Evidence Acquisition and Voluntary Disclosure*

Denis Shishkin †

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Abstract

A sender seeks hard evidence to persuade a receiver to accept a project by designing a quality test. Testing is not perfectly reliable and produces evidence only with some probability. If the sender obtains the evidence, she can choose to disclose it or pretend to not have obtained it. We show that when reliability is low, the sender chooses a pass/fail test which reveals whether the quality is above or below a threshold. Moreover, the equilibrium pass/fail threshold is always monotone in reliability but whether it is increasing or decreasing depends on whether the acquisition is overt or covert.

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†Department of Economics, University of California San Diego, dshishkin@ucsd.edu.

1 Introduction

Hard evidence is often sought and disclosed by one party (sender) to persuade another (receiver) to take a certain action. For example, pharmaceutical companies test new drugs and seek approval from the US Food and Drug Administration, startups build and test prototypes to secure funding from investors, sellers certify quality of their products to persuade consumers to buy them, etc. However, in many cases the receiver may be uncertain about whether the sender has obtained the evidence. In the above examples, it could be that by the time of the final decision the testing results may not have come back or came back inconclusive. In this case, even if the sender has evidence, she might be able to pretend to not have obtained any evidence. In other words, she can conceal sufficiently unfavorable evidence by claiming ignorance. This paper studies the trade-off that arises due to the conflict between the sender's preferences over disclosures before and after she obtains the evidence.

Consider the following example. An entrepreneur has a project of unknown quality. She can seek verifiable information about the quality to persuade an investor to fund the project. Before obtaining the evidence, she may prefer detailed information about the quality to be released to the investor, regardless of its contents. This is the case if, for example, evidence about moderately low quality gives the entrepreneur at least some chance to secure funding. But suppose that disclosure is voluntary and the investor is uncertain about whether the entrepreneur is informed. Then, if the entrepreneur learns that the quality is low, she may prefer not to disclose such evidence. This limits the investor's learning about low-quality projects. Therefore, the entrepreneur takes into account her future disclosure incentives when deciding what information to seek. We show that this may substantially affect which information is sought in the first place.

In principle, when the state and message spaces are rich and information acquisition is costless, one might expect to see complex communication between the agents. In reality, however, senders often rely on coarse verifiable information. In many cases, it is as simple as a *pass/fail test*, that is, a signal that reveals only whether the state of the world is sufficiently good. For example, sellers obtain certifications that their products have high enough quality, job candidates take professional exams with pass or fail grades, etc. This paper shows that the mere opportunity to conceal information as described above in equilibrium can lead to acquisition of simple information structures such as a pass/fail test.

To study these interactions, we consider a communication game between a sender (she) and a receiver (he). The state of the world is continuous and unknown to both players. The sender wants the receiver to take a certain action, but the receiver takes the action only if his expectation of the state exceeds his privately known outside option drawn from a unimodal distribution. The sender chooses what information to acquire, for example, by designing a partially informative test about the state but such testing is not perfectly reliable. In particular, with probability ρ, which we refer to as *reliability*, she will obtain hard evidence about the test results. Even if she obtains the evidence, she can then voluntarily disclose it or pretend to not have obtained it. Otherwise, she cannot prove that she is uninformed. We distinguish between two versions of the model. In the case of *overt* acquisition, the sender's choice of information is always observed. In the case of *covert* acquisition, the receiver cannot detect in the non-disclosure event whether the sender deviates ex-ante. The covert case captures situations in which the sender cannot commit to which evidence she will seek in contrast to the overt case.

Our main results (Theorems 1 and 2) characterize the equilibrium evidence structures in the overt and covert cases and show that they are essentially unique. The first key implication of the characterization is that low reliability leads to simplicity of the equilibrium evidence structure chosen by the sender. In particular, we show that if ρ is below a certain cutoff, the equilibrium structure takes the form of a *pass/fail test*: it reveals only whether the state is above or below a certain threshold. Otherwise, when ρ is above the threshold, it takes the form of a two-sided censorship, which is similar to a pass/fail test, but also perfectly reveals some intermediate states (see Figure 1). For example, with discrete states, two-sided censorship can be represented by a five-star rating system such that the 5-star grade correspond to the states below some lower threshold, 1-star grade – to the states above some upper threshold, and each of 2-, 3-, and 4-star grades – to some intermediate state.

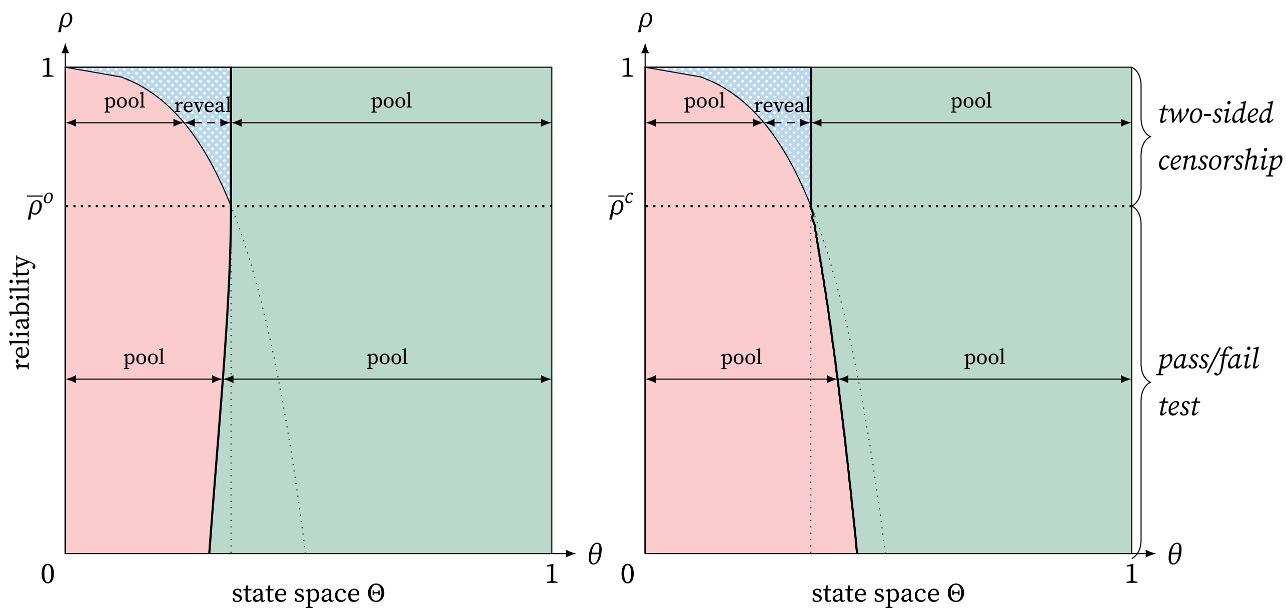
Second, we show that the equilibrium pass/fail threshold is monotone in reliability. However, while it is increasing under overt acquisition, it is decreasing under covert acquisition. In other words, whether the sender publicly or privately acquires the evi-

(a) Pass/fail test

(b) Two-sided censorship

FIGURE 1: Two types of equilibrium evidence structures.

dence affects how testing standards react to improvements in reliability. In addition, we show that covert equilibrium pass/fail threshold is always strictly higher than the overt one and that the difference between them shrinks as reliability improves.



(a) Overt acquisition

(b) Covert acquisition

FIGURE 2: Equilibrium evidence structures for the uniformly distributed state and triangular distribution of the receiver's outside option with the peak at 3/5.

Figure 2 illustrates the key features of the equilibrium evidence structures for the uniformly distributed state and the receiver's outside option following the triangular distribution the peak at 3/5. For each reliability level $\rho \in (0, 1]$, the corresponding horizontal line segment illustrates the optimal partition of the state space. When reliability is low ($\rho < \overline{\rho}$), there is a pass/fail threshold, such that states are pooled above and below it. When reliability is high ($\rho > \overline{\rho}$), the states are pooled above the upper threshold, pooled below the lower threshold, and fully revealed otherwise.

To get some intuition for these results, note that information is affected by three key forces arising due to voluntary disclosure, information design, and the covert nature of the acquisition. Note that a prevalent feature of the equilibrium signal is the upper and lower pooling of the states. First, because the sender does not want to reveal bad news, this prevents the receiver from learning detailed information about low states. Hence, the voluntary disclosure force drives the lower pooling region. Second, because the sender is uncertain about the receiver's cutoff for action and distribution of the receiver outside options is unimodal, there are increasing returns to disclosing more (less) information about low (high) states. In particular, the Bayesian persuasion literature[1] has established that a sender with full commitment and convex-concave indirect utility over posterior means will choose an *upper censorship* of the state, that is, a signal which reveals all states below a certain threshold and pools all states above it. In other words, the information design force drives the imprecision of information about high states which leads to the pooling of high states.

Moreover, *whether* and *how* these forces interact depends on the level of reliability and the third force—whether acquisition is covert. Note that for high reliability, only the lower pooling region is affected by reliability. In fact, in this case we show that the lower threshold corresponds to the disclosure threshold in the voluntary disclosure game with full information (conditional on being reliable) and the constant upper threshold corresponds to the information design game with full commitment (which coincides with $\rho = 1$). That is, for $\rho > \overline{\rho}$ the two thresholds are determined independently by the two forces and do not interact. Notably, the equilibria of the overt and covert cases coincide which can be interpreted as the robustness of the overt equilibrium structure to ex-ante deviations in the acquisition strategy which cannot be detected in the event of non-disclosure. We show that such robustness of the overt equilibrium signal holds because the only additional benefit from an ex-ante covert deviation compared to the overt case comes from a lower would-be non-disclosure receiver's posterior that turns out to be minimized by equilibrium two-sided censorship.[2]

In contrast, under $\rho < \overline{\rho}$, the equilibrium signal is a pass/fail test and the thresh-

[1]See, for example, Alonso and Câmara (2016a), Kolotilin (2018), and, for a recent characterization, Kolotilin, Mylovanov, and Zapechelnyuk (forthcoming).

[2]As explained in Section 3.4, this is related to the minimum principle of DeMarzo, Kremer, and Skrzypacz (2019) which is both necessary and sufficient for covert equilibria in the case of uniformly distributed outside option.

old is determined jointly by the interaction between voluntary disclosure and information design. In the overt case, the receiver fully observes the sender's ex-ante choice and, hence, solving for equilibria boils down to an optimization problem. We show that such sender's costless acquisition problem that takes into account voluntary disclosure can be reformulated as a costly information design problem. That is, her ex-ante expected value from seeking an evidence structure is proportional to her perfect-reliability commitment value from actually choosing a distribution of R's posteriors minus the 'concealment costs' arising due to strategic non-disclosure of bad news. We first show that the solution to this costly information design problem shares similarities with the full-commitment ($\rho = 1$) case in that the solution must be disclosure-equivalent to an upper-censorship. Then, focusing on equilibria in which the sender does not acquire more information than needed given her strategic concealment yields a two-sided censorship or a pass/fail test depending on whether the disclosure threshold is above or below the upper pooling threshold. We then show that the concealment costs feature substitutability between reliability and the testing standards implying that the overt equilibrium pass/fail threshold is increasing in reliability.

In the covert case, the sender's ex-ante choice is unobservable and, thus, solving for equilibria instead involves a fixed-point problem with respect to the sender's ex-ante choice π and the receiver's non-disclosure posterior $x_\varnothing$: (i) π must be best-responding to $x_\varnothing$ and (ii) $x_\varnothing$ must be Bayes-consistent given π. We show that the problem of finding the sender's best response to any given $x_\varnothing$ is equivalent to an auxiliary information design problem in which the sender's indirect utility is modified but again leads to an upper censorship solution. Finally, we show that the best-responding upper pooling threshold is increasing in $x_\varnothing$ and does not depend on reliability. At the same time, higher reliability leads to more skeptical receiver and so to a lower Bayes consistent non-disclosure posterior. This implies that the covert equilibrium pass/fail threshold is decreasing in reliability.

Related literature. This paper is related to the literature on disclosure of verifiable information (for a survey, see Milgrom, 2008).[3] The seminal works of Grossman (1981),

[3]Coarseness of information is also a common feature in cheap-talk models (Crawford and Sobel, 1982) with exogenously given soft information. There coarseness follows from partially aligned preferences of the players. In our model, information is hard, acquired endogenously, and the sender has state-independent preferences. See Pei (2015) and Argenziano, Severinov, and Squintani (2016) on in-

Milgrom (1981), and Milgrom and Roberts (1986) study disclosure under complete provability, that is when the sender can prove any true claim. The key insight of those papers is that complete provability implies "unraveling", which leads to full information revelation in equilibrium.[4] Our model is based on the approach of Dye (1985) and Jung and Kwon (1988), in which evidence is obtained with some probability and there is partial provability: if the sender is uninformed, she cannot prove this.[5]

The paper contributes to the literature endogenizing the sender's endowment of evidence in voluntary disclosure games. In Matthews and Postlewaite (1985), the sender makes a binary evidence acquisition decision before playing a voluntary disclosure game under complete provability. Lizzeri (1999) and Ali, Haghpanah, Lin, and Siegel (2021) study disclosure of verifiable information designed by a profit-maximizing monopolistic intermediary. Gentzkow and Kamenica (2017) study overt costly acquisition of evidence in a disclosure model where each type can perfectly self-certify and show that one or more sender(s) disclose everything they acquire.[6] Kartik, Lee, and Suen (2017) study a multi-sender disclosure game, where senders can invest in higher reliability, while taking the evidence structure as given. Ben-Porath, Dekel, and Lipman (2021) study a mechanism design problem with privately informed agents who can acquire evidence about their types.

Some recent papers endogenize the sender's evidence in the Dye (1985) framework. Kartik, Lee, and Suen (2017) study a multi-sender disclosure game, where senders can invest in higher reliability, while taking the evidence structure as given. Dasgupta, Krasikov, and Lamba (2022) study hard information design in a monopolistic screening model.

Bertomeu, Cheynel, and Cianciaruso (2021) study a closely related problem, in which the firm is maximizing its expected valuation by choosing an asset measurement system, subject to strategic withholding and disclosure costs. Their costless disclosure case can be mapped into our overt case, where the density of the receiver's outside op-

formation acquisition in a cheap-talk model.

[4] For a recent generalization, see Hagenbach, Koessler, and Perez-Richet (2014).

[5] Other approaches in which unraveling fails include costly disclosure (Jovanovic, 1982; Verrecchia, 1983) and multidimensional disclosure Shin (1994); Dziuda (2011). Okuno-Fujiwara, Postlewaite, and Suzumura (1990) provide sufficient conditions for unraveling in two-stage games, where in the first stage players can disclose private information, and give examples in which unraveling does not happen.

[6] Escudé (2019) provides an analogous result in a single-sender setting with covert costless acquisition and partial verifiability.

tion is increasing so that it would be optimal to acquire a fully-informative evidence structure for any reliability. DeMarzo, Kremer, and Skrzypacz (2019) study a problem that can be related to our covert case with the uniform outside option and where the sender's choice can be across any constrained collection of experiments of potentially heterogeneous reliability.[7] They show that an experiment is an equilibrium one if and only if satisfies the 'minimum principle', that is, it must minimize the Bayes-consistent receiver's non-disclosure posterior. Notably, their results imply that there show that there is always an equilibrium with 'simple tests' equivalent to our pass/fail tests. In their model, the sender's indirect utility over posterior means is linear and, therefore, she is ex-ante indifferent between all information structures, so the equilibrium condition boils down to the ex-ante incentive compatibility. In contrast to this paper, in Bertomeu, Cheynel, and Cianciaruso (2021) and DeMarzo, Kremer, and Skrzypacz (2019) the information design force is either trivial or absent. Complementary to their findings, we provide support for pass/fail test to be the unique optimum (up to outcome equivalence) in environments with the convex-concave sender's indirect utility and constant reliability across experiments in both overt and covert cases.

This paper also contributes to the literature on Bayesian persuasion and information design (for a survey, see Kamenica, 2019). In the special case of our model when the sender is known to possess the evidence ($\rho = 1$), the unraveling argument applies, and both the overt and covert optimal evidence acquisition problems become equivalent to Bayesian persuasion (Aumann and Maschler, 1995; Kamenica and Gentzkow, 2011). In particular, a number of papers (Alonso and Câmara, 2016b; Kolotilin, Mylovanov, Zapechelnyuk, and Li, 2017; Kolotilin, 2018; Dworczak and Martini, 2019) have shown in similar settings that upper censorship is optimal if the receiver's type distribution is unimodal.[8] Information structures equivalent to our pass/fail test and two-sided censorship also appear in Kolotilin (2018) in cases when the distribution of the receiver's type is not unimodal. There, pass/fail test can be optimal because of a particular shape of the receiver's type distribution (e.g. bimodal), rather than the interaction between the design and disclosure incentives.

[7] Ben-Porath, Dekel, and Lipman (2018) study a related voluntary disclosure problem, in which there is an ex-ante covert choice between risky projects, which, in our setting, would translate into a choice between priors.

[8] Moreover, Kolotilin, Mylovanov, and Zapechelnyuk (forthcoming) show that the converse also holds.

A standard assumption in this literature is that the sender commits to a signal, whose realization is directly observed by the receiver, while in our model it is voluntarily disclosed by the sender.[9] Some recent works also relax the assumption that the receiver directly observes signal realizations. In Felgenhauer (2019), the sender designs experiments sequentially at a cost and can choose when to stop experimenting and which outcomes to disclose. Nguyen and Tan (2019) study a model of Bayesian persuasion with costly messages, where a special case of the cost function corresponds to verifiable disclosure of hard evidence studied in this paper. The difference is that their sender can choose not only a signal about the state, but also the reliability. In contrast, ρ is exogenous in our model. If it could be chosen by the sender, she would set $\rho = 1$ and obtain her full-commitment payoff.

2 Model

Setup. There are two players: a sender (S, she) and a receiver (R, he). The state of the world is $\theta \in \Theta = [0, 1]$, unknown by both players, who share a prior belief with a CDF $\overline{F}$, a full-support density $\bar{f}$ and a mean x_0. R has a privately known outside option $\omega \in \Omega = [0, 1]$ drawn from some unimodal distribution independent of θ. In particular, assume that its CDF G admits a strictly quasiconcave full-support density g with a peak at some $\hat{\omega} > x_0$.[10] R either accepts ($a = 1$) or rejects ($a = 0$) and has a utility $u_R(a, \theta, \omega) = a(\theta - \omega)$. That is, R prefers to accept if and only if his expectation of the state is at least as high as his outside option. S has a utility $u_S(a) = a$ and so always wants R to accept.

The timing of the game is as follows.

1. S decides which evidence to seek (at no cost). Formally, S chooses a *test*, i.e., a measurable mapping $\pi\colon \Theta \to \Delta M$, where $M = [0, 1]$ is the message space.[11]
2. Nature draws outside option ω from G, state θ from $\overline{F}$, message m from $\pi(\theta)$, and

[9]See also Onuchic (2021) for a model in the sender can commit to a disclosure rule for realizations of an exogenously given signal.

[10]The assumption $\hat{\omega} > x_0$ always makes equilibrium communication informative and can be interpreted as the conflict between the players' preferences being sufficiently large for a given G. Otherwise, if the conflict is small ($\hat{\omega} \leqslant x_0$), then, for some parameters of the model, equilibrium communication will be uninformative. In addition, it will always be uninformative if g is close enough to Dirac $\delta_{\hat{\omega}}$.

[11]For any compact metrizable Y, let ΔY denote the set of all Borel probability measures endowed with weak* topology.

the set of available messages $\hat{M}$ as follows:

- With probability $\rho \in (0, 1]$, $\hat{M} = \{m, \varnothing\}$ which is interpreted as S obtaining a proof that the realized message is m;
- With probability $1 - \rho$, $\hat{M} = \{\varnothing\}$, which can be interpreted either as S not being able to prove which outcome realized or that S has not learned the outcome of the experiment at all.

3. S observes $\hat{M}$ and chooses $\hat{m} \in \hat{M}$. That is, even if S obtains evidence, she can choose whether to disclose it or claim to not have obtained it.[12]
4. We distinguish between two variants of the game, depending on whether the evidence structure chosen by S is observed by R:
   - Under *covert evidence acquisition*, R observes $\hat{m}$ and, if $\hat{m} \neq \varnothing$, also observes π. Then he updates his belief and chooses an action;
   - Under *overt evidence acquisition*, R observes $\hat{m}$ and π, updates his belief and chooses an action.

Note that in both variants of the game, R observes π if S discloses evidence. This assumption enables the 'hard evidence' interpretation of information. That is, if S discloses a piece of evidence certifying some statement about the state, such a certificate must also include a (non-falsifiable) description of the test that generated it.[13]

We refer to the probability ρ as *reliability* of the testing environment and assume—for the main part of the paper—that it is fixed and commonly known. In many settings, this is motivated by the uncertainty about how long collecting evidence will take. Then, if there is a point in time at which the parties are planning to sign a contract, S might or might no be able to obtain the evidence by this deadline independently from the chosen test π.

There exist a number of interpretations of the payoff environment. First, as described above, ω can be interpreted as a single receiver's private information. Second,

[12]In principle, there can be many 'cheap-talk' messages that are always available to S. However, in this environment, any cheap-talk communication is uninformative because S's payoff is strictly increasing in R's posterior mean. Thus, it is without loss of generality to assume there is a unique 'cheap-talk' message $\hat{m} = \varnothing$ which can be interpreted as a S's claim that she does not have any proof.

[13]An alternative but equivalent formulation of this conceptual assumption is that each test is a mapping $\pi \colon \Theta \to \Delta(M \times \Pi)$ such that each "extended message" (m', π') also encodes the description of the experiment, i.e. $\pi(M \times \{\pi\}|\theta) = 1$ for all θ and π. In this formulation, R would observe π only through the extended message in the event of disclosure.

the set Ω can be viewed as a population of receivers. In this interpretation, S publicly discloses evidence and aims to maximize the mass of those who accept. Third, consider a setting in which R does not have a private type, but the action space is continuous. r For example, suppose that R is taking an action $a \in A = [0,1]$ to match the state ($u_R(a,\theta) = -(a-\theta)^2$), and S has a state-independent utility function that is convex-concave in the action, i.e. $u_S = G$.[14] Then such a model is strategically equivalent to the one we study.[15]

We study perfect Bayesian equilibria of the game. However, because of the assumptions on the players' preferences, the analysis is amenable to the belief-based approach as explained below. Since it is straightforward to recover the players' actual strategies from beliefs, it will be convenient to abstract away from strategies in the main text of the paper.[16]

Belief-based approach. We now describe a framework that will be convenient for analyzing the equilibria of the game. It relies on the representation of information structures with convex functions, which has proven to be useful in information design (Gentzkow and Kamenica, 2016; Kolotilin, 2018). Although it might not seem as the most intuitive way of representing information, investing into this framework pays off. In particular, this is because this approach turns out to be also convenient for the analysis of voluntary disclosure. A unified treatment of all aspects of the model then allows to solve for the equilibria under both overt and covert acquisition. Following the belief-based approach, we analyze the game by identifying the outcomes with R's posterior-belief means, since they allow to recover players' behavior and payoffs.

Fix any R's posterior belief $\beta \in \Delta\Theta$ with the mean $x^\beta := \int_\Theta \theta \, \mathrm{d}\beta(\theta)$. Then, the best response of R with an outside option ω is given by $a^\omega(\beta) := \mathbf{1}\{x^\beta \geqslant \omega\}$ for all $\omega \neq x^\beta$. Therefore, S's interim expected payoff is given by the probability R accepts given x^β, i.e.

$$\int_\Omega u_S(a^\omega(\beta)) \, \mathrm{d}G(\omega) = G(x^\beta).$$

[14] Dworczak and Martini (2019) provide an example of a continuous-action game in which the sender's objective is convex-concave.

[15] To see why, note that G measures S's indirect utility as a function of the induced posterior mean in either interpretation, the belief-based approach section below elaborates on this.

[16] Appendix A.1 presents a formal definition of an equilibrium.

In other words, one can think of R's outside-option CDF G as S's *indirect utility function* defined on the set of R's *posterior means*. For readability, we will use x to denote a posterior mean and X to denote the corresponding set (despite it coinciding with $\Theta = [0, 1]$ by construction).

Information structures as integral CDFs. Because ultimately only posterior means matter, each test π can be associated with a *posterior-mean distribution*, which we will identify with the corresponding CDF F_π.[17] Without loss of generality, since all relevant distributions have support inside $[0, 1]$ and cannot have mass at 0, we treat CDFs as functions on $[0, 1]$. We will further identify each posterior-mean distribution with the corresponding *integral CDF* (ICDF), which is an increasing convex function I_π defined as the antiderivative of the CDF F_π[18]

$$\begin{aligned} I_\pi \colon [0, 1] &\to [0, 1], \\ x &\mapsto \int_0^x F_\pi. \end{aligned}$$

Clearly, the CDF is pinned down by ICDF as its right derivative $F_\pi = I'_\pi$.[19]

To illustrate the approach, consider two extreme tests: fully informative $\overline{\pi}$ and uninformative $\underline{\pi}$. As $\overline{\pi}$ fully reveals the state, all posteriors are degenerate at the corresponding states, and the posterior-mean distribution then coincides with the prior $\overline{F}$, i.e.,

$$F_{\overline{\pi}} = \overline{F}.$$

As $\underline{\pi}$ reveals no information, there is a unique posterior that is equal to the prior $\overline{F}$. This means that the corresponding posterior-mean distribution is degenerate at the prior mean x_0, i.e., its CDF is a step function with a step at x_0,

$$F_{\underline{\pi}} = \mathbf{1}_{[x_0, 1]}.$$

[17] That is, let $\beta \colon M \to \Delta\Theta$ be the belief map, i.e. any measurable map that satisfies the Bayes rule, $\int_{\hat{\Theta}} \pi(\hat{M}|\cdot)\,\mathrm{d}\overline{F} = \int_\Theta \int_{\hat{M}} \beta(\hat{\Theta}|\cdot)\,\mathrm{d}\pi(\cdot|\theta)\,\mathrm{d}\overline{F}(\theta)$ for all Borel $\hat{\Theta}, \hat{M} \subseteq [0, 1]$. Then the posterior-mean CDF corresponding to (any consistent) β is given by $F_\pi(x) := \int_\Theta \pi(\{m \in M \colon x^{\beta(m)} \leqslant x\}|\cdot)\,\mathrm{d}\overline{F}$.

[18] We omit the variable of integration whenever it is not ambiguous, adopting the following notation: $\int_a^b f := \int_a^b f(x)\,\mathrm{d}x$, $\int_a^b f\,\mathrm{d}g := \int_a^b f(x)\,\mathrm{d}g(x)$.

[19] Throughout the paper, for any convex $I \colon [0, 1] \to [0, 1]$, let $I'(x)$ denote the right derivative of I at x for all $x \in [0, 1)$ and $I'(1) := 1$.

Let $\bar{I}$ and $\underline{I}$ denote the corresponding ICDFs of $\overline{\pi}$ and $\underline{\pi}$, respectively. Figure 3 below illustrates them for a uniformly distributed prior. In this case, integrating $\overline{F}(x) = x$ and $\underline{F}(x) = \mathbf{1}_{[\frac{1}{2},1]}(x)$ yields quadratic fully-informative ICDF $\bar{I}(x) = \frac{x^2}{2}$ and piece-wise linear uninformative ICDF $\bar{I}(x) = [x - \frac{1}{2}]^+$, where $[\cdot]^+ := \max(\cdot, 0)$.

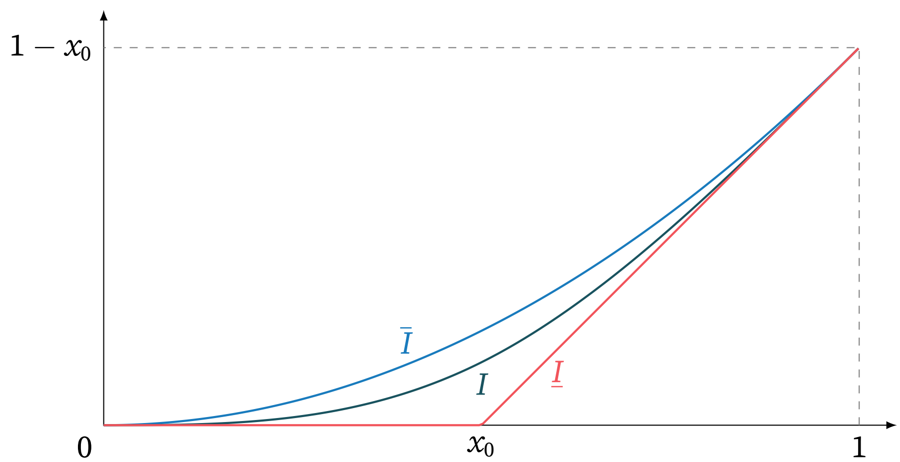


FIGURE 3: ICDFs of evidence structures corresponding to full information $\bar{I}$, no information $\underline{I}$, and some partial information I, for a uniform prior.

We can now describe the space of all possible information structures. As is well known, Blackwell informativeness order on information structures translates into *mean-preserving spreads* over distributions of posterior means.[20] Formally, an ICDF J is *more informative* than a posterior-mean ICDF I if and only if $J(x) \geqslant I(x)$ for all $x \in [0, 1]$ with equality at $x = 1$.

If $I = I_\pi$ is generated by some test π, then I is convex as an antiderivative of an increasing function and $\bar{I} \geqslant I \geqslant \underline{I}$ as every π is more informative than $\underline{\pi}$ and less informative than $\overline{\pi}$. Gentzkow and Kamenica (2016) and Kolotilin (2018) showed that the converse also holds, that is for any convex I such that $\bar{I} \geqslant I \geqslant \underline{I}$, there exists a test π such that $I_\pi = I$. Thus, we can define the set of all feasible posterior-mean ICDFs as

$$\mathcal{I} := \{I \colon [0, 1] \to [0, 1] \colon I \text{ convex and } \bar{I} \geqslant I \geqslant \underline{I}\}.$$

Finally, in addition to the informativeness partial order $\geqslant$ on $\mathcal{I}$, we also define the following strict informativeness binary relation $>$ as an asymmetric part of $\geqslant$. That is,

[20] Rothschild and Stiglitz (1970) prove equivalence in the context of a risk averter's preferences over monetary lotteries and Leshno, Levy, and Spector (1997) provide a corrected proof of their result. Blackwell and Girshick (1954) prove a decision-theoretic equivalence result in the finite case.

J is *strictly more informative* than I if and only if $J > I$, i.e., $J \geqslant I$ and $J \neq I$. In the current setting, this notion has the following interpretation: J is strictly more informative than I if and only if R is ex-ante strictly better off having posterior-mean ICDF J than I for any strictly increasing CDF of the outside option (see Corollary 10 in Appendix A.2).

The benefit of this approach is that it allows to treat all information structures in a unified way. The next section presents the analysis of the game and shows how elements of $\mathcal{I}$ can represent relevant information structure (and hence behavior) at different stages of the game, including evidence structure chosen by S, information acquired by S (i.e, distribution of R's posterior means under truthful disclosure), and disclosed information (i.e, distribution of R's posterior means under strategic disclosure).

3 Analysis

In this section, we characterize the equilibria of the game. First, we analyze an auxiliary disclosure game in which the evidence structure is fixed and commonly known. Then, we characterize the resulting S value and show that the ex-ante acquisition problem can be stated as an optimization problem in the overt case and as a fixed-point problem in the covert case. Finally, we characterize the equilibrium evidence acquisition.

3.1 Voluntary disclosure

In this section, we analyze an auxiliary disclosure game in which the evidence structure I is fixed and commonly known. Analyzing this game is useful to understand the on-path R beliefs and S disclosure decisions. Moreover, in the overt case, I is always observed by R and so the auxiliary game can be treated as a subgame of the main game.

Fix any feasible posterior-mean ICDF $I \in \mathcal{I}$ with the corresponding CDF $F := I'$. Then, let $x_\varnothing \in X$ denote *non-disclosure R's posterior mean* and note that it is strictly optimal for S (not) to disclose x if and only if it is above (below) $x_\varnothing$ since her interim payoff function G is strictly increasing.[21] In order for $x_\varnothing$ to be Bayes-consistent, it must equal R's posterior mean conditional on non-disclosure given S disclosure strategy and

[21] Here, it is without loss of generality of payoffs, equilibrium evidence structure, and all subsequent results to assume that S does not disclose when indifferent.

F. Thus, combining it with the aforementioned optimality of S's $x_\varnothing$-threshold strategy we obtain the following equation

$$x_\varnothing = \frac{(1-\rho)}{1-\rho+\rho F(x_\varnothing)} x_0 + \frac{\rho F(x_\varnothing)}{1-\rho+\rho F(x_\varnothing)} \mathbb{E}_F(x | x \leqslant x_\varnothing). \tag{1}$$

The following lemma provides a convenient way to solve this fixed-point problem in terms of the ICDF approach.

Lemma 1. *A non-disclosure posterior mean $x_\varnothing \in X$ is Bayes-consistent*[22] *with the $x_\varnothing$-threshold disclosure strategy if and only if it solves*

$$I(x_\varnothing) = \frac{1-\rho}{\rho}(x_0 - x_\varnothing). \tag{2}$$

Proof. Integrate by parts to obtain

$$\mathbb{E}_F(x|x \leqslant x_\varnothing) = \frac{1}{F(x_\varnothing)} \int_0^{x_\varnothing} x \, \mathrm{d}F(x) = x_\varnothing - \frac{I(x_\varnothing)}{F(x_\varnothing)},$$

then plug it into (1) and rearrange the terms. $\square$

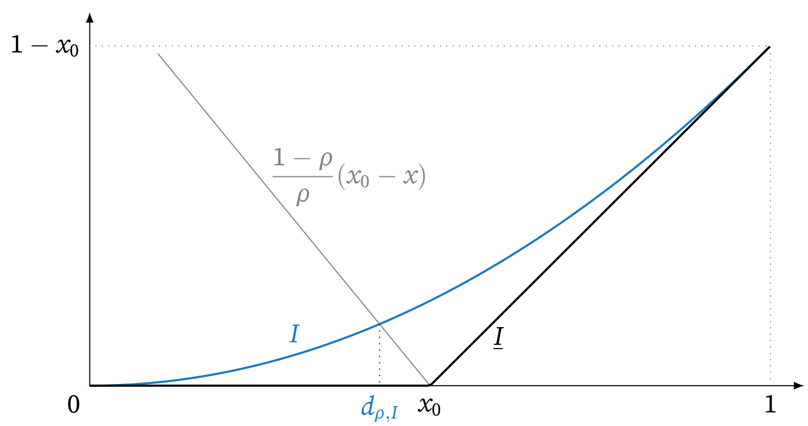


FIGURE 4: Construction of the disclosure threshold $d_{\rho,I}$.

Figure 4 illustrates that the equilibrium posterior mean must be at the intersection of the ICDF and the straight line whose slope depends on ρ. Lemma 1 immediately implies the following unraveling, uniqueness, and comparative statics results, which can be also clearly seen from the geometric representation of Figure 4.

[22]To account for the cases when Bayes rule applies vacuously, we say that '$x_\varnothing$ is Bayes-consistent' if and only if it solves (1) with both sides multiplied by the probability of non-disclosure $1-\rho+\rho F(x_\varnothing)$.

Corollary 1 (Unraveling). *If $\rho = 1$, then S discloses all (except, possibly, the lowest) realizations of an evidence structure.*

Proof. Suppose $\rho = 1$ and $x_\varnothing$ is the S disclosure threshold. Then x is also a Bayes-consistent non-disclosure posterior mean which, by Lemma 1, is the root of I and, therefore, $x \leqslant \min \operatorname{supp} I$. □

Corollary 2 (Uniqueness). *There exists a unique solution to* (1) *if and only if $\rho \neq 1$ or $I(x) > 0$ for all $x > 0$.*

Proof. By Lemma 1, the set of solutions is given by the roots of a function

$$\xi_{\rho,I} \colon [0,1] \to \mathbb{R}$$
$$x \mapsto \rho I(x) + (1-\rho)(x - x_0),$$

which is continuous and strictly increasing. Then, the 'if' part follows from the Intermediate Value Theorem since, for any $I \in \mathcal{I}$, $\xi_{\rho,I}(0) \geqslant 0$ and $\xi_{\rho,I}(x_0) \leqslant 0$, with the first (second) inequality strict if and only if $\rho = 1$ ($\min \operatorname{supp} I = 0$). Finally, the 'only if' part holds because $\xi_{1,I} = I$. □

Let $d_{\rho,I}$ denote the equilibrium *disclosure threshold* for given ρ and I. By Corollary 2, the only technicality preventing global uniqueness is that R—as a Bayesian—may in principle assign a posterior mean strictly below the support of I in the off-path event of non-disclosure. In this case, we assume $d_{1,I} \coloneqq \lim_{\rho \nearrow 1} d_{\rho,I} = \min \operatorname{supp} I$ for convenient continuity in ρ. [23]

Corollary 3 (More information leads to more disclosure). *The disclosure threshold $d_{\rho,I}$ is strictly decreasing in ρ, and decreasing in I with respect to the informativeness order $\geqslant$.*

Proof. Since $(\rho, I) \mapsto \xi_{\rho,I}$ is strictly increasing in ρ and increasing in I (with respect to $\geqslant$), it follows that the root of $\xi_{\rho,I}$ is strictly decreasing in ρ and decreasing in I. □

It ought be noted that the results similar to or exactly as stated in Corollaries 1, 2 and 3 are well-known in related settings. In particular, Corollary 1 follows from the unraveling principle (see e.g. Milgrom, 1981; Grossman, 1981). Intuitively, when $\rho = 1$, R is certain that S has evidence and so R's full skepticism incentivizes S to fully disclose.

[23] While this is the least permissive in terms of equilibria outcomes, it turns out to be without loss of generality.

Similar uniqueness and comparative statics results were established in Propositions 1 and 2 in Jung and Kwon (1988) and (see also Proposition 1 in Acharya, DeMarzo, and Kremer, 2011) for continuous distributions and in Corollary 2 and Proposition 2 of DeMarzo, Kremer, and Skrzypacz (2019) for general distributions. Intuitively, when S is less informed, R's skepticism is more 'muted' which allows S to credibly conceal more evidence in equilibrium. We establish these results using the ICDF approach which provides a simple representation (2) for the R non-disclosure posterior which, in equilibrium, coincides with the S's disclosure threshold. We will then rely on this representation to obtain characterizations of equilibrium evidence structures.

It will be useful to think about S's strategic disclosure of information as a garbling. For some S chosen evidence structure $I \in \mathcal{I}$, what is the resulting distribution $I_\rho^D \in \mathcal{I}$ corresponding to the *disclosed* evidence structure or, equivalently, the actual distribution of R posteriors? Since S does not disclose either if she is uninformed or if the realized evidence is below $d_{\rho,I}$ and otherwise R's posterior mean equals exactly the realized evidence, the direct computation yields the *disclosed CDF*

$$I_\rho^{D\prime}(x) = \begin{cases} 0, & x < d_{\rho,I} \\ 1 - \rho + \rho F(x), & x \geqslant d_{\rho,I} \end{cases}$$

which gives the following simple expression for the *disclosed ICDF*

$$I_\rho^D(x) := \left[\rho I(x) + (1-\rho)(x - x_0)\right]^+. \tag{Discl}$$

It is also useful to contrast this with the case of mandatory (i.e. truthful) disclosure in which case S reveals all information she acquired. In this case, R's information is exactly the chosen evidence structure I with probability ρ and no information otherwise. Thus, the corresponding *acquired ICDF* is simply the convex combination of I and $\underline{I}$

$$\begin{aligned} I_\rho^A(x) &:= \rho I(x) + (1-\rho)\underline{I}(x) \\ &= \rho I(x) + (1-\rho)(x - x_0)^+. \end{aligned}$$

Figure 5 illustrates these transformations. As can be clearly seen, we have $\bar{I} \geqslant I \geqslant I_\rho^A \geqslant I_\rho^D \geqslant \underline{I}$. In words, first, the evidence structure I S seeks must be less informative than $\bar{I}$ by Bayes plausibility. Second, I_ρ^A represents the information available for S to disclose which is a garbling of I since S's acquisition technology is not perfectly reliable. Next, I_ρ^D represents the information S will actually strategically disclose which is bounded by mandatory disclose from above and by no information from below. Note that the kink of ICDF represents a mass point of the corresponding posterior-mean distribution. Hence I^A_ρ has a $(1-\rho)$-mass point at x_0 due to imperfect reliability, and I^D_ρ has a $(1-\rho+\rho F(d_{\rho,I}))$-mass point at the posterior mean $d_{\rho,I}$ representing the event of non-disclosure. Finally, note also that I^D_ρ differs from I^A_ρ below x_0 because—compared to I^A_ρ—under voluntary disclosure, all the mass below $d_{\rho,I}$ is combined with the $(1-\rho)$-mass point at x_0 into a single mass point at $d_{\rho,I}$.

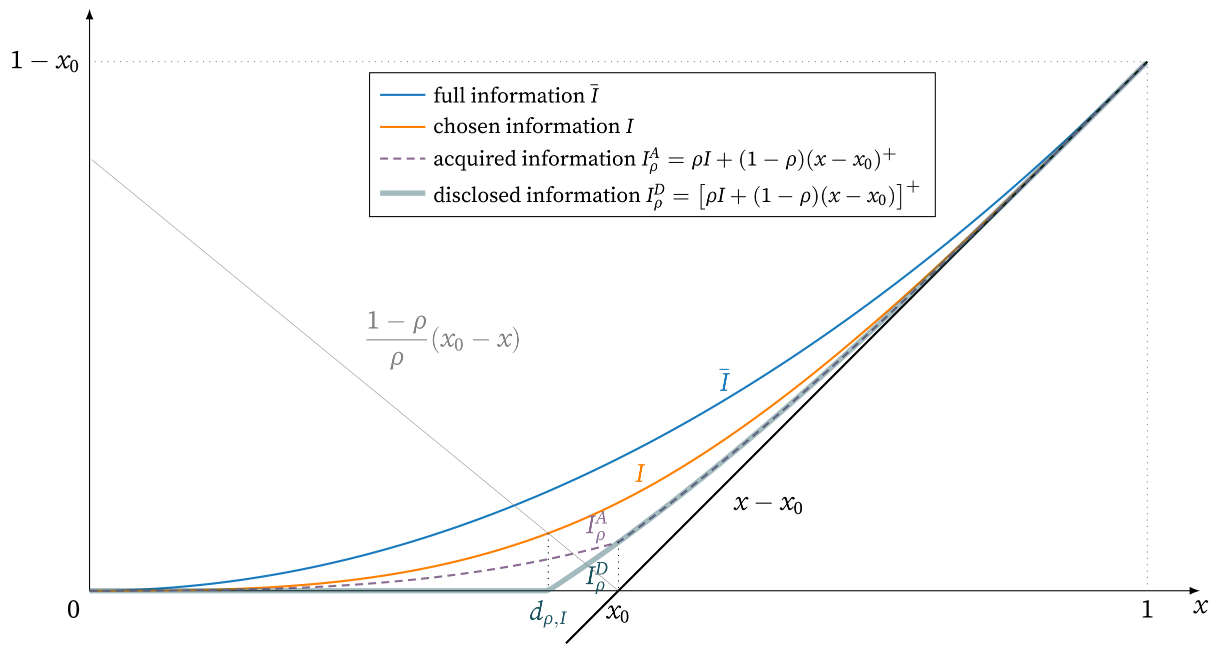


FIGURE 5: Construction of the disclosure threshold $d_{\rho,I}$ and the transformations I^A_ρ and I^D_ρ.

3.2 Equilibrium Evidence Acquisition

In this section, we endogenize the evidence structure as S's ex-ante choice.

We begin with a few definitions. Say that I is an *(c-) o-equilibrium structure* if there exists a PBE[24] of the (covert) overt acquisition game in which S chooses π which induces $I_\pi = I$. Now fix some $x_\varnothing \in X$ and define S value from choosing $I \in \mathcal{I}$ given R's non-

[24]We formally define these equilibria concepts in terms of beliefs and strategies in Appendix A.1.

disclosure belief mean is $x_\varnothing$ and S discloses realizations $x > x_\varnothing$ as

$$v_\rho(I|x_\varnothing) := [1-\rho+\rho I'(x_\varnothing)]\, G(x_\varnothing) + \rho \int_{x_\varnothing}^{1} G \, \mathrm{d}I' - G(x_0).$$

Note that we normalize the S no-information value to zero by subtracting the no-information payoff $G(x_0)$. This definition enables the following preliminary characterization of equilibria.

Lemma 2. *For any evidence structure $I^* \in \mathcal{I}$,*

(i) I^* is an o-equilibrium structure if and only if

$$I^* \in \underset{I\in\mathcal{I}}{\operatorname{argmax}}\, v_\rho(I|d_{\rho,I}), \tag{Overt}$$

(ii) I^* is an c-equilibrium structure if and only if

$$I^* \in \underset{I\in\mathcal{I}}{\operatorname{argmax}}\, v_\rho(I|d_{\rho,I^*}), \tag{Covert}$$

Proof. Omitted. All omitted proofs are given in Appendix A.2. □

Notice that the two seemingly similar (Overt) and (Covert) programs are actually quite different because (Overt) is an optimization problem and (Covert) is a fixed-point problem. Conceptually, in the overt case, S can commit to the way information is acquired (up to reliability) but not to the way it is disclosed. That is, whatever I she chooses will lead to a Bayes-consistent R's non-disclosure posterior mean $d_{\rho,I}$. Then, since the disclosure subgame for each I has a unique outcome, S simply needs to find the best such outcome across all feasible evidence structures.

In contrast, in the covert case, deviating to another evidence structure I is not detected by R and so his non-disclosure posterior remains at d_{ρ,I^*}—the same as on the equilibrium path. Therefore, S's chosen evidence structure maximizes her payoff for a fixed non-disclosure posterior mean which in turn needs to be Bayes-consistent.

An alternative way to understand the (Overt) objective is to relate it to the S full-commitment problem in which she is able to directly design R's information. Define the S indirect value function over R's posterior-mean ICDF as

$$\begin{aligned} v &\colon \mathcal{I} \to \mathbb{R}, \\ I &\mapsto v_1(I|0) = \int_0^1 G \, \mathrm{d}I' - G(x_0), \end{aligned}$$

so that the S full-commitment problem can be written as

$$\max_{I \in \mathcal{I}} \int_0^1 G \, \mathrm{d}I' = \max_{\mathcal{I}} v. \tag{FC}$$

Then, Lemma 1 and the (Discl) transformation imply that the (Overt) objective at the ICDF is equal to the (FC) objective evaluated at the same ICDF transformed under voluntary disclosure, that is

$$\begin{aligned} v_\rho(I|d_{\rho,I}) &= [1-\rho+\rho I'(d_{\rho,I})]\, G(d_{\rho,I}) + \rho \int_{d_{\rho,I}}^1 G \, \mathrm{d}I' - G(x_0) \\ &= \int_0^1 G(x) \, \mathrm{d}\big[\rho I(x) + (1-\rho)(x-x_0)\big]^{+\prime} - G(x_0) \\ &= v(I_\rho^D) \end{aligned}$$

As a corollary of the Lemma 2, we also obtain that both o- and c-equilibria exist by standard continuity/convexity/compactness arguments.

Corollary 4. *For any $\rho \in (0,1]$, an o-equilibrium and a c-equilibrium exist.*

3.3 Benchmark: Perfect Reliability

Before we characterize the equilibrium evidence structure, it will be instructive to look at the extreme case of $\rho = 1$, that is, when S always obtains the evidence she seeks. Recall that in this case, a standard unraveling argument applies (Corollary 1), that is, S fully discloses whatever she acquires due to R being fully skeptical, hence, $I_1^D = I$.

Then, both (Overt) and (Covert) programs reduce to (FC). This means that two out of three forces affecting R's information—voluntary disclosure and observability of acquisition strategy—are irrelevant in this case and an equilibrium is characterized by a pure information design problem. Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017), Kolotilin (2018) study a model of Bayesian persuasion with R's private payoff type which is similar to the above. In particular, they show that if the distribution of R types is unimodal, the optimal signal simply reveals (pools) all states below (above) some threshold.[25]

Intuitively, when the distribution of outside options is unimodal, the S ex-post payoff function G is convex below and concave above $\hat{\omega}$. Therefore, when the state is low

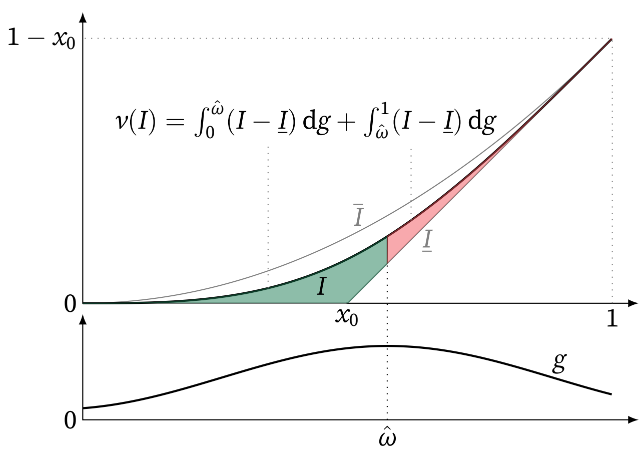
[25]Optimality of upper censorship in similar settings also appears in Alonso and Câmara (2016b) and Dworczak and Martini (2019).

(high), more information benefits (hurts) S because her indirect utility function G is convex-concave. It then turns out that it is optimal for S to choose a signal which take a simple form of upper censorship of the prior distribution.

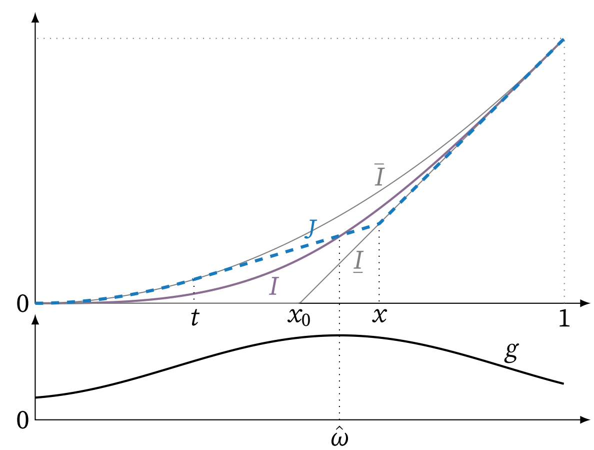
To formalize this intuition, it will be useful to rewrite the objective function v by integrating by parts twice as follows

$$v(I) = \int_0^1 G \, \mathrm{d}I' - G(x_0) = \int_0^1 G \, \mathrm{d}(I' - \underline{I}') = \int_0^1 (I - \underline{I}) \, \mathrm{d}g.$$

Such an integral representation implies that the S's value can be visualized as the 'area' between I and $\underline{I}$ 'weighted' by the measure induced by the density of R's outside option G as illustrated in Figure 6a. Because G is increasing (decreasing) below (above) the mode



(a) $v(I)$ as the sum of the positive and the negative parts.



(b) Lemma 4: J is an improvement over I.

FIGURE 6: Graphical representation of v and Lemma 4.

$\hat{\omega}$, it induces a positive (negative) measure on the corresponding interval. Thus, the S value $v(I)$ is the sum of the positive part $\int_0^{\hat{\omega}} (I - \underline{I}) \, \mathrm{d}g$ and the negative part $\int_{\hat{\omega}}^1 (I - \underline{I}) \, \mathrm{d}g$. This implies that S benefits from more information at the bottom and less information at the top.

The above decomposition of value leads to the following two lemmas. Fix any $I, J \in \mathcal{I}$ and $\hat{\omega} \in [0, 1]$. Call J a *pivoted* I if J is weakly above I on $[0, \hat{\omega}]$ and weakly below I on $[\hat{\omega}, 1]$ and $I \neq J$. Call J an *S-improvement over* I if $v(J) - v(I) = \int (J - I) \, \mathrm{d}g > 0$ for all strictly quasiconcave g with a peak at $\hat{\omega}$.

Lemma 3. *For any $I, J \in \mathcal{I}$, J is an S-improvement over I if and only if J is a pivoted I.*

In other words, S always prefers more information (higher ICDF) below $\hat{\omega}$ and less information (lower ICDF) above $\hat{\omega}$.

The next lemma characterizing the possibility of an S-improvement is instrumental both to solving the (FC) and the general problem with imperfect reliability. Call I a t *upper censorship* if it is induced by a test which pools (reveals) all states above (below) some threshold $t \in \Theta$.

Lemma 4. *For any $I \in \mathcal{I}$ there exists some $t \in \Theta$ such that either t upper censorship coincides with or is an S-improvement over I.*

Proof. Fix any $I \in \mathcal{I}$. By Lemma 3, we need to construct a t upper censorship J which either coincides with I or is a pivoted I. The construction is shown in Figure 6b and the formal proof is given in the Appendix of Lipnowski, Ravid, and Shishkin (2021), Lemma 5. Take the line tangent to $\bar{I}$ going through the point $(\hat{\omega}, I(\hat{\omega}))$, and let $(t, \bar{I})$ and $(x, \underline{I})$ be the points of tangency with $\bar{I}$ and intersection with $\underline{I}$, respectively. Let J be equal to the tangent line on $[t, x]$, to $\bar{I}$ on $[0, t]$, and to $\underline{I}$ on $[x, 1]$. It is easy to verify that J is a t upper censorship and that either $I \neq J$ and then J is an S-improvement, or $I = J$, then no improvement exists. □

In particular, the above lemma implies that a structure cannot be S-improved if and only if it is an upper censorship. Since v is simultaneously the (FC) objective, and (Overt) and (Covert) objective under $\rho = 1$, we immediately obtain the following characterization of the perfect reliability case.

Corollary 5. *If $\rho = 1$, then there exists some $t_1^* \in [0, \hat{\omega}]$ such that the following four statements are equivalent for any $I^* \in \mathcal{I}$:*

(i) I^* is an o-equilibrium structure.

(ii) I^* is a c-equilibrium structure.

(iii) I^* solves the (FC) problem.

(iv) I^* is the t_1^* upper censorship.

Proof. First, note that by Lemma 4, any solution to the (FC) problem is an upper censorship since otherwise it could be S-improved. Second, (i) is equivalent to (iii) since $v_1(I|d_{1,I}) = v(I_1^D) = v(I)$. Third, the formal proof in the appendix shows that (ii) is equivalent to (i) due to unraveling under perfect reliability (Corollary 1) and uniqueness of the threshold t_1^* is due to the strict quasiconcavity of g. Finally, any optimal threshold is below $\hat{\omega}$ because the $\hat{\omega}$ upper censorship is strictly better for S than any t upper censorship with $t > \hat{\omega}$ since the $\hat{\omega}$ upper censorship has the same positive part of the value decomposition and smaller negative part in the absolute sense. □

3.4 Benchmark: Uniformly Distributed Outside Option

Another useful benchmark is the case of uniformly distributed ω.[26] In this case, S's indirect payoff function G is affine and so she has no information design motive because she is ex-ante indifferent between all evidence structures. Consequently, *every feasible* evidence structure is an o-equilibrium one.

Observation 1. *If R's outside option is uniformly distributed, then every $I^* \in \mathcal{I}$ is an o-equilibrium structure.*

Proof. Plugging in $G(x) = x$ for all $x \in [0, 1]$ in the (Overt) objective, we obtain

$$\nu_\rho(I|d_{\rho,I}) = \nu(I^D) = \int_0^1 x \, \mathrm{d}(I_\rho^D - \underline{I})'(x) = 0,$$

since $\int_0^1 x \, \mathrm{d}I'(x) = x_0$ for all $I \in \mathcal{I}$ including I_ρ^D and $\underline{I}$. □

As for the covert case, a related model to our covert acquisition model with a uniform outside option was studied by DeMarzo, Kremer, and Skrzypacz (2019). Their results imply that c-equilibria are characterized by the *minimum principle* which can be stated in our setting as

$$I^* \in \operatorname*{argmin}_{I \in \mathcal{I}} d_{\rho,I}. \tag{MP}$$

Intuitively, in the absence of information design incentives for S, I^* is a c-equilibrium structure if and only if S has no profitable deviation ex-ante. The only way S may benefit from a deviation to I is if I has a lower corresponding non-disclosure posterior mean $d_{\rho,I}$. Hence no deviation exists if and only if I^* is minimal in the sense of (MP).

Observation 2. *If R's outside option is uniformly distributed, then $I^* \in \mathcal{I}$ is a c-equilibrium structure if and only if I^* satisfies* (MP).

[26]With the exception of this subsection, this case is ruled out in the model as the corresponding density is not strictly quasiconcave.

Proof. Plugging in $G(x) = x$ for all $x \in [0,1]$ in the (Covert) objective, we obtain

$$\begin{aligned} v_\rho(I|x_\varnothing) &= [1-\rho+\rho I'(x_\varnothing)]\, x_\varnothing + \rho(1-I'(x_\varnothing))\mathbb{E}_I(x|x > x_\varnothing) - x_0 \\ &= \rho\left(I(x_\varnothing) - \frac{1-\rho}{\rho}(x_0 - x_\varnothing)\right). \end{aligned}$$

That is, (Covert) is equivalent to $I^* \in \operatorname{argmax}_{I\in\mathcal{I}} I(d_{\rho,I^*})$ which is equivalent to (MP). □

3.5 Overt-Covert Equivalence

Clearly, the minimum principle no longer characterizes c-equilibria if the outside option is non-uniform because S in this case is not ex-ante indifferent between all information structures. However, the following result shows that although the minimum principle is no longer necessary in the non-uniform case, the combination of (MP) and (Overt) optimality is sufficient to satisfy (Covert).

To state this result, we first make the following difference-in-difference comparison between deviations in the overt and covert cases.

Lemma 5. *The net benefit to S from an ex-ante deviation in the covert case is higher (lower) than that in the overt case if and only if the corresponding non-disclosure posterior is higher (lower).*

Proof. Consider a deviation from I^* to I. The desired difference between covert and overt cases is[27]

$$\begin{aligned} &[v_\rho(I^*|d_{\rho,I^*}) - v_\rho(I|d_{\rho,I^*})] - [v_\rho(I^*|d_{\rho,I^*}) - v_\rho(I|d_{\rho,I})] \\ &= (1-\rho)\,[G(d_{\rho,I}) - G(d_{\rho,I^*})] + \rho\int_{d_{\rho,I^*}}^{d_{\rho,I}} I'\,\mathrm{d}g, \end{aligned}$$

which has the same sign as $d_{\rho,I} - d_{\rho,I^*}$. □

This lemma immediately implies that the minimum principle is sufficient for an o-equilibrium to be c-equilibrium. This is because the only way S may benefit from deviating to I *in addition* to her gain from the same deviation in the overt case is from 'hiding' a lower Bayes-consistent non-disclosure posterior.

Corollary 6. *If the set of o-equilibria satisfying the minimum principle* (MP) *is non-empty, then this set coincides with the set of c-equilibria.*

[27] For any $a, b \in [0,1]$, we follow the notational convention $\int_a^b F\,\mathrm{d}g := \int_0^b F\,\mathrm{d}g - \int_0^a F\,\mathrm{d}g$.

Proof. Suppose I^* is an o-equilibrium which satisfies (MP) and take any $I \in \mathcal{I}$. We will show that I is a o-equilibrium satisfying (MP) if and only if it is a c-equilibrium. By Lemma 5, it is weakly beneficial to covertly deviate from I to I^* because it is weakly beneficial to overtly deviate from I to I^* and there is an additional non-negative benefit since I^* satisfies the minimum principle. But then I is a c-equilibrium if and only if both of these non-negative benefits are zero which is equivalent to I being an o-equilibrium satisfying (MP). $\square$

3.6 Overt Acquisition

Now we turn to the general overt case when testing may be imperfectly reliable. We start by introducing the following class of information structures that nests the upper censorship defined in Section 3.3.

Call an evidence structure $I \in \mathcal{I}$ *a* (θ_l, θ_h) *two-sided censorship of* $J \in \mathcal{I}$ if it is a garbling of J which perfectly reveals all realizations in $[\theta_l, \theta_h]$, pools ones above θ_h, and also pools ones below θ_l. Formally, I is the lowest ICDF that coincides with J on $[\theta_l, \theta_h]$ as illustrated in Figure 7. This class includes three important special cases: I is *a* θ_h *upper censorship* of J if $\theta_l = 0$; *a* θ_l *lower censorship* of J if $\theta_h = 1$; and *a* θ *pass/fail test* of J if $\theta_l = \theta_h = \theta$. Whenever $J = \bar{I}$ in the above notions, we will omit saying 'of $\bar{I}$' for brevity.

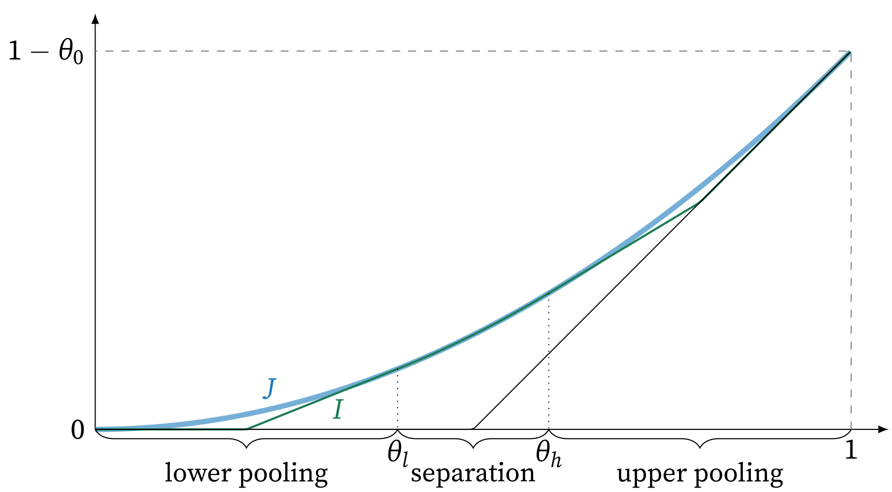


FIGURE 7: Two-sided censorship.

A test inducing a two-sided censorship can be interpreted as a grading system that assigns the PASS grade to the states above the upper cutoff, the FAIL grade to the states below the lower cutoff, and has a variety of intermediate grades corresponding exactly to each state in between. In addition, if $\theta_l = 0$ and $\theta_h = 1$, both pooling intervals are empty, which corresponds to the fully informative structure $\bar{I}$. And if $\theta_l = \theta_h \in \{0, 1\}$, then all states are pooled, which corresponds to the uninformative structure $\underline{I}$.

In order to state the main results, we now introduce a notion of disclosure-equivalence to address the multiplicity of equilibrium evidence structures that naturally arises in the model.

Definition 1. Call I and J *disclosure-equivalent* if their (Discl) transforms coincide, that is, $I^D_\rho = J^D_\rho$.

To illustrate this definition, suppose I is an equilibrium evidence structure induced by a test which is perfectly informative about states below some $x \leqslant d_{\rho,I}$. Then, although S obtains precise information about states below x, she will end up not disclosing any of the corresponding realizations of I. Now consider J which is a x lower censorship of I, that is, J pools all realizations of I below x. But then this is observationally equivalent from R's perspective since the same realizations of I and J are disclosed and so it does not matter whether S learns more or less bad news which will be concealed anyway. As a result, disclosure equivalence affects neither the Bayes-consistent nondisclosure posterior, nor whether (MP) is satisfied, nor whether (Overt) and (Covert) equilibrium conditions are satisfied.

It is also easy to verify that Definition 1 and the (Discl) transformation imply I and J are disclosure equivalent if and only if they coincide on $[d_{\rho,I}, 1]$. This implies that the disclosure-equivalence class of any I has the least informative element given by the $d_{\rho,I}$ lower censorship of I. From now on, we will focus on such least informative equilibria structures. The reason for such a selection from disclosure-equivalence classes is three-fold. First, it is straightforward to construct a disclosure-equivalence class from the least informative structure and so it is easy to recover all equilibria.[28] Second, this selection can be seen as a 'revelation principle': for every equilibrium of the game, there exists a 'canonical' outcome-equivalent equilibrium, in which there is a unique bad-news realization which is the only one not disclosed by S. Third, one can also view this as a selection based on vanishing Blackwell-monotone cost of information.

[28] Indeed, the set of all structures disclosure-equivalent to I is the $\geqslant$-interval between the $d_{\rho,I}$ lower-censorship of I and pointwise maximum over all structures coinciding with I on $[d_{\rho,I}, 1]$.

The following theorem provides a characterization of o-equilibria.

Theorem 1. *There exists $\overline{\rho}^o \in [0,1)$ such that any o-equilibrium evidence structure is disclosure equivalent to a pass/fail test if $\rho < \overline{\rho}^o$, and to the $(d_{\rho,\bar{I}}, t_1^*)$ two-sided censorship if $\rho > \overline{\rho}^o$.*

Moreover, for all except countably many $\rho < \overline{\rho}^o$, the equilibrium pass/fail threshold $t_\rho^o \in (0, d_{\overline{\rho}^o,\bar{I}})$ is unique and increasing in ρ.

Recall that in isolation, the voluntary disclosure force leads to pooling at the bottom and the information design force leads to pooling at the top of the state space as evident from Sections 3.1 and 3.3, respectively. Theorem 1 then demonstrates that whether and how these two forces interact depends on reliability. When reliability is above $\overline{\rho}$, the interaction between the two forces is trivial and optimal evidence structure is a two-sided censorship of the state. The lower threshold $d_{\rho,\bar{I}}$ is not affected by the design of the evidence structure and coincides with the disclosure threshold under fully-revealing evidence structure. Moreover, the upper threshold t_1^* is unaffected by voluntary disclosure: it stays constant and coincides with the optimal upper threshold that the sender would use under $\rho = 1$. In other words, the optimal structure is a straightforward combination of the two forces.

However, when reliability is below $\overline{\rho}$, the interaction between the two forces becomes non-trivial and the sender switches to a pass/fail test. From the ex-ante perspective, S prefers more information at the bottom and, therefore, voluntary disclosure hurts S because it induces pooling of low states. In other words, while she would want to commit to reveal low states, she cannot if disclosure is voluntary. When ρ drops below $\overline{\rho}$, it becomes optimal to design evidence structure in order to reduce the ex-ante loss from lower pooling. This is achieved by a pass/fail test, as it allows to reduce the lower pooling interval by enlarging the upper pooling interval because, under pass/fail test, S discloses if and only if she passes the test.

Moreover, as reliability falls so does the total probability of disclosure, if the signal is kept the same. Then, it is optimal to lower the pass/fail test threshold in order to enlarge the upper pooling interval and increase the probability of disclosure conditional on obtaining evidence thereby compensating for falling total probability of disclosure.

Formally, the result is based on two observations.

Corollary 7 (of Lemma 4). *Every o-equilibrium structure is disclosure-equivalent to an up-*

per censorship.

Proof. Take any o-equilibrium structure I. By Lemma 4, there exists an S-improvement upper-censorship J. Then, $J - I$ is non-negative on $[0, \hat{\omega}]$ and non-positive on $[\hat{\omega}, 1]$ and so is $J^D_\rho - I^D_\rho$. If $J^D_\rho = I^D_\rho$, then I is disclosure equivalent to an upper censorship. Otherwise, J^D_ρ is an S-improvement over I^D_ρ and therefore I does not maximize $v(I^D_\rho) = v_\rho(I|d_{\rho,I})$ which contradicts with it being an o-equilibrium structure, by Lemma 2. $\square$

This observation suggests that the information design force has a similar effect as we observed in the case of perfect reliability in Section 3.3. It allows to relax the (Overt) program to a one-dimensional optimization over upper censorship threshold $t \in \Theta$. But then every upper censorship is disclosure-equivalent to either a two-sided censorship or to a pass/fail test depending on whether the optimal upper censorship threshold t^o_ρ is above or below the corresponding non-disclosure posterior.

Second, we demonstrate that the objective function in the optimization with respect to the upper censorship threshold t has the increasing differences property so that the set of optima is increasing in ρ in the sense of strong set order. Moreover, it has strictly increasing marginal differences for low values of t which implies that the set of optima is either equal to $\{t^*_1\}$ or weakly below t^*_1 it with every selection strictly increasing in ρ. Denoting by $\overline{\rho}^o$ the switching point, this implies that the o-equilibrium upper censorship threshold is unique for all $\rho > \overline{\rho}^o$ and all except possibly a countable subset of $\rho \leqslant \overline{\rho}^o$.

3.7 Covert Acquisition

We now turn to the case in which the S acquisition strategy is unobserved by R unless S discloses evidence.

Notice that with high reliability the unique o-equilibrium satisfies the minimum principle. Intuitively, the o-equilibrium two-sided censorship is disclosure equivalent to the t^*_1 upper-censorship which is perfectly informative about low states. This implies R is least skeptical—for a given ρ—about non-disclosure and has the lowest possible Bayes-consistent non-disclosure posterior. Hence, it is also a c-equilibrium structure as S cannot benefit from R's 'fixed' non-disclosure posterior by deviating (Lemma 5). In other words, with high reliability, not only information design and voluntary disclosure do not interact, but also the covertness of acquisition has no impact because S chooses a relatively detailed test.

In contrast, with low reliability, o-equilibrium pass/fail test fails the minimum principle, because the threshold is always strictly below the minimal non-disclosure posterior $d_{\rho,\bar{I}}$. So there is an additional benefit to S from deviating from such a signal to some structure with a lower corresponding non-disclosure posterior. Thus, covertness of the acquisition may now also affect the equilibrium information and break the overt-covert equivalence. The next result shows that this is indeed the case. Perhaps surprisingly, c-equilibria retain the same structural property of being pass/fail test as o-equilibria but reliability has the opposite effect on the pass/fail threshold.

Theorem 2. *There exists $\overline{\rho}^c \in [0, \rho^o]$ such that every c-equilibrium evidence structure is disclosure equivalent to a pass/fail test if $\rho \leqslant \overline{\rho}^c$, and to the $(d_{\rho,\bar{I}}, t_1^*)$ two-sided censorship if $\rho > \overline{\rho}^c$.*

Moreover, for all $\rho < \overline{\rho}^c$, the c-equilibrium pass/fail threshold $t_\rho^c \in (d_{\overline{\rho}^c,\bar{I}}, x_0)$ is unique and strictly decreasing in ρ.

To get some intuition for this result, it is useful to think about equilibria in terms of two variables: R's non-disclosure posterior mean $x_\varnothing$ and S's choice of information I. Then, solving the (Covert) program is equivalent to finding a pair $(I, x_\varnothing)$ such that S is best-responding to R's non-disclosure belief (i.e., I maximizes $v_\rho(\cdot|x_\varnothing)$), and that the R's belief is Bayes consistent (i.e., $x_\varnothing = d_{\rho,I}$).

Next, note that the following analogue of Corollary 7 holds in the covert case.

Corollary 8 (of Lemmas 4 and 5). *Every c-equilibrium structure is disclosure-equivalent to an upper censorship.*

Proof. Note that if J is an S-improvement of I, then $d_{\rho,J} < d_{\rho,I}$. Then, the argument is virtually identical to the proof of Corollary 7. That is, if I is a c-equilibrium which is not disclosure-equivalent to its S-improvement upper censorship J, then J_ρ^D is an S-improvement over I_ρ^D. Then, by Lemma 5, the benefit from deviating from I to J is strictly positive, because it would be a strict improvement in the overt case and J has a lower corresponding non-disclosure posterior. □

Unfortunately, Corollary 8 does mean that one may restrict the space of S acquisition strategies to the class of upper censorship because it is has no implications off the equilibrium path. However, it turns out that Lemma 4 is powerful enough to guarantee sufficiency of upper censorship class both on- and off-path.

Corollary 9 (of Lemma 4). *For all $x_\varnothing \in [0, x_0]$, every maximizer of $v_\rho(I|x_\varnothing)$ coincides with an upper-censorship on $[x_\varnothing, 1]$ and the set of maximizers is independent of ρ.*

Proof. For any $x_\varnothing \in [0, x_0]$, notice that the objective $v_\rho(I|x_\varnothing)$ can be rewritten as

$$\begin{aligned}
\operatorname*{argmax}_{I\in\mathcal{I}} v_\rho(I|x_\varnothing) &= \operatorname*{argmax}_{I\in\mathcal{I}} \; [1-\rho+\rho I'(x_\varnothing)]\, G(x_\varnothing) + \rho\int_{x_\varnothing}^1 G\,\mathrm{d}I' - G(x_0) \\
&= \operatorname*{argmax}_{I\in\mathcal{I}} \; \int_0^1 G_{\vee x_\varnothing}\,\mathrm{d}(I' - \underline{I}') \\
&= \operatorname*{argmax}_{I\in\mathcal{I}} \; \int_0^1 (I_{\vee x_\varnothing} - \underline{I})\,\mathrm{d}g
\end{aligned}$$

where we define $J_{\vee x_\varnothing}(x) := \max\{J(x), J(x_\varnothing)\}$ for all $x \in X, J \in \mathcal{I}$ and $\mathcal{I}_{\vee x_\varnothing} := \{J_{\vee x_\varnothing} : J \in \mathcal{I}\}$. It follows immediately that the set of maximizers is independent of ρ. Then, the definition of the function v and the notion of S-improvement can be readily extended to $\mathcal{I}_{\vee x_\varnothing}$. The rest of the argument is very similar to the proof of Corollary 7.

Take any solution I of the above program and consider it's upper censorship S-improvement J as constructed in the proof of Lemma 4. Then, $J - I$ is nonnegative on $[0, \hat\omega]$ and non-positive on $[\hat\omega, 1]$ and so is $J_{\vee x_\varnothing} - I_{\vee x_\varnothing}$. If $J_{\vee x_\varnothing} = I_{\vee x_\varnothing}$, then I and J coincide on $[x_\varnothing, 1]$ and we are done. Otherwise, $J_{\vee x_\varnothing} \neq I_{\vee x_\varnothing}$ and so $J_{\vee x_\varnothing}$ is an S-improvement over $I_{\vee x_\varnothing}$, hence $v(J_{\vee x_\varnothing}) > v(I_{\vee x_\varnothing})$, which contradicts with I solving the program. □

In other words, when S is choosing a test under some fixed non-disclosure belief $x_\varnothing$, she can always guarantee herself a payoff of $G(x_\varnothing)$ by non-disclosing. Hence, best-responding to $x_\varnothing$ is equivalent to solving the full-commitment problem with the ex-post payoff function $G_{\vee x_\varnothing}$ which is equal to G truncated from below at $G(x_\varnothing)$. Therefore, Lemma 4 implies that pivoting the ICDF on $[x_\varnothing, 1]$ constitutes an S-improvement to which only upper censorship are immune to.

Next, the (Covert) problem reduces to finding a pair $(t, x_\varnothing)$ such that $x_\varnothing = d_{\rho, I_t}$ and the t upper censorship I_t is a best response to $x_\varnothing$, i.e. $t \in \Theta$ maximizes $\int_0^1 G_{\vee x_\varnothing}\,\mathrm{d}I_t'$. Figure 8 illustrates the graphs of the best-response and the Bayes-consistency mappings and their intersections for various reliability levels. It is easy to show that the S best response is unique, continuous in $x_\varnothing$, equals t_1^* for $x_\varnothing \leqslant t_1^*$ and strictly increasing otherwise. But then since the Bayes consistency mapping $t \mapsto d_{\rho, I_t}$ is continuous, equal to $d_{\rho, \bar{I}}$ below the diagonal and strictly decreasing above. The intersection of the graphs of the best-response and Bayes-consistency mappings then exists and unique. Moreover, since higher reliability does not change the best response and lowers the Bayes consistent non-disclosure belief, the intersection is constant when $d_{\rho,\bar{I}} \leqslant t_1^*$ and strictly decreasing otherwise. Therefore, one can define $\overline{\rho}^c := \inf\{\rho \in (0,1]\colon d_{\rho,\bar{I}} \leqslant t_1^*\}$ to obtain the result.

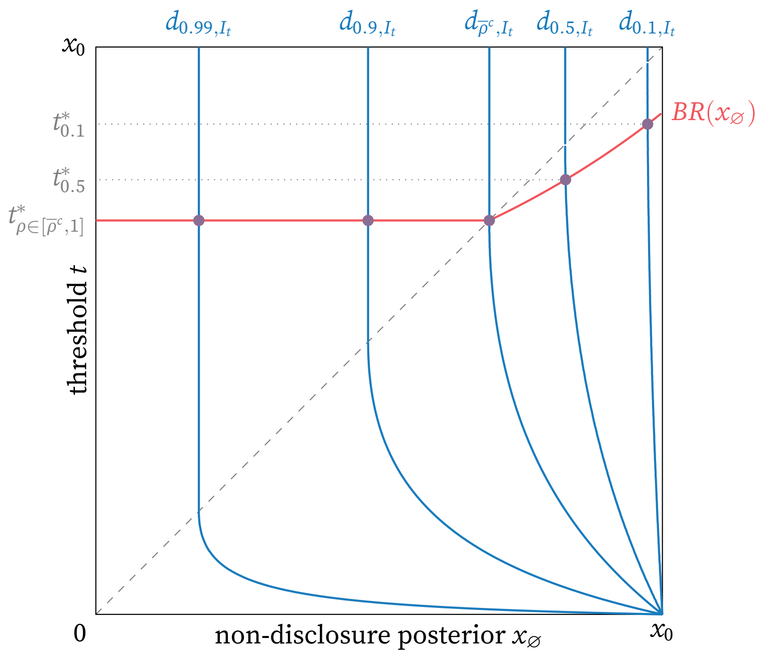


FIGURE 8: C-equilibrium thresholds obtained for various reliability levels as intersections of the best-response and Bayes-consistency mappings for the case of uniform θ and triangularly distributed ω with the peak at $3/5$.

4 Conclusion

This paper studies overt and covert acquisition of hard information subject to an exogenous reliability constraint. Despite the fact that the sender lacks full commitment, we show how tools from information design can be adapted to fully characterize the equilibrium evidence structure without putting parametric restrictions on a rich environment. The main results demonstrate how each of the main forces—information design, voluntary disclosure, and covert/overt nature of acquisition—contribute to the equilibrium structure. When the reliability is high, the three forces do not interact:

the sender acquires essentially the same signal (upper censorship) as under full commitment and the nature of acquisition is irrelevant. When the reliability is low, the equilibrium signal takes a very simple form of a pass/fail test with the threshold jointly determined by the three forces. In particular, the pass/fail threshold is monotone in reliability but whether it is increasing or decreasing depends on whether acquisition is overt or covert.

Our analysis under the assumptions of costless acquisition and exogenous reliability may also shed some light at cases when these assumptions may not hold. First, if acquiring a test comes at some Blackwell-monotone cost, our results suggest that in some cases it may have little impact on the equilibrium structure. In particular, when reliability is low, if a pass/fail test—a very coarse information structure—arises in the costless case, then the sender would have no incentives to choose a more informative and complex structure in the costly acquisition case. Second, suppose the sender could make an investment in reliability. Then our results can be seen as deriving the value of reliability which can then be compared to the cost of investment. Alternatively, suppose the sender could jointly choose among tests with various reliability levels which are all below some technological limit $\rho^{\max}$. Since, in the overt case, it could be easily shown that the sender always strictly benefits from higher reliability and so she would prefer tests with reliability equal to $\rho^{\max}$ which could then be interpreted the exogenous reliability in our model.

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A Appendix

A.1 Equilibrium Definition

Let Π be the set of all tests, i.e., measurable mappings $\Theta \to \Delta M$ endowed with the discrete σ-algebra. For any convex measurable space Y, given a probability measure $\nu \in \Delta Y$, let $\mathbf{E}\nu := \int_Y y \, \mathrm{d}\nu(y) \in Y$ denote the barycenter of ν.

Under both overt and covert acquisition, an equilibrium consists of four objects: an S testing strategy $\pi \in \Pi$, an S's disclosure strategy (in terms of the probability of disclosure) $\delta \colon M \times \Pi \to [0,1]$, a R's belief map $\beta \colon (M \cup \{\varnothing\}) \times \Pi \to \Delta\Theta$, and a R's acceptance strategy (in terms of the probability of acceptance) $\alpha \colon (M \cup \{\varnothing\}) \times \Omega \times \Pi \to [0,1]$. For convenience, for all $\pi \in \Pi$ and $\omega \in \Omega$, denote $\delta_\pi := \delta(\cdot, \pi), \beta_\pi := \beta(\cdot, \pi), \alpha_\pi^\omega := \alpha(\cdot, \omega, \pi), u_R^\omega := u_R(\cdot, \omega)$ and let $\ell_{\pi,\rho,\delta} \colon \Theta \to \Delta(M \cup \{\varnothing\})$ denote the likelihood function of the experiment π with reliability ρ, i.e., for all Borel $M' \subseteq M$, $\ell_{\pi,\rho,\delta}(M'|\theta) := \rho \int_{M'} \delta_\pi \, \mathrm{d}\pi(\cdot|\theta)$.

Now, an overt-acquisition equilibrium, or *o-equilibrium*, is a tuple $(\pi^*, \delta, \alpha, \beta)$ of measurable mappings such that, for all $m \in M, \omega \in \Omega, \pi \in \Pi$,

$$\beta_\pi \text{ is derived from Bayes rule given } \mu_0 \text{ and } \ell_{\pi,\rho,\delta}. \tag{o-Bayes}$$

$$\operatorname{supp} \alpha_\pi^\omega(m) \subset \operatorname*{argmax}_{a \in [0,1]} \int_\Theta u_R^\omega(a, \theta) \, \mathrm{d}\beta_\pi(\theta|m), \tag{R-IC}$$

$$\delta_\pi(m) \in \operatorname*{argmax}_{d \in [0,1]} \int_\Omega (d\alpha_\pi^\omega(m) + (1-d)\alpha_\pi^\omega(\varnothing)) \, \mathrm{d}G(\omega), \tag{S-IC}$$

$$\pi^* \in \operatorname*{argmax}_{\pi' \in \Pi} \int_\Theta \int_{M \cup \{\varnothing\}} \int_\Omega \alpha_{\pi'}^\omega(m) \, \mathrm{d}G(\omega) \, \mathrm{d}\ell_{\pi',\rho,\delta}(m|\theta) \, \mathrm{d}\overline{F}(\theta), \tag{Ex-Ante}$$

The definition of a covert-acquisition equilibrium, or *c-equilibrium*, is equivalent, except condition (o-Bayes) is replaced with

$$\beta_\pi \text{ is derived from Bayes rule given } \mu_0 \text{ and } \begin{cases} \ell_{\pi,\rho,\delta}, & \text{on } M, \\ \ell_{\pi^*,\rho,\delta}, & \text{on } \{\varnothing\}. \end{cases} \tag{c-Bayes}$$

That is, in contrast to the overt case, R's beliefs depend only on the on-path S's choice of π^* in the event of non-disclosure as she then cannot detect S's ex-ante deviations.

A.2 Proofs

A.2.1 Proof of Lemma 2

First, note that equilibrium conditions (R-IC) and (S-IC) both involve maximizing a linear function on $[0, 1]$ and are, therefore, equivalent to

$$\alpha_{\pi}^{\omega}(m) = \mathbf{1}(\mathbf{E}\beta_{\pi}(m) \geqslant \omega) \tag{R-IC$'$}$$

for all $m \in M$ and $\omega \in \Omega$ (except, possibly, for $\mathbf{E}\beta_{\pi}(m) = \omega$), and

$$\delta_{\pi}(m) = \mathbf{1}(\mathbf{E}\beta_{\pi}(m) \geqslant \mathbf{E}\beta_{\pi}(\varnothing)) \tag{S-IC$'$}$$

for all $m \in M$ (except, possibly, when $\mathbf{E}\beta_{\pi}(m) = \mathbf{E}\beta_{\pi}(\varnothing)$).

Then, one can rewrite the (Ex-Ante) objective function as

$$\begin{aligned}
&\int_{\Theta} \int_{M \cup \{\varnothing\}} \int_{\Omega} \alpha_{\pi'}^{\omega}(m) \,\mathrm{d}G(\omega) \,\mathrm{d}\ell_{\pi,\rho,\delta}(m|\theta) \,\mathrm{d}\overline{F}(\theta) \\
&= \int_{\Theta} \int_{M \cup \{\varnothing\}} G(\mathbf{E}\beta_{\pi'}(m)) \,\mathrm{d}\ell_{\pi',\rho,\delta}(m|\theta) \,\mathrm{d}\overline{F}(\theta) \\
&= \rho \int_{\Theta} \int_{M} G(\mathbf{E}\beta_{\pi'}(m)) \,\mathrm{d}\left[\delta_{\pi'}(m)\pi'(m|\theta)\right] \,\mathrm{d}\overline{F}(\theta) \\
&\quad + G(\mathbf{E}\beta_{\pi'}(\varnothing)) \left(1 - \rho + \rho \int_{\Theta} \int_{M} (1 - \delta_{\pi'}(m)) \,\mathrm{d}\pi'(m|\theta) \,\mathrm{d}\overline{F}(\theta)\right) \\
&= \rho \int_{\Theta} \int_{\{m \in M \colon \mathbf{E}\beta_{\pi'}(m) > \mathbf{E}\beta_{\pi'}(\varnothing)\}} G(\mathbf{E}\beta_{\pi'}(m)) \,\mathrm{d}\pi'(m|\theta) \,\mathrm{d}\overline{F}(\theta) \\
&\quad + G(\mathbf{E}\beta_{\pi'}(\varnothing)) \left(1 - \rho + \rho \int_{\Theta} \pi'(\{m \in M \colon \mathbf{E}\beta_{\pi'}(m) \leqslant \mathbf{E}\beta_{\pi'}(\varnothing)\}|\theta) \,\mathrm{d}\overline{F}(\theta)\right).
\end{aligned}$$

By the definition of $F_{\pi'}$, the (Ex-Ante) objective function can be further rewritten as

$$\rho \int_{\mathbf{E}\beta_{\pi'}(\varnothing)}^{1} G(x) \,\mathrm{d}F_{\pi'} + G(\mathbf{E}\beta_{\pi'}(\varnothing))\left[1 - \rho + \rho F_{\pi'}(\mathbf{E}\beta_{\pi'}(\varnothing))\right] = \nu_{\rho}(I_{\pi'}|\mathbf{E}\beta_{\pi'}(\varnothing))$$

To sum up, $(\pi^*, \delta, \alpha, \beta)$ satisfies (R-IC), (S-IC), and (Ex-Ante) if and only if it satisfies (R-IC$'$) and (S-IC$'$), and

$$\pi^* \in \operatorname*{argmax}_{\pi' \in \Pi} \nu_{\rho}(I_{\pi'}|\mathbf{E}\beta_{\pi'}(\varnothing)). \tag{Ex-Ante$'$}$$

Now consider the overt case. Since (o-Bayes) implies $\mathbf{E}\beta_{\pi'}(\varnothing) = d_{\rho,I_{\pi'}}$ for all $\pi' \in \Pi$, if I^* is an o-equilibrium structure then

$$I^* \in \operatorname*{argmax}_{I \in \mathcal{I}} \nu_{\rho}(I|d_{\rho,I}),$$

which is exactly the (Overt) program.

Next, consider the covert case. Note that (c-Bayes) implies, for all $\pi'' \in \Pi$, $\mathbf{E}\beta_{\pi''}(\varnothing) = d_{\rho,I_{\pi^*}}$. Hence, if I^* is a c-equilibrium structure then

$$I^* \in \operatorname*{argmax}_{I \in \mathcal{I}} v_\rho(I|d_{\rho,I^*}),$$

which is exactly the (Covert) program.

Finally, to show the sufficiency of the two programs for the corresponding equilibria, we use $\{I_{\pi'} : \pi' \in \Pi\} = \mathcal{I}$. Given a solution I^* to the (Overt) (respectively, Covert) program, take any $\pi^* \in \Pi$ such that $I_{\pi^*} = I^*$, any β_π satisfying (o-Bayes) (respectively, c-Bayes) and let α and δ be defined as in (R-IC′) and (S-IC′) to construct a profile $(\pi^*, \delta, \alpha, \beta)$ which is an o-equilibrium (c-equilibrium, respectively) by the above arguments. □

A.2.2 Proof of Corollary 4

Endow $\mathcal{I}$ with a topology corresponding (induced under $\mu \mapsto (t \mapsto \int_0^t \mu[0,x]\mathrm{d}x)$) to the weak* topology on $\Delta[0,1]$. Then, the (Overt) program admits a solution since it has a compact domain and a continuous objective.

For the covert case, consider the correspondence

$$\begin{aligned} \Phi\colon \mathcal{I} \times [0,1] &\rightrightarrows \mathcal{I} \times [0,1] \\ (I, x_\varnothing) &\mapsto \operatorname*{argmax}_{I' \in \mathcal{I}} v_\rho(I'|x_\varnothing) \times \{d_{\rho,I}\}. \end{aligned}$$

Note that $x_\varnothing \mapsto \operatorname{argmax}_{I' \in \mathcal{I}} v_\rho(I'|x_\varnothing)$ is non-empty-, convex-, compact-valued and upper-hemicontinuous by Berge's Theorem and $I \mapsto d_{\rho,I}$ is a continuous mapping by Lemma 1. Therefore, Φ is a non-empty-, convex-, compact-valued and upper-hemicontinuous correspondence on a compact and convex domain and, thus, it admits a fixed point by the Kakutani-Glicksberg-Fan theorem. □

A.2.3 Proof of Lemma 3

We first establish the following result which implies both the equivalence between pivoting and S-improvements of Lemma 3 and the equivalence between strict informativeness and (ex-ante) R-improvements.

Lemma 6. *Let $z_1, \dots, z_{2k-1} \in [0,1]$ and $Z_i^+ := [z_{2i}, z_{2i-1}]$, $Z_i^- := [z_{2i-1}, z_{2i}]$ for all $i = 1, \dots, k$, where $z_0 := 0, z_{2k} = 1$. Then, for any $I, J \in \mathcal{I}$ the following statements are equivalent:*

(i) J is weakly above (below) I on each Z_i^+ (Z_i^-) and $J \neq I$,

(ii) $\int_0^1 (J - I)\,\mathrm{d}h > 0$ for all $h\colon [0,1] \to \mathbb{R}$ strictly increasing (decreasing) on each Z_i^+ (Z_i^-).

Proof. First, suppose J is weakly above (below) I on each Z_i^+ (Z_i^-) and let $h\colon [0,1] \to \mathbb{R}$ be strictly increasing (decreasing) on each Z_i^+ (Z_i^-). Then, $\int_0^1 (J-I)\,\mathrm{d}h \geqslant \int_N (J-I)\,\mathrm{d}h$ for any interval $N \subseteq [0,1]$. If $J \neq I$, then there exists $\varepsilon > 0$, $i \in \{0, \dots, 2k\}$ and $0 \leqslant \underline{z} < \overline{z} \leqslant 1$ such that either $J - I > \varepsilon$ on $[\underline{z}, \overline{z}] \subseteq Z_i^+$ or $I - J > \varepsilon$ on $[\underline{z}, \overline{z}] \subseteq Z_i^-$. In both cases, we have $h(\overline{z}) \neq h(\underline{z})$ and hence

$$\int_0^1 (J - I)\,\mathrm{d}h \geqslant \int_{\underline{z}}^{\overline{z}} (J - I)\,\mathrm{d}h \geqslant \varepsilon |h(\overline{z}) - h(\underline{z})| > 0.$$

Second, suppose $\int_0^1 (J-I)\,\mathrm{d}h > 0$ for all $h\colon [0,1] \to \mathbb{R}$ strictly increasing (decreasing) on each Z_i^+ (Z_i^-), which immediately implies $J \neq I$. Next, take any $x \in [0,1]$ and suppose $x \in Z_i^+$ for some i. Define $h^x := \mathbf{1}_{[x,1]}\colon [0,1] \to \mathbb{R}$ and and consider a sequence of ramp functions $h_n^x := [1 - n(x - \mathbf{id})^+]^+ : [0,1] \to \mathbb{R}$. Note that $h_n^x \to h^x$ in the sense of weak convergence and, by assumption, $\int_0^1 (J - I)\,\mathrm{d}h^x > 0$ for all n. Hence,

$$J(x) - I(x) = \int_0^1 (J - I)\,\mathrm{d}h^x = \lim_{n \to \infty} \int_0^1 (J - I)\,\mathrm{d}h_n^x \geqslant 0$$

Finally, the case of $x \in Z_i^-$ for some i is analogous with $h^x := \mathbf{1}_{[0,x)}$ and $h_n^x := [1 - n(\mathbf{id} - x)^+]^+$. □

Proof of Lemma 3. The result follows immediately from Lemma 6 by letting $k = 1, z_1 = \hat{\omega}$. □

Now note that Lemma 6 is related to the following strict version of the Blackwell-Rothschild-Stiglitz Theorem.

Corollary 10. *For all $I, J \in \mathcal{I}$, the following statements are equivalent*

(i) J is strictly more informative than I: $J > I$,

(ii) J is an R-improvement over I: $w(J) - w(I) = \int_0^1 (J-I)\,\mathrm{d}G > 0$ for all strictly increasing G.

Proof. The result follows immediately from Lemma 6 by letting $k = 1, z_1 = 1$. □

A.2.4 Towards the Proofs of Theorems 1 and 2

In this section, we consider the relaxed maximization over upper censorship thresholds as suggested by Corollaries 7 and 8. We establish some properties of its objective function and introduce some notation which will be used in the proofs of the main results.

Properties of upper censorships. Fix any $t \in \Theta$. Let I_t denote the t upper censorship of $\bar{I}$, $F_t := I_t'$ be the corresponding CDF, $x_t := \int_t^1 \theta \,\mathrm{d}\bar{F}(\theta)/(1-\bar{F}(t))$ be the conditional mean of the upper pooling. That is,

$$I_t(x) = \begin{cases} \bar{I}(x), & x \leqslant t, \\ \bar{I}(t) + \bar{F}(t)(x-t), & x > t, \end{cases} \quad F_t(x) = \begin{cases} \bar{F}(x), & x \leqslant t, \\ \bar{F}(t), & x \in (t, x_t), \\ 1, & x \geqslant x_t, \end{cases} \quad x_t = \frac{x_0 + \bar{I}(t) - t\bar{F}(t)}{1-\bar{F}(t)}$$

Given our assumptions on $\bar{I}$, x_t for $t = 0$ defined this way is consistent with our notation x_0 for the prior mean. Moreover, $x_t > t$ for all $t \in [0,1)$ and x_t is strictly increasing in t. In particular, this implies that, by the Intermediate Value Theorem, there exists a unique $\underline{t} \in (0, \hat{\omega})$ such that

$$x_{\underline{t}} = \hat{\omega}.$$

Clearly, $I_t(x)$, $F_t(x)$, and x_t are almost everywhere continuous and differentiable in t. In particular,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} I_t(x) &= \mathbf{1}_{[t,x_t]}(x)\bar{f}(t)(x-t), && \text{for all } (t,x) \in [0,1]^2 \text{ such that } x \neq x_t, \\ \tfrac{\mathrm{d}}{\mathrm{d}t} F_t(x) &= \mathbf{1}_{[t,x_t]}(x)\bar{f}(t), && \text{for all } (t,x) \in [0,1]^2 \text{ such that } x \notin \{t, x_t\}, \\ \tfrac{\mathrm{d}}{\mathrm{d}t} x_t &= \frac{(x_t - t)\bar{f}(t)}{1-\bar{F}(t)}, && \text{for all } t \in [0,1). \end{aligned}$$

Next, fix any $\rho \in (0,1]$. Then, the Bayes-consistent non-disclosure posterior equals

$$d_{\rho,I_t} = \begin{cases} \dfrac{\rho[\bar{F}(t)t - \bar{I}(t)] + (1-\rho)x_0}{1-\rho(1-\bar{F}(t))}, & \text{if } t \leqslant d_{\rho,\bar{I}}, \\ d_{\rho,\bar{I}}, & \text{if } t > d_{\rho,\bar{I}}. \end{cases}$$

and is differentiable in t and ρ everywhere except at $(t,\rho)=(0,1)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}d_{\rho,I_t} = -\frac{\rho\bar{f}(t)\left[d_{\rho,\bar{I}}-t\right]^+}{1-\rho(1-\overline{F}(t))} \leqslant 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\rho}d_{\rho,I_t} = -\frac{I_t(d_{\rho,I_t})+x_0-d_{\rho,I_t}}{1-\rho(1-\overline{F}(t)\wedge\overline{F}(d_{\rho,I_t}))} < 0,$$

and satisfies

$$t > d_{\rho,\bar{I}} \iff t > d_{\rho,I_t},$$
$$t < d_{\rho,\bar{I}} \iff t < d_{\rho,I_t}.$$

For convenience, we extend this mapping by continuity: $d_{0,I_t} := \lim_{\rho\to 0} d_{0,I_t} = x_0$.

Properties of the relaxed (FC) objective with the truncated G. We begin with the following lemma which will be key in both overt and covert cases. For any $t \in \Theta$, let $\underline{v}_{x_\varnothing}(t)$ denote the expected full-commitment S payoff from choosing ICDF I_t of R's posterior means given optimal disclosure and some fixed R's non-disclosure posterior $x_\varnothing \in [0,x_0]$, that is define

$$\underline{v}\colon \Theta\times[0,x_0]\to\mathbb{R},$$
$$(t,x_\varnothing)\mapsto \underline{v}_{x_\varnothing}(t) := \int_0^1 G_{\vee x_\varnothing}\,\mathrm{d}F_t - G(x_0)$$

where $G_{\vee x_\varnothing}(x) := \max\{G(x),G(x_\varnothing)\}$ for all $x\in X$. Note that $\underline{v}_{x_\varnothing}$ is exactly what S is maximizing when she is best-responding to R's non-disclosure belief in the covert case. In addition, for the overt case, we will use the fact that the function $\underline{v}_0(t) = v(I_t)$ coincides with the (relaxed) full-commitment objective.

Lemma 7. *The function $\underline{v}$ has the following properties:*

(Cont) $\underline{v}_{x_\varnothing}$ *is continuous for all* $x_\varnothing\in[0,x_0]$,

(Incr) $\underline{v}_{x_\varnothing}$ *is strictly increasing on* $[0,\underline{t}]$ *for all* $x_\varnothing\in[0,x_0]$,

(SQC) $\underline{v}_{x_\varnothing}$ *is strictly quasiconcave with the peak in* $(0,\hat{\omega})$ *for all* $x_\varnothing\in[0,x_0]$,

(ZMD) $\underline{v}$ *has zero marginal differences on* $\{(t,x_\varnothing)\in[0,1]\times[0,x_0]\colon x_\varnothing\leqslant t\}$,

(SIMD) $\underline{v}$ *has strictly increasing marginal differences on* $\{(t,x_\varnothing)\in(0,1)\times(0,x_0)\colon x_\varnothing > t\}$.

Proof. First, (Cont) holds since $\overline{F}, G, t\mapsto x_t$ are continuous and we have

$$\underline{v}_{x_\varnothing}(t) = \int_0^t G_{\vee x_\varnothing}\bar{f} + (1-\overline{F}(t))G(x_t) - G(x_0).$$

Second, (ZMD) and (SIMD) both follow from the observation

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}x_\varnothing}\underline{v}_{x_\varnothing}(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}x_\varnothing}\int_0^1 G_{\vee x_\varnothing}\,\mathrm{d}F_t \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}x_\varnothing}\left[F_t(x_\varnothing)G(x_\varnothing)+\int_{x_\varnothing}^1 G\,\mathrm{d}F_t\right] \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}x_\varnothing}\left[1-\int_{x_\varnothing}^1 F_t\,\mathrm{d}G\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[F_t(x_\varnothing)g(x_\varnothing)\right] \\
&= \mathbf{1}_{[t,x_t]}(x_\varnothing)\bar{f}(t)g(x_\varnothing) \\
&= \mathbf{1}\{0<t<x_\varnothing<1\}.
\end{aligned}$$

Third, we show (Incr) and (SQC). Denote the tangent lines to G and $G_{\vee x_\varnothing}$ at t as

$$\begin{aligned}
G^T(s|t) &:= G(t)+g(t)(s-t), \\
G^T_{\vee x_\varnothing}(s|t) &:= G_{\vee x_\varnothing}(t)+\mathbf{1}_{[x_\varnothing,1]}(t)g(t)(s-t).
\end{aligned}$$

for all $s\in[0,1]$. Fix any $x_\varnothing\in[0,x_0]$ and note that strict convexity (concavity) of G below (above) $\hat{\omega}$ implies

$$\begin{aligned}
&\text{for all } t,s\in[0,\hat{\omega}],\, t\neq s\colon && G(s)>G^T(s|t), \\
&\text{for all } t,s\in[\hat{\omega},1],\, t\neq s\colon && G(s)<G^T(s|t),
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\text{for all } t,s\in[0,\hat{\omega}],\, s\neq t && G_{\vee x_\varnothing}(s)\geqslant G^T_{\vee x_\varnothing}(s|t), && \text{(wConvexity)} \\
&\text{for all } t,s\in[0,\hat{\omega}],\, s\neq t,\, s\vee t\geqslant x_\varnothing && G_{\vee x_\varnothing}(s)> G^T_{\vee x_\varnothing}(s|t), && \text{(sConvexity)} \\
&\text{for all } t,s\in[\hat{\omega},1],\, s\neq t\colon && G_{\vee x_\varnothing}(s)< G^T_{\vee x_\varnothing}(s|t). && \text{(sConcavity)}
\end{aligned}$$

By definition of $\underline{v}_{x_\varnothing}$, we have for all $t\neq x_\varnothing$

$$\begin{aligned}
\underline{v}_{x_\varnothing}(t) &= \int_0^1 G_{\vee x_\varnothing}\,\mathrm{d}F_t - G(x_0) = \int_0^t G_{\vee x_\varnothing}\,\mathrm{d}\bar{F}+(1-\bar{F}(t))G(x_t)-G(x_0), \\
\underline{v}'_{x_\varnothing}(t) &= \bar{f}(t)\left[G_{\vee x_\varnothing}(t)-G(x_t)-g(x_t)(t-x_t)\right] = \bar{f}(t)\left[G_{\vee x_\varnothing}(t)-G^T(t|x_t)\right].
\end{aligned}$$

Thus, to show (Incr), it is sufficient to show that $G_{\vee x_\varnothing}(t)>G^T(t|x_t)$ for all $t\in[0,\underline{t}]$. But $t\in[0,\underline{t}]$ implies $0\leqslant t<x_t\leqslant\hat{\omega}$, and, hence, the desired inequality holds for all $t\in[0,\underline{t}]$ by (sConvexity).

Similarly, to prove (SQC), it is sufficient to show that for all $0 < t_1 < t_2 < 1$,

$$G_{\vee x_\varnothing}(t_2) \geqslant G^T_{\vee x_\varnothing}(t_2|y_2) \implies G_{\vee x_\varnothing}(t_1) > G^T_{\vee x_\varnothing}(t_1|y_1), \tag{3}$$

$$G_{\vee x_\varnothing}(t_1) \leqslant G^T_{\vee x_\varnothing}(t_1|y_1) \implies G_{\vee x_\varnothing}(t_2) < G^T_{\vee x_\varnothing}(t_2|y_2), \tag{4}$$

where $y_1 := x_{t_1} < x_{t_2} =: y_2$.

To prove (3) and (4), consider the following exhaustive cases:

1. Suppose $y_1 \leqslant \hat{\omega}$. Then, $t_1, y_1 \in (0, \hat{\omega}]$ and $t_1 \neq y_1 > x_0 \geqslant x_\varnothing$. Hence, both (3) and (4) hold since (sConvexity) implies the conlusion of (3) always holds and the premise of (4) never holds.
2. Suppose $t_2 \geqslant \hat{\omega}$. Then, $t_2, y_2 \in [\hat{\omega}, 1)$ and $y_2 > t_2$. Hence, both (3) and (4) hold since (sConcavity) implies the conlusion of (4) always holds and the premise of (3) never holds.
3. Suppose $t_2 < \hat{\omega} < y_1$. Then, $0 < t_1 < t_2 < \hat{\omega}$ and $x_\varnothing \leqslant x_0 \leqslant \hat{\omega} < y_1 < y_2 \leqslant 1$. First, to prove (3), suppose $G_{\vee x_\varnothing}(t_2) \geqslant G^T_{\vee x_\varnothing}(t_2|y_2)$. Note that

$$g(y_1) > g(y_2) > \frac{G(\hat{\omega}) - G_{\vee x_\varnothing}(t_2)}{\hat{\omega} - t_2} > g(t_2), \tag{5}$$

where the first inequality follows monotonicity of g on $[\hat{\omega}, 1]$, the third — from (sConvexity) for $t = t_2, s = \hat{\omega}$ and the second — from the premise of (3) and (sConcavity) for $t = y_2, s = \hat{\omega}$ as $G_{\vee x_\varnothing}(t_2) - g(y_2)(t_2 - y_2) \geqslant G(y_2) > G(\hat{\omega}) - g(y_2)(\hat{\omega} - y_2)$. Now, the conclusion of (3) follows from

$$\begin{aligned}
G_{\vee x_\varnothing}(t_1) &\geqslant g(t_2)(t_1 - t_2) + G_{\vee x_\varnothing}(t_2) && \text{by (wConvexity) for } t = t_2, s = t_1 \\
&\geqslant g(t_2)(t_1 - t_2) + G(y_2) + g(y_2)(t_2 - y_2) && \text{by the premise of (3)} \\
&> g(y_2)(t_1 - t_2) + G(y_2) + g(y_2)(t_2 - y_2) && \text{by (5)} \\
&= g(y_2)(t_1 - y_2) + G(y_2) \\
&> g(y_2)(t_1 - y_2) + G(y_1) - g(y_2)(y_1 - y_2) && \text{by (sConcavity) for } t = y_2, s = y_1 \\
&= g(y_2)(t_1 - y_1) + G(y_1) \\
&> g(y_1)(t_1 - y_1) + G(y_1). && \text{by (5)}
\end{aligned}$$

Finally, to prove (4), suppose $G_{\vee x_\varnothing}(t_1) \leqslant G^T_{\vee x_\varnothing}(t_1|y_1)$ and note that

$$\frac{G(\hat{\omega}) - G_{\vee x_\varnothing}(t_1)}{\hat{\omega} - t_1} > g(y_1) > g(y_2), \tag{6}$$

where the second inequality follows from monotonicity of g on $[\hat{\omega}, 1]$ and the first — from the premise of (4) and (sConcavity) for $t = y_1, s = \hat{\omega}$ as $G_{\vee x_\varnothing}(t_1) - g(y_1)(t_1 - y_1) \geqslant G(y_1) > G(\hat{\omega}) - g(y_1)(\hat{\omega} - y_1)$. Thus, the conclusion of (4) follows from

$$\begin{aligned}
& g(y_2)(t_2 - y_2) + G(y_2) \\
& > g(y_2)(t_2 - y_2) + G(y_1) - g(y_2)(y_1 - y_2) && \text{by (sConcavity) for } t = y_2, s = y_1 \\
& = g(y_2)(t_2 - y_1) + G(y_1) \\
& > g(y_1)(t_2 - y_1) + G(y_1) && \text{by (6)} \\
& > g(y_1)(t_2 - y_1) + G(\hat{\omega}) - g(y_1)(\hat{\omega} - y_1) && \text{by (sConcavity) for } t = y_1, s = \hat{\omega} \\
& = g(y_1)(t_2 - \hat{\omega}) + G(\hat{\omega}) \\
& > \frac{G(\hat{\omega}) - G_{\vee x_\varnothing}(t_1)}{\hat{\omega} - t_1}(t_2 - \hat{\omega}) + G(\hat{\omega}) && \text{by (6)} \\
& > G_{\vee x_\varnothing}(t_2). && \text{chordal slopes} \uparrow\text{: } \tfrac{G(\hat{\omega}) - G_{\vee x_\varnothing}(t_2)}{\hat{\omega} - t_2} > \tfrac{G(\hat{\omega}) - G_{\vee x_\varnothing}(t_1)}{\hat{\omega} - t_1}
\end{aligned}$$

Finally, the peak is positive by (Incr) and strictly below $\hat{\omega}$ since $\underline{v}'_{x_\varnothing}(s) = G_{\vee x_\varnothing}(t) < G(x_t) - g(x_t)(x_t - s)$ for all $s \geqslant \hat{\omega}$ by (sConcavity). □

Properties of the relaxed (Overt) objective. Now consider the optimization problem

$$\max_{t \in \Theta} \tilde{v}_\rho(t), \tag{Overt$'$}$$

where

$$\begin{aligned}
\tilde{v}\colon \Theta \times (0,1] &\to \mathbb{R}, \\
(t, \rho) &\mapsto \tilde{v}_\rho(t) := \frac{v_\rho(I_t | d_{\rho, I_t})}{\rho}.
\end{aligned}$$

Lemma 8. *The function $\tilde{v}$ has the following properties:*

(CD) $\tilde{v}_\rho$ *is continuous and (Lebesgue-a.e.) differentiable for all* $\rho \in (0, 1]$,

(Incr) $\tilde{v}_\rho$ *is strictly increasing on* $[0, \underline{t}]$ *for some* $\underline{t} \in (0, \hat{\omega})$ *and all* $\rho \in (0, 1]$,

(SQC) $\tilde{v}_1$ *is strictly quasiconcave with the peak in* $(0, \hat{\omega})$,

(Diff) $\tilde{v}_1 - \tilde{v}_\rho$ *is strictly increasing on* $[0, d_{\rho, \bar{I}}]$, *and constant on* $[d_{\rho, \bar{I}}, 1]$.

(ID) $\tilde{v}$ *has increasing differences,*

(SIMD) $\tilde{v}$ *has strictly increasing marginal differences on* $(0, d_{\rho', \bar{I}}) \times (0, \rho']$ *for all* $\rho' \in (0, 1]$,

Proof. First, (CD) follows from continuity and differentiability of $I_t(x)$ and d_{ρ,I_t} in t.

Second, (Incr) and (SQC) follows directly from (Incr) and (SQC) of Lemma 7 and the fact that $\underline{\nu}_0 = \tilde{\nu}_1$.

Third, we establish (Diff). We have

$$\begin{aligned}
\tilde{\nu}_\rho(t) &= \frac{\nu_\rho(I_t|d_{\rho,I_t})}{\rho} \\
&= \frac{\nu\left([I_t]^D_\rho\right)}{\rho} \\
&= \frac{1}{\rho}\int_0^1 \left(\left[\rho I_t(x) + (1-\rho)(x-x_0)\right]^+ - \underline{I}(x)\right) \mathrm{d}g(x) \\
&= \int_{d_{\rho,I_t}}^1 \left[I_t(x) + \frac{(1-\rho)}{\rho}(x-x_0) - \frac{\underline{I}(x)}{\rho}\right] \mathrm{d}g(x).
\end{aligned}$$

and hence by Lemma 1

$$\begin{aligned}
\tilde{\nu}_1(t) - \tilde{\nu}_\rho(t) &= \int_0^1 (I_t - \underline{I})\,\mathrm{d}g - \int_{d_{\rho,I_t}}^1 \left[I_t(x) + \frac{(1-\rho)}{\rho}(x-x_0) - \frac{\underline{I}(x)}{\rho}\right] \mathrm{d}g(x) \\
&= \int_0^{d_{\rho,I_t}} I_t\,\mathrm{d}g + \int_{d_{\rho,I_t}}^{x_0} \frac{(1-\rho)}{\rho}(x_0-x)\,\mathrm{d}g(x) \\
&= \int_0^{x_0} I_t(x) \vee \frac{(1-\rho)}{\rho}(x_0-x)\,\mathrm{d}g(x).
\end{aligned}$$

Thus, $\tilde{\nu}_1 - \tilde{\nu}_\rho$ is strictly increasing on $[0, d_{\rho,\bar{I}}]$ and constant on $[d_{\rho,\bar{I}}, 1]$ because so is $t \mapsto I_t(x) \vee \frac{(1-\rho)}{\rho}(x_0 - x) = I_t(x)$ for all $x \in [t, d_{\rho,I_t}]$ (since $d_{\rho,I_t} > d_{\rho,\bar{I}} > t$ if $t < d_{\rho,\bar{I}}$ and $d_{\rho,I_t} = d_{\rho,\bar{I}}$ otherwise).

Finally, notice that (ID) and (SIMD) are equivalent to $\tilde{\nu}_1 - \tilde{\nu}_\rho$ having increasing differences everywhere and strictly increasing marginal differences on $(0, d_{\tilde{\rho},\bar{I}}) \times (0, \tilde{\rho}]$ for all $\tilde{\rho} \in (0,1]$. To prove this, consider

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\rho\,\mathrm{d}t}\left[\tilde{\nu}_\rho(t) - \tilde{\nu}_1(t)\right] &= -\frac{\mathrm{d}^2}{\mathrm{d}\rho\,\mathrm{d}t}\left[\int_0^{d_{\rho,I_t}} I_t\,\mathrm{d}g + \int_{d_{\rho,I_t}}^{x_0} \frac{(1-\rho)}{\rho}(x_0-x)\,\mathrm{d}g(x)\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\int_t^{d_{\rho,I_t}} \bar{f}(t)(x-t)\,\mathrm{d}g(x) + \frac{\mathrm{d}d_{\rho,I_t}}{\mathrm{d}t}\left(I_t(d_{\rho,I_t}) - \frac{(1-\rho)}{\rho}(x_0 - d_{\rho,I_t})\right)\right] \\
&= -\frac{\mathrm{d}d_{\rho,I_t}}{\mathrm{d}\rho}\bar{f}(t)(d_{\rho,I_t} - t)^+ g'(d_{\rho,I_t}),
\end{aligned}$$

where $I_t(d_{\rho,I_t}) - \frac{(1-\rho)}{\rho}(x_0 - d_{\rho,I_t}) = 0$ by the definition of d_{ρ,I_t}. Therefore, $\tilde{\nu}_\rho$ has the desired properties because, for all $\tilde{\rho} \in (0,1]$, $t \in (0,1)$, the terms $g'(d_{\tilde{\rho},I_t})$, $-\frac{\mathrm{d}d_{\tilde{\rho},I_t}}{\mathrm{d}\tilde{\rho}}$ and $d_{\tilde{\rho},I_t} - t$ are (strictly) positive (for $d_{\tilde{\rho},\bar{I}} > t$). $\square$

A.3 Proof of Theorem 1

Denote the solution correspondence of the relaxed (Overt′) program as

$$T^o\colon (0,1] \to [0,1]$$
$$\rho \mapsto \operatorname*{argmax}_{t\in[0,1]} \tilde{v}_\rho(t),$$

and note that $T^o_\rho := T^o(\rho) = \operatorname{argmax}_{[0,1]} \tilde{v}_\rho = \operatorname{argmax}_{[0,1]} \rho\tilde{v}_\rho = \operatorname{argmax}_{t\in[0,1]} v_\rho(I_t|d_{\rho,I_t})$. Hence, by Lemma 2 and Corollary 7, I is an o-equilibrium evidence structure if and only if I is disclosure equivalent to $I_{t^o_\rho}$ for some $t^o_\rho \in T^o_\rho$.

It is then sufficient to show that there exists $\overline{\rho}^o \in [0,1]$ such that

(i) T^o is non-empty-valued, compact-valued, and upper hemicontinuous,

(ii) T^o is increasing in the strong set order,

(iii) $T^o_\rho = \{t^*_1\}$ for all $\rho > \overline{\rho}^o$, where $\{t^*_1\} := \operatorname{argmax}_{[0,1]} \tilde{v}_1$,

(iv) $0 < t^o_\rho < d_{\rho,\bar{I}}$ for all $t^o_\rho \in T^o_\rho \setminus \{t^*_1\}$,

(v) $\overline{\rho}^o < 1$,

(vi) every selection from T^o is strictly increasing on $(0, \overline{\rho}^o]$.

Now we show that all these properties follow from the properties of $\tilde{v}$ shown in Lemma 8. First, (CD) implies (i) by Berge's Maximum Theorem. Second, (ID) implies (ii) by the Weak Monotone Comparative Statics Theorem (Topkis, 1978, Theorem 6.1). Third, since (SQC) implies $T^o_1 = \{t^*_1\}$, we define $\overline{\rho}^o := \inf\{\rho \in (0,1]\colon T^o_\rho = \{t^*_1\}\} \in [0,1]$ so that (iii) automatically holds.

Fourth, (iv) holds because (Incr) implies $\tilde{v}_\rho$ is strictly increasing below $\underline{t} > 0$ and (SQC) and (Diff) imply $\operatorname{argmax}_{[d_{\overline{\rho}^o,\bar{I}},1]} \tilde{v}_\rho = \{t^*_1\}$. Moreover, the same properties imply

$$T^o_\rho \cap [d_{\rho,\bar{I}},1] = \operatorname*{argmax}_{[0,1]} \tilde{v}_\rho \cap [d_{\rho,\bar{I}},1] \subseteq \operatorname*{argmax}_{[d_{\rho,\bar{I}},1]} \tilde{v}_\rho = \operatorname*{argmax}_{[d_{\rho,\bar{I}},1]} \tilde{v}_1 = \{t^*_1\}$$

and so for ρ close enough to 1, we have $t^*_1 \in [d_{\rho,\bar{I}},1]$ (since $d_{\rho,\bar{I}} \underset{\rho\to 1}{\to} d_{1,\bar{I}} = \min\operatorname{supp}\bar{I} = 0$) which implies (v) by upper hemicontinuity of T^o.

Finally, we prove (vi) by contradiction.[29] Suppose some selection from T^o is not strictly increasing on $(0,\overline{\rho}^o]$. Since T^o is increasing in the strong set order, this means

[29] Although the logic here is very similar to Edlin and Shannon (1998), their Strict Monotonicity Theorem 1 is not directly applicable here due to the fact that the strictly increasing marginal differences property (SIMD) holds on a contracting domain $[0, d_{\rho,\bar{I}}]$.

there exist $0 < \rho_1 < \rho_2 \leqslant \overline{\rho}^o$, $t \in T^o_{\rho_1} \cap T^o_{\rho_2}$. Since $t \in (0,1)$, we have $\tilde{v}'_{\rho_1}(t) = \tilde{v}'_{\rho_2}(t) = 0$. If $t \neq t_1^*$, then $t < d_{\rho,\bar{I}}$ by (iv) and so we get a contradiction with the implication of (SIMD)

$$0 = \tilde{v}'_{\rho_2}(t) - \tilde{v}'_{\rho_1}(t) = \int_{\rho_1}^{\rho_2} \frac{\mathrm{d}^2}{\mathrm{d}\rho\,\mathrm{d}t}\tilde{v}_\rho(t)\mathrm{d}\rho > 0.$$

If $t = t_1^*$, then by definition of $\overline{\rho}^o$, there exists $s \in T^o_{\rho_1} \setminus \{t\}$ such that $s < d_{\rho_2,\bar{I}} \leqslant t$ and so by (SIMD) we get

$$\tilde{v}_{\rho_1}(s) - \tilde{v}_{\rho_1}(t) \geqslant \tilde{v}_{\rho_2}(s) - \tilde{v}_{\rho_2}(t) + \int_{\rho_1}^{\rho_2}\int_s^t \frac{\mathrm{d}^2}{\mathrm{d}\rho\,\mathrm{d}t}\tilde{v}_\rho(t)\,\mathrm{d}t\,\mathrm{d}\rho > 0,$$

which is a contradiction with $t \in T^o_{\rho_1}$. □

A.3.1 Proof of Theorem 2

By Lemma 2 and Corollary 9, I is a c-equilibrium evidence structure if and only if I is disclosure equivalent to I_{t^*} such that

$$t^* \in \operatorname*{argmax}_{t\in[0,1]} v_\rho(I_t | d_{\rho,I_{t^*}}) \tag{Covert$'$}$$

The proof of the theorem follows directly from the following two claims establishing the properties of the (Covert$'$) fixed-point program.

Claim 1. The S best response correspondence

$$\begin{aligned} BR\colon\, [0, x_0] &\to [0,1] \\ x_\varnothing &\mapsto \operatorname*{argmax}_{t\in[0,1]} \underline{v}_{x_\varnothing}(t). \end{aligned}$$

has the following properties:

(i) BR is a singleton-valued and, thus, can be treated as a function,

(ii) BR is continuous,

(iii) $BR(x_\varnothing) = t_1^*$ for all $x_\varnothing \in [0, t_1^* \wedge x_0]$,

(iv) $BR(x_\varnothing) \in [t_1^*, t_1^* \vee x_\varnothing]$ for all $x_\varnothing \in [t_1^*, x_0]$,

(v) BR is strictly increasing on $[t_1^*, x_0]$ if $t_1^* < x_0$.

Proof. First, by Berge's Maximum Theorem, we have $|BR(x_\varnothing)| \geqslant 1$ and upper hemicontinuity of BR. Second, we have $|BR(x_\varnothing)| \leqslant 1$ due to strict quasiconcavity of $\underline{v}_{x_\varnothing}$. Thus, we have (i) and (ii).

Third, since by (ZMD) and (SIMD), we have $\underline{v}_{x_\varnothing} - \underline{v}_0$ strictly increasing on $[0, x_\varnothing]$ and constant on $[x_\varnothing, 1]$. Recall that $\underline{v}_0 = \tilde{v}_1$ is strictly quasiconcave with the peak $t_1^* \in (0, \hat{\omega})$. Therefore, $\underline{v}_{x_\varnothing} = (\underline{v}_{x_\varnothing} - \underline{v}_0) + \tilde{v}_1$ is strictly increasing on $[0, t_1^*]$ and strictly decreasing on $[t_1^* \vee x_\varnothing, 1]$, which immediately implies (iii) and (iv). In addition, this implies that maximizers are always interior, that is,

$$BR(x_\varnothing) \in [t_1^*, t_1^* \vee x_\varnothing] \subset (0, \hat{\omega} \vee x_0) \subseteq (0, 1) \implies \underline{v}'_{x_\varnothing}(BR(x_\varnothing)) = 0 \text{ for all } x_\varnothing \in [0, x_0].$$

Fourth, to show (v), suppose, by contradiction, there exist $t_1^* \leqslant x_1 < x_2 \leqslant x_0$ such that $BR(x_1) \geqslant BR(x_2)$. Note that BR is weakly increasing by the Weak Monotone Comparative Statics Theorem (Topkis, 1978) because (ZMD) and (SIMD) imply increasing differences. Thus, $BR(x_1) = BR(x_2) = t \in (t_1^*, x_1)$ and $\underline{v}'_{x_1}(t) = \underline{v}'_{x_2}(t) = 0$. But then $\int_{x_1}^{x_2} \frac{\mathrm{d}^2}{\mathrm{d}x_\varnothing\, \mathrm{d}t} \underline{v}_{x_\varnothing}(t) = \underline{v}'_{x_2}(t) - \underline{v}'_{x_1}(t) = 0$ which contradicts (SIMD). □

Claim 2. There exists $\overline{\rho}^c \in [0, 1]$ such that the fixed-point correspondence

$$\begin{aligned} T^c \colon (0,1] &\to [0,1] \\ \rho &\mapsto T^c_\rho := \{t \in [0,1] \colon t \in BR(d_{\rho, I_t})\} \end{aligned}$$

has the following properties:

(a) T^c is a singleton-valued and, thus, can be treated as a function,
(b) T^c is continuous,
(c) $T^c_\rho = t_1^*$ for all $\rho \in [\overline{\rho}^c, 1]$,
(d) $T^c_\rho < d_{\rho, \bar{I}}$ for all $\rho \in [\overline{\rho}^c, 1]$,
(e) T^c is strictly increasing on $(0, \overline{\rho}^c)$,
(f) $\overline{\rho}^c \in [0, \overline{\rho}^o]$.

Proof. Define $\overline{\rho}^c := \inf\{\rho \in (0,1] \colon d_{\rho,\bar{I}} \leqslant t_1^*\} \leqslant \overline{\rho}^o$. For any $\rho \in (0, 1]$, define a function

$$\begin{aligned} \tilde{d}_\rho \colon [0,1] &\to [0, x_0) \\ t &\mapsto d_{\rho, I_t}. \end{aligned}$$

First, fix any $\rho \geqslant \overline{\rho}^c$ so that $d_{\rho,\bar{I}} \leqslant t_1^*$. Then, Claim 1 implies $T^c_\rho = \{t_1^*\}$ because

$$\begin{aligned} \{BR\left(d_{\rho,I_t}\right) \colon t \in [0, t_1^*]\} &= BR\left([d_{\rho,\bar{I}}, x_0]\right) \subseteq [t_1^*, 1], \\ \{BR\left(d_{\rho,I_t}\right) \colon t \in (t_1^*, 1]\} &= BR\left(t_1^* \wedge x_0\right) = \{t_1^*\}. \end{aligned}$$

Second, fix any $0 < \rho < \overline{\rho}^c$ so that $x_0 > d_{\rho,I_t} \geqslant d_{\rho,\bar{I}} > t_1^*$ for all $t \in [0,1]$. Then, we have $T_\rho^c \subseteq [t_1^*, x_0]$ since

$$\{BR\,(d_{\rho,I_t}) \,:\, t \in [0,1]\} = BR\left([d_{\rho,\bar{I}}, x_0]\right) \subseteq BR\,([t_1^*, x_0]) \subseteq [t_1^*, x_0].$$

Now let $\bar{t} = BR(x_0) > t_1^*$ and note T_ρ^c is the set of the roots of the function

$$\begin{aligned} \Delta_\rho \colon [t_1^*, \bar{t}] &\to \mathbb{R} \\ t &\mapsto BR^{-1}(t) - d_{\rho,I_t} \end{aligned}$$

is continuous, strictly increasing, and has $\Delta_\rho(t_1^*) = t_1^* - d_{\rho,I_{t_1^*}} < 0$ and $\Delta_\rho(\bar{t}) = x_0 - d_{\rho,\bar{I}} > 0$. Hence, by the Intermediate Value Theorem, there exists a unique root of Δ_ρ continuous in ρ. Moreover, since $\frac{\mathrm{d}}{\mathrm{d}\rho}\Delta_\rho = -\frac{\mathrm{d}}{\mathrm{d}\rho} d_{\rho,I_t} > 0$ for all $t \in [0,1], \rho \in (0,1]$, the root is strictly decreasing in ρ, which completes the proof of the claim. □

□