intercal - \mathbf{V}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| + \left|\left|\mathbf{V}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right|\right| \\
&\leq \frac{\varepsilon}{8} + \left|\left|\mathbf{V}\right|\right| \cdot \left|\left|(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\right|\right| \cdot \left|\left|\mathbf{U}^\intercal - \overline{\mathbf{U}}^\intercal\right|\right| + \left|\left|\mathbf{V} - \overline{\mathbf{V}}\right|\right| \cdot \left|\left|(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\right|\right| \cdot \left|\left|\overline{\mathbf{U}}^\intercal\right|\right| \\
&\leq \frac{\varepsilon}{8} + \left|\left|(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\right|\right| \cdot \left(\left|\left|\mathbf{U} - \overline{\mathbf{U}}\right|\right| + \left|\left|\mathbf{V} - \overline{\mathbf{V}}\right|\right|\right).
\end{aligned}
$$

Observe that since $\overline{s}_k > 0$ for $1 \leq k \leq m$ and $\overline{\mu} > 0$, there exists $\overline{K}$ such that

$$
\left|\left|(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\right|\right| \leq \max_{1\leq k\leq m} \left\{\frac{\overline{s}_k}{\overline{s}_k^2 + \overline{\mu}}\right\} \leq \overline{K}.
$$

$\mathbf{U}_r$ and $\mathbf{V}_r$ are the r-dimensional best approximation of $\mathbf{U}$ and $\mathbf{V}$, respectively. Then, by the spectral theorem, we obtain $||\mathbf{U} - \mathbf{U}_r|| \leq s_{r+1}$ and $||\mathbf{V} - \mathbf{V}_r|| \leq s_{r+1}$. Also, by the Wedin $\sin\theta$ theorem, $||\mathbf{U}_r - \overline{\mathbf{U}}||$ and $||\mathbf{V}_r - \overline{\mathbf{V}}||$ become arbitrarily small when the network size is sufficiently large. Thus, since s_{m+1} is arbitrarily small for sufficiently large n, with probability at least $1 - \varepsilon$,

$$
\max\{||\mathbf{U} - \overline{\mathbf{U}}||, ||\mathbf{V} - \overline{\mathbf{V}}||\} \leq s_{m+1} + \frac{\varepsilon}{32\overline{K}} \leq \frac{\varepsilon}{16\overline{K}}.
$$

Hence, with probability at least $1 - \varepsilon$, it follows that

$$
\left|\left| \left(\mathbf{V}(\mathbf{S}^2 + \mu\mathbf{I})^{-1}\mathbf{S}\,\mathbf{U}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}\,\overline{\mathbf{U}}^\intercal\right) \frac{\mathbf{b}^*}{\sqrt{n}}\right|\right| < \frac{\varepsilon}{2}.
$$

For the second term on the right-hand side of inequality (A.2), we use

$$
\left|\left|(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}^2\right|\right| \leq \sup_{1\leq k\leq r} \left\{\frac{\overline{s}_k^2}{\overline{s}_k^2 + \overline{\mu}}\right\} \leq 1.
$$

Then, again, by applying facts of the matrix convergences of $||\mathbf{U} - \overline{\mathbf{U}}||$, $||\mathbf{V} - \overline{\mathbf{V}}||$, $||\mathbf{S} - \overline{\mathbf{S}}||$, it easily follows that

$$
\left|\left| \left(\mathbf{V}(\mathbf{S}^2 + \mu\mathbf{I})^{-1}\mathbf{S}^2\,\mathbf{V}^\intercal - \overline{\mathbf{V}}(\overline{\mathbf{S}}^2 + \overline{\mu}\mathbf{I})^{-1}\overline{\mathbf{S}}^2\,\overline{\mathbf{V}}^\intercal\right) \frac{\mathbf{b}^0}{\sqrt{n}}\right|\right| < \frac{\varepsilon}{2}.
$$

Therefore, the statement is proven for the case of $\alpha_i = 0$ for all i.

We consider cases of $\alpha_i \in [0, \overline{\alpha}]$ for some i. Let $\mathbf{T}^0$ and $\overline{\mathbf{T}}^0$ be the influence matrices when $\alpha_i = 0$ for all i. Then, it suffices to show that $||\mathbf{T} - \mathbf{T}^0||$ and $||\,\overline{\mathbf{T}} - \overline{\mathbf{T}}^0\,||$ are arbitrarily small with a high probability if n is sufficiently large. In what follows, we provide a proof for $||\mathbf{T} - \mathbf{T}^0||$ as a proof for the other norm is analogous.

We rewrite the updating rule equation (2.1) as

$$
\mathbf{b}_i = \frac{\alpha_i\mathbf{b}_i^0 + (1-\alpha_i)\sum_{j=1}^n \mathbf{A}_{ij}\mathbf{b}_j^0}{\alpha_i + (1-\alpha_i)d_i(\mathbf{A})} = \widetilde{\alpha}_i\mathbf{b}_i^0 + (1-\widetilde{\alpha}_i)\sum_{j=1}^n \frac{\mathbf{A}_{ij}}{d_i(\mathbf{A})}\mathbf{b}_j^0, \tag{A.2}
$$

where $\widetilde{\alpha}_i = \frac{\alpha_i}{\alpha_i + (1-\alpha_i)d_i(\mathbf{A})}$. Then, by letting $\widetilde{\boldsymbol{\alpha}} = diag(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n) \neq \mathbf{0}$, we have

$$
||\mathbf{T} - \mathbf{T}^0|| = ||\widetilde{\boldsymbol{\alpha}}\mathbf{I} + (\mathbf{I} - \widetilde{\boldsymbol{\alpha}})\mathbf{T}^0 - \mathbf{T}^0|| \leq ||\widetilde{\boldsymbol{\alpha}}|| \cdot ||\mathbf{I} - \mathbf{T}^0|| \leq 2||\widetilde{\boldsymbol{\alpha}}||.
$$

Note that since $\widetilde{\alpha}_i = \frac{\alpha_i}{\alpha_i + (1-\alpha_i)d_i(\mathbf{A})} \leq \frac{1}{\inf_{1\leq i\leq n} d_i(\mathbf{A})}$ for all i, it follows that $||\widetilde{\boldsymbol{\alpha}}|| \leq \frac{1}{\inf_{1\leq i\leq n} d_i(\mathbf{A})}||\mathbf{I}||$. Recall that $d_i(\mathbf{A})$ diverges with probability one by the assumptions in the main text. Therefore, for any give $\varepsilon > 0$, $||\mathbf{T} - \mathbf{T}^0|| < \varepsilon$ with probability at least $1 - \varepsilon$. ■

Proof of Proposition 1

Proof. By the discussion in the main text, a proof of the proposition is straightforward. ■

Proof of Proposition 2

Proof. By the discussion in the main text, a proof of the proposition is straightforward. ■

Proof of Proposition 3

Proof. As a block matrix representation, we can rewrite $\overline{\mathbf{T}}$ by

$$\begin{pmatrix} \frac{p_s}{kp_s+p_d}\mathbf{1}_k\mathbf{1}_k^\top & \frac{p_d}{kp_s+p_d}\mathbf{1}_k \\ \frac{p_d}{p_s+kp_d}\mathbf{1}_k^\top & \frac{p_s}{p_s+kp_d}\mathbf{1}_1 \end{pmatrix}.$$

Note that the sum of eigenvectors of $\overline{\mathbf{T}}$ equals the trace of $\overline{\mathbf{T}}$. Since there are two non-zero eigenvalues, and the largest eigenvalue is one, it follows that

$$1+\overline{\lambda}_2 = trace(\overline{\mathbf{T}}) = k\frac{p_s}{kp_s+p_d} + \frac{p_s}{p_s+kp_d} = k\frac{p_s}{kp_s+p_d} + \left(1-\frac{kp_d}{p_s+kp_d}\right).$$

Thus, we obtain

$$\overline{s}_2 = |\overline{\lambda}_2| = k\left(\frac{p_s}{kp_s+p_d} - \frac{p_d}{p_s+kp_d}\right) = \frac{(p_s-p_d)(p_s+p_d)}{(p_s+kp_d)(kp_s+p_d)},$$

and it is strictly decreasing in k. With the above calculations, the proposition is proven. ■

Proof of Proposition 4

Proof. We first find $\overline{\mathbf{T}}$ by

$$\begin{pmatrix} \frac{p_{11}}{p_{11}+p_{12}} & \frac{p_{12}}{p_{11}+p_{12}} \\ 0 & 1 \end{pmatrix}.$$

The corresponding singular value decomposition of $\overline{\mathbf{T}} = \mathbf{USV}$ is

$$\mathbf{U} = \begin{pmatrix} \frac{1+\lambda}{\sqrt{2+\lambda(4+4\lambda+\sqrt{4+8\lambda(1+\lambda)}}} & \frac{-\lambda-\sqrt{1+2\lambda(1+\lambda)}}{\sqrt{2+\lambda(4+4\lambda+\sqrt{4+8\lambda(1+\lambda)}}} \\ \frac{1+\lambda}{\sqrt{2+\lambda(4+4\lambda-2\sqrt{1+2\lambda(1+\lambda)}}} & \frac{-\lambda+\sqrt{1+2\lambda(1+\lambda)}}{\sqrt{2+\lambda(4+4\lambda-2\sqrt{1+2\lambda(1+\lambda)}}} \end{pmatrix},$$

$$\mathbf{S} = \begin{pmatrix} \frac{\sqrt{1+\lambda+\lambda^2+\sqrt{1+2\lambda(1+\lambda)}}}{1+\lambda} & 0 \\ 0 & \frac{\sqrt{1+\lambda+\lambda^2-\sqrt{1+2\lambda(1+\lambda)}}}{1+\lambda} \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} \frac{2}{\sqrt{4+\frac{-2-2\lambda+\sqrt{4+8\lambda(1+\lambda)}}{\lambda^2}}} & -\frac{2(1+\lambda+\sqrt{1+2\lambda(1+\lambda)})}{\lambda\sqrt{4+\frac{4(1+\lambda+\sqrt{1+2\lambda(1+\lambda)})^2}{\lambda^2}}} \\ \frac{2}{\sqrt{4+\frac{-2-2\lambda+\sqrt{4+8\lambda(1+\lambda)}}{\lambda^2}}} & \frac{2(-1-\lambda+\sqrt{1+2\lambda(1+\lambda)})}{\lambda\sqrt{4+\frac{4(1+\lambda+\sqrt{1+2\lambda(1+\lambda)})^2}{\lambda^2}}} \end{pmatrix}.$$

We show that $\overline{s}_1 - \overline{s}_2$ monotonically decreases as λ increases. To this end, we calculate that

$$\overline{s}_1^2 - \overline{s}_2^2 = \frac{2\sqrt{1+2\lambda(1+\lambda)}}{(1+\lambda)^2}, \tag{A.3}$$

and its derivative with respect to λ is strictly negative for all λ. Furthermore, the right-hand side of (A.3) converges to zero. Therefore, the gap of the two singular values monotonically decreases and converges to zero as λ increases.

We finally show that $\mathbf{v}_2^1 > \mathbf{v}_1^1$ and the difference monotonically decreases but does not converge to zero as $\lambda \to \infty$. It easily follows that $\mathbf{v}_2^1 > 0$ and $\mathbf{v}_1^1 > 0$. Then,

$$(\mathbf{v}_2^1)^2 - (\mathbf{v}_1^1)^2 = \frac{1+\lambda}{\sqrt{1+2\lambda(1+\lambda)}} > 0. \tag{A.4}$$

The derivative of the right-hand side of (A.4) with respect to λ is strictly negative. Moreover, by applying L'Hôpital's rule, it follows that the right-hand side converges to $\frac{1}{\sqrt{2}}$, which implies that the gap of $\mathbf{v}_2^1$ and $\mathbf{v}_1^1$ does not converge to zero. By repeating the above procedure, the properties for $\mathbf{v}_2^2$ and $\mathbf{v}_1^2$ in the main text are proven. ■

Proof of Proposition 5

Proof. Since $\mathbf{T}^\infty$ is a rank-one matrix, it suffices to find an expression of $\mathbf{v}^1$. Note that since $\mathbf{v}^1$ is an eigenvector of $(\mathbf{T}^\infty)^\intercal\mathbf{T}^\infty$, $\mathbf{v}^1$ satisfies $(\mathbf{T}^\infty)^\intercal\mathbf{T}^\infty\mathbf{v}^1 = \lambda\mathbf{v}^1$ for some $\lambda > 0$. By letting $\mathbf{d} = (d_1(\mathbf{A})_1, \ldots, d_n(\mathbf{A})_n)^\intercal$, we have

$$\begin{aligned}
(\mathbf{T}^\infty)^\intercal\mathbf{T}^\infty\mathbf{d} &= \frac{1}{\left(\sum_{k=1}^n d_k(\mathbf{A})\right)^2}\begin{pmatrix} d_1(\mathbf{A}) & \cdots & d_1(\mathbf{A}) \\ & \ddots & \\ d_n(\mathbf{A}) & \cdots & d_n(\mathbf{A}) \end{pmatrix}\begin{pmatrix} d_1(\mathbf{A}) & \cdots & d_n(\mathbf{A}) \\ & \ddots & \\ d_1(\mathbf{A}) & \cdots & d_n(\mathbf{A}) \end{pmatrix}\mathbf{d} \\
&= \frac{n}{\left(\sum_{k=1}^n d_k(\mathbf{A})\right)^2}\begin{pmatrix} d_1(\mathbf{A})d_1(\mathbf{A}) & \cdots & d_1(\mathbf{A})d_n(\mathbf{A}) \\ & \ddots & \\ d_n(\mathbf{A})d_1(\mathbf{A}) & \cdots & d_n(\mathbf{A})d_n(\mathbf{A}) \end{pmatrix}\mathbf{d} \\
&= \frac{n\mathbf{d}^\intercal\mathbf{d}}{\mathbf{1}^\intercal\mathbf{d}\mathbf{d}^\intercal\mathbf{1}}\mathbf{d}.
\end{aligned}$$

Hence, $\mathbf{d}$ is an eigenvector of $(\mathbf{T}^\infty)^\intercal\mathbf{T}^\infty$, and its entries are all positive. This result implies that $\mathbf{v}^1 = \frac{\mathbf{d}}{||\mathbf{d}||}$ is the desired unit eigenvector. $\mathbf{v}_i^1$ is proportional to the degree of agent i.

Similarly, we can calculate $\mathbf{u}^1$ as an eigenvector of $\mathbf{T}^\infty(\mathbf{T}^\infty)^\intercal$. We observe

$$\mathbf{T}^\infty(\mathbf{T}^\infty)^\intercal\mathbf{1} = \frac{\sum_{k=1}^n d_k(\mathbf{A})^2}{\left(\sum_{k=1}^n d_k(\mathbf{A})\right)^2}\begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}\mathbf{1} = \frac{\mathbf{d}^\intercal\mathbf{d}}{\mathbf{1}^\intercal\mathbf{d}\mathbf{d}^\intercal\mathbf{1}}\mathbf{1}\mathbf{1}^\intercal\mathbf{1} = \frac{n\mathbf{d}^\intercal\mathbf{d}}{\mathbf{1}^\intercal\mathbf{d}\mathbf{d}^\intercal\mathbf{1}}\mathbf{1}.$$

$\mathbf{u}^1 = \frac{\mathbf{1}}{\sqrt{n}}$ is the desired unit eigenvector.

The unique non-zero singular value s_1 is $\frac{n\mathbf{d}^\intercal\mathbf{d}}{\mathbf{1}^\intercal\mathbf{d}\mathbf{d}^\intercal\mathbf{1}}$. Let $f : \mathbb{N}_0 \to \mathbb{R}_+$ be the degree distribution of the network, where $f(d)$ is the fraction of agents having degree d. Also, define $\widetilde{f} : \mathbb{N}_0 \to \mathbb{R}_+$ as $\widetilde{f}(d) = \frac{df(d)}{\langle d \rangle}$, where $\langle d \rangle$ is the average of the degree distribution f. Let $\langle \widetilde{d} \rangle$ be the average of $\widetilde{f}$. Then, the singular value can be rewritten as

$$s_1 = \frac{n\mathbf{d}^\intercal\mathbf{d}}{\mathbf{1}^\intercal\mathbf{d}\mathbf{d}^\intercal\mathbf{1}} = \frac{n\sum_{k=1}^n d_k(\mathbf{A})^2}{\left(\sum_{k=1}^n d_k(\mathbf{A})\right)^2} = \sum_{k=1}^n d_k(\mathbf{A})\frac{d_k(\mathbf{A})}{n\langle d \rangle}\frac{1}{\langle d \rangle} = \frac{\langle \widetilde{d} \rangle}{\langle d \rangle} = \frac{\langle d^2 \rangle}{\langle d \rangle^2}.$$

By plugging expressions into the expression of $\mathbf{b}'$, Proposition 5 follows. ■

Proof of Proposition 6

Proof. Figure 10 describes a situation in which given influencer 1's choice of θ_1, influencer 2 chooses θ_2 to minimize the distance between the resulting opinion vector $\mathbf{b}$ and her target belief $\mathbf{b}_2^*$. All vectors are represented by taking $\overline{\mathbf{u}}^1$ and $\overline{\mathbf{u}}^2$ as the principal axes.

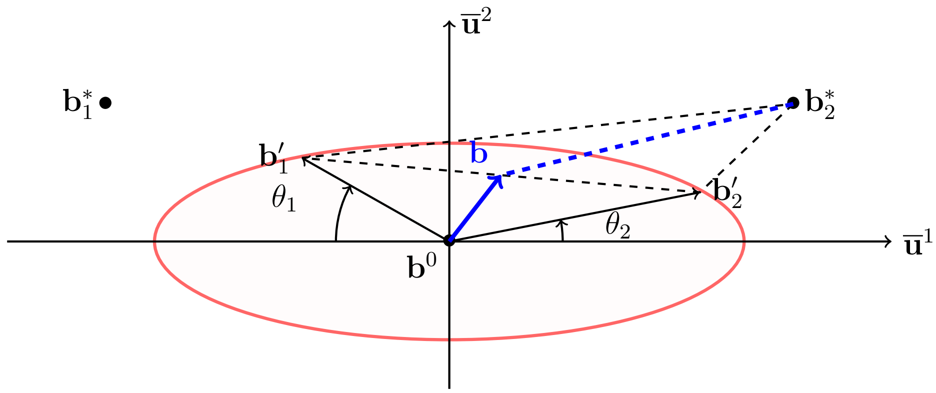


Figure 10: Illustration of best response of influence 2

For the calculation of influencer 2's best response with respect to influencer 1's choice, we apply the Apollonius theorem.[48] The Apollonius theorem states that $2(y^2 + w^2) = x^2 + z^2$, where $x = ||\mathbf{b}_2^* - \mathbf{b}'_1||$, $y = ||\mathbf{b}_2^* - \mathbf{b}||$, $z = ||\mathbf{b}_2^* - \mathbf{b}'_2||$, and $w = ||\mathbf{b} - \mathbf{b}'_2|| = ||\mathbf{b} - \mathbf{b}'_1|| = \frac{1}{2}||\mathbf{b}'_1 - \mathbf{b}'_2||$ by the assumption of the model. Since x is given by influencer 1's choice of θ_1, influencer 2's optimal choice of θ_2 minimizes $2y^2 = z^2 - 2w^2$. Consequently, the best responding θ_2 solves the first-order condition of $z\frac{\partial z}{\partial \theta_2} = 2w\frac{\partial w}{\partial \theta_2}$. Note that the right-hand side of the optimality condition captures the competition effect. When influence 1 is absent, influence 2 solves the first-order condition of $\frac{dz}{d\theta_2} = 0$. However, when influencer 2 competes against influencer 1, she has to exercise more extreme injection so as to make the resulting opinion $\mathbf{b}$ close to $\mathbf{b}_2^*$.

We now parameterize the vectors $\mathbf{b}'_2$ and $\mathbf{b}$ as functions of (θ_1, θ_2). Since $\mathbf{b}'_2$ is on the boundary of the ellipse, its coordinates on the $\overline{\mathbf{u}}^1$-axis and $\overline{\mathbf{u}}^2$-axis are $\overline{s}_1 \cos\theta_2$ and $\overline{s}_2 \sin\theta_2$, respectively. Similarly, the coordinates of $\mathbf{b}'_1$ are $\overline{s}_1 \cos(\pi - \theta_1) = -\overline{s}_1 \cos\theta_1$ and $\overline{s}_2 \sin(\pi - \theta_1) = \overline{s}_2 \sin\theta_1$. Thus, z and w are parameterized by

$$z = ||(\phi\cos t, \phi\sin t) - (\overline{s}_1\cos\theta_2, \overline{s}_2\sin\theta_2)||,$$
$$w = \frac{1}{2}||(-\overline{s}_1\cos\theta_1, \overline{s}_2\sin\theta_1) - (\overline{s}_1\cos\theta_2, \overline{s}_2\sin\theta_2)||,$$

where $\phi = ||\mathbf{b}_2^* - \mathbf{b}^0||$ is the Euclidean distance between $\mathbf{b}_2^*$ and $\mathbf{b}^0$, and $t = \measuredangle(\mathbf{b}_2^*, \overline{\mathbf{u}}^1)$ is the angle between $\mathbf{b}_2^*$ and $\overline{\mathbf{u}}^1$. Finally, a symmetric Bayes-Nash equilibrium (θ_1^*, θ_2^*) satisfies

$$z\frac{\partial z}{\partial \theta_2}\Big|_{(\theta_1^*, \theta_2^*)} = 2w\frac{\partial w}{\partial \theta_2}\Big|_{(\theta_1^*, \theta_2^*)} \quad \text{and} \quad \theta_1^* = \theta_2^* = \theta^*. \tag{A.5}$$

It suffices to show the existence of the solution to equation (A.5). To see why, suppose that there are two equilibria represented by $\underline{\theta}^* = \overline{\theta}^*$ with $\underline{\theta}^* < \overline{\theta}^*$. Then, given $\theta_1^* = \overline{\theta}^*$, influencer 2's indifference curve must be tangential to the ellipses in Figure 10 at the choice

[48] The Apollonius theorem can be proven by using the parallelogram law in elementary Euclidean geometry. We refer to a classic textbook by Godfrey and Siddons (1908) as a reference.

of $\theta_2^* = \overline{\theta}^*$. Similarly, given $\theta_1^* = \underline{\theta}^*$, influencer 2's indifference curve must be tangential to the ellipses in Figure 10 at the choice of $\theta_2^* = \underline{\theta}^*$; however, it is impossible because the slope of the ellipse at $\theta_1^* = \overline{\theta}^*$ is steeper than the slope of the ellipse at $\theta_2^* = \underline{\theta}^*$. Therefore, if a solution exists, then it must be unique.

By requiring $\theta_1 = \theta_2$ in equilibrium, the optimality condition is parameterized as

$$\overline{s}_1 \phi \cos t \sin \theta_2 + \overline{s}_2 \cos \theta_2 (-\phi \sin t + \overline{s}_2 \sin \theta_2)$$
$$= (\overline{s}_1 - \overline{s}_2)\phi \sin \theta_2 \cos t + \overline{s}_2 \phi \sin(\theta_2 - t) + \frac{\overline{s}_2^2}{2} \sin 2\theta_2 = 0.$$

On one hand, at $\theta_2 = 0$, the left-hand side of the above equation is less than or equal to zero. On the other hand, at $\theta_2 = t$, the left-hand side of the above equation is greater than or equal to zero. Thus, by the intermediate value theorem, there exists $\theta^* \in [0, \frac{\pi}{2}]$ such that the optimality condition holds. Therefore, a unique Bayes-Nash equilibrium exists. ■

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