Sponsored Link Auctions with Consumer Search *

Mingshi Kang†

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Abstract

I study a sponsored link auction game in which consumers search one set of products (*block*) before the other, and sellers compete in bids to place their product links to the first block. Consumers are assumed to be unaware of the products in the second block when searching in the first block, search in each block optimally according to Weitzman (1979) and update the current best option during the search. I characterize consumers' shopping outcomes with the block-by-block search behavior. Letting sellers choose product prices and auction bids together, I characterize the equilibrium of the complete information second price auction with two payment schemes: fixed payment and per-transaction payment. I find auction revenue and consumer surplus are larger under the fixed payment, and seller profits are larger under the per-transaction payment because the latter distorts the winner's pricing strategy. In the case that a social planner runs the platform, I find a consumer-optimal product positioning rule if sellers commit to prices before the position is allocated.

JEL Classification: D11, D44, D83, M37

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†Department of Economics, University of Western Ontario, London, ON, Canada, mkang94@uwo.ca.

1 Introduction

Amazon and eBay place "sponsored links" at the top of the webpages where customers can spot them immediately. Top positions on the product search results provide an instant visibility boost to customers. Sellers who own the sponsored link positions are determined by auctions.[1] Sponsored link auctions are important and fast-growing channels for online shopping platforms to collect revenues. Amazon reports auction revenues of 14.08 billion dollars in 2019 and 21.48 billion dollars in 2020.[2] Unlike auctions selling concrete items, sponsored link auctions sell advertisement positions. The value of a sponsored link position thus depends on how it increases the auction winner's profit by affecting consumers' search outcomes.

This paper contributes to the literature by introducing two novel elements in the sponsored link auctions. First, I assume consumers have partial information on product values, observe product prices during the search, and use the information to search optimally. Second, sellers in my model choose both product prices and auction bids optimally to maximize their profits. The questions are: How do sellers' pricing strategies and bidding strategies interact in such an environment. And how the interaction affects the surplus split between consumers, sellers, and the shopping platform. My main finding is that the answers depend on the payment rule of the auction. If the auction winner's payment is fixed in a lump-sum manner, sellers' pricing and bidding strategies are independent. However, if the auction winner pays to the platform every time a product is sold, product prices increase under the equilibrium. Thus, a shift from the fixed payment to the per-transaction payment decreases consumer surplus and increases the sellers' profits. Moreover, I find that sponsored link auction revenue decreases from such a change of the payment rules.

I consider an oligopoly market where the value of a product is separated into two parts: one is known prior search, the other needs to be discovered from search.[3] To be specific, a consumer's payoff from purchasing a product i is $v_i + z_i - p_i$, where v_i (prior value) and p_i

[1] The auction rules used by Amazon and eBay are different. Amazon uses second price auctions, and the payments are made based on the number of clicks of product links. eBay uses first price auctions, and the payments are made based on the number of transactions. Criteria about the quality of the product exist to be placed on a sponsored link.

[2] See Amazon's latest sale report, page 19. Sponsored link auction revenue is the last item in "Net sales".

[3] The definition of search good, and how it distinguishes from experienced good is well-documented by Nelson (1970) [16].

(product price) are known before the search but a search cost s_i has to be paid in order to learn z_i (match value).[4] The payoff setting fits into the online shopping context reasonably well, as the shopping procedure can be described as follows. First, a consumer chooses a shopping platform and type the keywords to describe the category of the product to buy. Second, based on the input keywords, the platform produces a webpage of multiple product links, with the corresponding product prices (p_i) and brief product descriptions (v_i). Third, the consumer sequentially clicks a set of links based on the prior information (v_i, p_i and belief on z_i), spends time (pays search cost s_i) to collect detailed information (learns z_i) on the product description page. If the product does not fit the consumer's preference, the consumer continues searching until a decision (purchase or quit) is made.[5]

Weitzman's (1979) [19] solution fully characterizes the optimal search rule in such an environment, with the optimal search order and the stopping rule. However, because a consumer is assumed to be aware of the existence of **all** products when the search starts, the consumer search is not affected by the product position. Awareness of all products initially thus makes product demands irrelevant to product positions on a webpage. Such an assumption is realistic in small markets, in which consumers view all the products immediately after they input the keywords. However, the search result of a particular keyword on Amazon usually consists of hundreds of links. Consumers need to read the search results in a top-to-down manner, scroll down, and flip pages to know the products' existence at lower positions.[6] With limited memory and the belief that links are sorted from high to low quality, most consumers only glance at the first few pages before stopping.

To capture the online search behavior and justify why sellers pay for sponsored link positions in the search framework, I introduce the concept of *block*, which is a set of adjacent products in the keyword search result. For example, the first block is the first page, or the set of products that a consumer can see immediately (top of the first page), which may consist of the sponsored links and the "best sellers." The second block consists of the products that

[4]This payoff is also used in Armstrong and Zhou (2011) [1], Haan et al. (2017) [12] and Choi, Dai and Kim (2018) [7].

[5]Quit the search is equivalent to taking an outside option, which can be purchasing from another platform or purchasing in-store.

[6]Joachims et al. [13] provide experimental evidence supporting the block-by-block search by monitoring browser users' eyeballs movement when they glance over google search results online. They find users "first scan the viewable results quite thoroughly before resorting to scrolling".

a consumer needs to scroll down or flip the webpage to see. Blocks are mutually exclusive, such that one product belongs to one block. I assume all consumers search block-by-block, apply Weitzman's optimal search rule within each block and update the current best option when switching blocks. For instance, when searching the second block, consumers use the value of the best product searched in the first block as the new outside option, conditional on it being better than the original outside option.

Intuitively, block-by-block search distorts both search order and stopping rule compared to the case without blocks (Weitzman). The search order is distorted since consumers are unaware of any product in the second block at the beginning of the search, even if some products in the second block have a high priority to search. The stopping rule is distorted since consumers update the best option according to the search outcome in the first block (a higher outside option value) and thus search fewer products in the second block. Thus, the block-by-block search behavior increases the demand for products in the first block, decreases the demand for products in the second block, and creates incentives for sellers to win the sponsored link position in the first block.

I focus on the environment with two blocks and one sponsored link position in the first block for tractability. Despite the limited number of blocks, the setup captures the critical features of sellers' pricing and bidding competition with consumer search. Having only two blocks is not an unreasonable abstraction from reality. First, to guarantee good users' experiences, Amazon and eBay impose a set of prerequisites, regarding product quality, on the participants of the sponsored link auction. Since the online platform ranks products from high quality to low quality in general, the sellers competing for the sponsored link position are likely from higher positions. Second, most consumers only search among the first few pages of a selling website.[7] Thus, focusing on the first two blocks does not affect the underlying implications the paper intends to study.

Characterization of demand functions relies on the eventual purchase theorem by Choi, Dai and Kim [7] (henceforth CDK), who summarize the conditions of purchase outcomes in Weitzman under a discrete choice formulation. I adapt CDK's eventual purchase theorem to the context of block-by-block search (Theorem 1). With the revised eventual purchase theorem, I characterize the product demands under any product position. By assuming

[7]Using detailed online browsing and transaction data of the book market, Delossantos et al. (2012) [9] find it is likely that consumers choose a fixed and small sample size to search when entering the market.

log-concave densities of product values, the demand functions are log-supermodular, and pricing equilibrium is uniquely pinned down by first-order conditions (Theorem 2). Hence, the equilibrium profit of any seller is uniquely determined under any auction outcome. Sellers then bid for the sponsored link position to increase their equilibrium profits.

I study second price auctions with two payment schemes: fixed payment and per-transaction payment. Under fixed payment, the winner pays the second-highest bid once in a lump-sum manner. Under per-transaction payment, the winner pays the second-highest bid whenever a product is sold. Since sellers can update their bids frequently and experiment with them, the sponsored link auction is treated as a complete information game. Under both payment schemes, the unique pure strategy pricing equilibrium exists given the winner's identity and the corresponding bid payment (Theorem 2 applies). This is because the bid payment is independent of the winner's pricing strategy under the fixed payment, and can be regarded as the additional marginal cost paid by the winner to produce a product under per-transaction payment. Conditional on all sellers set price optimally, I characterize the pure strategy Nash equilibrium of the bidding game, under which no seller finds it profitable to unilaterally revise the bid to change the position (Theorem 3 and Theorem 4).

In the comparative statics analysis with symmetric sellers, I find the fixed payment leads to a higher auction revenue and consumer surplus, while the per-transaction payment provides more seller profits (Theorem 5). The reason is that the per-transaction payment distorts the winner's optimal pricing rule and increases the prices of all products. Since a seller's profit increases if a competitor sets a higher price, and a product's equilibrium price increases in its marginal cost, non-winners in the per-transaction payment auction have incentives to increase their bids to increase the winner's marginal cost.

The paper also provides the consumer optimal positioning of products if sellers commit to price before the position is allocated (Theorem 6). Intuitively speaking, the platform needs to help consumers find a high payoff product with a low search cost. It turns out that a low search cost can be achieved by allocating products with low match value uncertainties into the first block. Thus, allocating all products with high expected values into a higher position may fail to be optimal, because search friction prevent consumers to purchase the best product in general. Surprisingly, low match value uncertainty can be more important: Under an extreme situation where there is no uncertainty in the match values of all products in block 1, consumers need not pay search costs to learn the block-1 match values. Thus, consumer surplus equals that without blocks, which is the highest attainable surplus in a block-by-block search environment.

Compiani et al. (2021) [8] find a consumer optimal positioning, called "diamonds in rough," in a different setup. Instead of assuming consumers search block-by-block, they assume that the reservation value, which determines the optimal search order, has an additive separable term decreasing in product position (larger for higher position). They find to maximize consumer surplus, platforms need to put products whose utility indexes exceed search indexes into higher positions. Their search environment is more general as the utility index and the search index are independent. In my model, subject to Weitzman's search framework, the counterparts of the utility index and search index are correlated through the prior and match values. Moreover, their model does not consider search cost explicitly. So the results are not directly comparable.

A mass body of literature studies sponsored link auctions under different setups, but most papers are under the context that the auction is run by online search engines (Google and Yahoo!). Some of the differences between my paper and past papers are caused by the different auction context, especially that product prices are not observable on Google's search results. So I emphasize that the paper is not arguing that the sponsored link auction model with flexible search behavior is a superior one. Rather, I think it provides a different perspective to analyze the auction when consumers have more or less prior information. Varian (2007) [20], and Edelman, Ostrovsky, and Schwarz (2007) [11] study the generalized second price auction, in which the winner of the k's position pays the $k-1$th highest bid. Unlike how I model block-by-block positions, positions in their paper are strictly ranked, and a higher position always have a higher number of clicks. Both papers assume a seller's payoff from a click depends only on the seller's identity but not on the seller's position, and the number of clicks received by a seller depends only on the seller's position but not on the seller's identity. And they characterize a local condition for the Nash equilibrium and find truth-telling is not an equilibrium in the generalized second price auction. Börgers et al. (2013) [4] extend the generalized second price auction by allowing both the value of a click and the expected number of clicks to be seller and position specific. They find a multiplicity of Nash equilibria exist in general. By explicitly modeling consumer search behavior and letting sellers compete in prices, I capture the "allocative externality" that is not considered by the previous literature. Allocative externality exists when a seller's payoff depends not only on the position but also on other sellers' identities. For example, a seller gets more clicks if other sellers on the same webpage are not well known than if the sellers around are well known.

Chen and He (2006) [5] and Athey and Ellison (2011) [2] incorporate search behavior into sponsored link auction games in a different context. They assume the value of any link is draw from the same distribution, and consumers have no prior information on the value before clicking a link. Search is driven by consumers' belief in the links' qualities, which are the probabilities of satisfying a consumer's need, independent of the links' realized values. With no prior information on product values and prices, consumers always search for the first product that satisfies their needs. Identical value distribution implies that consumers have the same expected payoff from any link and that all sellers set the same equilibrium price and receive the same profit from a transaction. So the interaction between price and bid studied in this paper is not valid in their setup. By allowing heterogeneous consumer preferences and sellers' strategic pricing, the paper analyzes consumer surplus from a different perspective and studies how price and bid interact in the sponsored link auction.

The rest of the paper is organized as follows. Section 2 introduces the model and consumer's optimal search rule. Section 3 characterizes product demands in two blocks. Section 4 establishes the existence and uniqueness of pricing strategy under any position outcome. The existence of pure strategy Nash equilibrium of the bidding game is established in Section 5. Section 6 analyzes comparative statics that compares different bid payment schemes. Section 7 provides the consumer optimal positioning of products. The last section concludes.

2 The Model

A market consists of n sellers indexed by $i \in N := \{1, \dots, n\}$. Each seller i can produce a product at a marginal cost c_i and zero fixed cost. All products are in the same category and are horizontally differentiated. Each seller produces one type of product, so the index i also refers to seller i's product.

To sell their products, sellers need to put product links on an online shopping platform. A platform allocates all sellers into two *blocks*, indexed by $k \in \{1, 2\}$. Let $N_k \subset N$ be the set of sellers in block k, with $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. I call (N_1, N_2) the *position* of sellers. Sellers in the same block share the same position and the size of the two blocks are fixed, such that $|N_1| = n_1$ and $|N_2| = n_2$.

The platform sells one position in N_1 through a second price auction. Before the auction, the platform determines the *status quo* position of sellers, (N_1^0, N_2^0), and a positioning function $l : N_2^0 \to N_1^0$. Since the size of the two blocks are fixed, the positioning function l decides which seller in N_1^0 to be moved to block 2 if $i \in N_2^0$ wins. If seller $i \in N_2^0$ wins the auction, the position is $(N_1, N_2) = (N_1^0 + i - l(i), N_2^0 - i + l(i))$.[8] If seller $i \in N_1^0$ wins, the position stays the same as the *status quo*, i.e., $(N_1, N_2) = (N_1^0, N_2^0)$. Without loss of generality, let $N_1^0 = \{1, \dots, n_1\}$ and $N_2^0 = \{n_1 + 1, \dots, n\}$, so the index i is generated such that the first n_1 sellers are in N_1^0 and the rests are in N_2^0.

position	seller
N_1^0	1
	2
N_2^0	3
	4

position	seller
N_1	1
	3
N_2	2
	4

Table 1: status quo position (left) and the position when seller 3 wins the auction with $l(3) = 2$ (right).

After positions are determined by the auction, all sellers simultaneously announce their prices. Define the price of product i as p_i. If $i \in N_1^0$ wins the auction, the price vector of products in block 1 is $\boldsymbol{p}_1 := (p_1, \dots, p_{n_1})$ and that in block 2 is $\boldsymbol{p}_2 := (p_{n_1+1}, \dots, p_n)$. Else, if $i \in N_2^0$ wins the auction, the price vectors are $\boldsymbol{p}_1 := \left(p_1, \dots, p_{l(i)-1}, p_i, p_{l(i)+1}, \dots, p_{n_1}\right)$ and $\boldsymbol{p}_2 := \left(p_{n_1+1}, \dots, p_{i-1}, p_{l(i)}, p_{i+1}, \dots, p_n\right)$.

There is a unit mass of consumers with unit demands. A consumer's random utility from consuming product i is $V_i + Z_i$, where V_i is the prior value for product i, and Z_i is the additional value the consumer can discover from visiting product i's selling page. For each consumer, the realization of V_i and Z_i, denoted as v_i and z_i, are drawn from distributions $F_i(\cdot)$ and $G_i(\cdot)$ independently, with continuously differentiable densities $f_i(\cdot)$ and $g_i(\cdot)$, and supports $[\underline{v}_i, \overline{v}_i]$ and $[\underline{z}_i, \overline{z}_i]$. For all i, $F_i(\cdot)$ and $G_i(\cdot)$ are common knowledge among consumers and sellers.[9] I define $\boldsymbol{v}_k$ and $\boldsymbol{z}_k$ as the vectors of prior value and match

[8] I use $+$ and $-$ as set operators: $(N_1^0 + i - l(i), N_2^0 - i + l(i)) = (N_1^0 \cup \{i\} \setminus \{l(i)\}, N_2^0 \setminus \{i\} \cup l(i))$.

[9] A single consumer does not make decisions based on F_i, since v_i is realized from the consumer's point

value in block $k \in \{1,2\}$ in the same way as how $\boldsymbol{p}_k$ is defined.

Consumers are assumed to search in the follow way:

1. A consumer starts the search and sees all products in block 1. The consumer knows $\boldsymbol{v}_1$ and $\boldsymbol{p}_1$ immediately.

2. The consumer can visit any product $i \in N_1$'s selling page and learn z_i at a search cost s_i on the first visit. Based on the realized z_i, the consumer either ends the search in block 1, or continues searching in block 1. Recall a product is costless after the first visit.

3. If the consumer finds no more product in block 1 worth visiting, he/she sees all the products in block 2 and knows $\boldsymbol{v}_2$ and $\boldsymbol{p}_2$ immediately. The consumer repeats the search process until either he/she makes a purchasing decision or decides to quit the search.

The underlying assumption of the search behavior is that consumers are unaware of products in block 2 when search in block 1. Searching block 1 before block 2 is non-strategic, as block 1 is always shown to consumers before block 2.[10]

Let $\widetilde{N}$ be the set of products whose links are clicked by a consumer before the purchase decision is made. The consumer who eventually purchases product i gets a payoff:

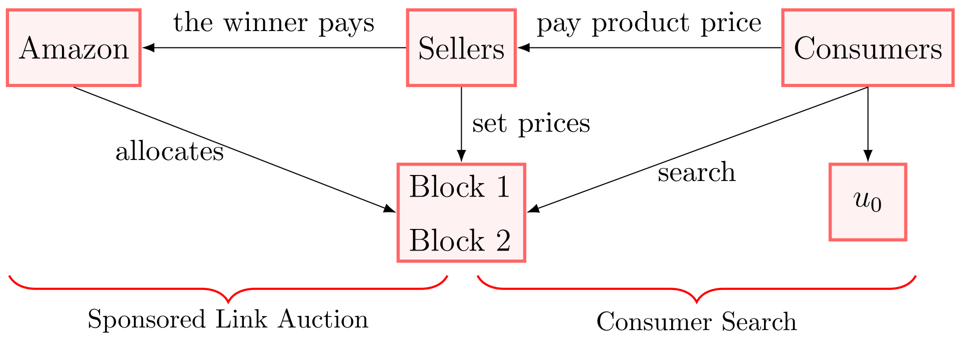
$$U(v_i, z_i, p_i, \widetilde{N}) = v_i + z_i - p_i - \sum_{j \in \widetilde{N}} s_j.$$

If a consumer chooses to purchase nothing from the platform after search, the payoff is $u_0 - \sum_{j \in \widetilde{N}} s_j$, where u_0 is the outside option value and is assumed to be identical to all consumers. Consumers search the two blocks to maximize their expected payoffs, under the restriction that they can search block 2 only if they finish searching in block 1.

The diagram below demonstrate how the platform, sellers and consumers interact in the model.

of view. Thus, assuming F_i is unknown to consumers does not change any result in this paper.

[10] If consumers strategically choose which block to search first based on the prior information, sellers in block 1 are restricted to set their price low enough to make consumers optimal to search block 1.



Weitzman (1979) [19] characterizes the optimal search rule under the environment where consumers are aware of all products initially (without blocks). When two blocks exist, the optimal search rule in each block remains the same as that in Weitzman. The key difference is that when searching in the second block, consumers update the outside option according to the search outcome from the first block. The following lemma states the optimal search strategy.

Lemma 1 *For any consumer, given* $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$*, the optimal search strategy is as follows: for each* i*, let* z_i^* *be determined by:*

$$s_i = \int_{z_i^*}^{\overline{z}_i} (1 - G_i(z_i)) dz_i. \tag{1}$$

- *Order: In each block* $k \in \{1, 2\}$*, the consumer clicks the products in the descending order of* $v_i + z_i^* - p_i$*.*

- *Stopping: At any point during the search, let* $\widetilde{N}_k \subseteq N_k$ *be the set of visited products in block* k*.*[11] *If* $k = 1$*, the consumer stops searching block 1 and switches to block 2 if and only if*

$$\max\left\{u_0, \max_{i \in \widetilde{N}_1}\{v_i + z_i - p_i\}\right\} > \max_{j \in N_1 \setminus \widetilde{N}_1} \{v_j + z_j^* - p_j\}.$$

  If $k = 2$*, the consumer stops searching and takes the current best option if and only if*

$$\max\left\{u_0, \max_{i \in \widetilde{N}_1 \cup \widetilde{N}_2} \{v_i + z_i - p_i\}\right\} > \max_{j \in N_2 \setminus \widetilde{N}_2} \{v_j + z_j^* - p_j\}.$$

Lemma 1 shows consumer's optimal search rule depends on product price and two types of value: *true value*, $v_i + z_i$, and *reservation value*, $v_i + z_i^*$. I define the corresponding *net value* as the value minus price, e.g., the *net true value* of product i is $v_i + z_i - p_i$. The

[11] Since the consumer always search block 1 before searching block 2, $\widetilde{N}_1$ is fixed when searching block 2.

ordering rule in Lemma 1 says, for any consumer with realized value $(\boldsymbol{v}_k, \boldsymbol{z}_k)_{k=1}^2$ and price $(\boldsymbol{p}_k)_{k=1}^2$, the product with a higher net reservation value in each block has a higher priority to search, as it is more likely to have a higher net true value. Given block k is under search, the stopping rule says that a consumer stops searching block k if the value of the current best option is larger than the *net reservation values* of all products not visited in block k. Since $s_i = \mathbb{E}[\max\{Z_i - z_i^*, 0\}]$ (by Eq. (1) and integration by parts), the stopping rule can be intuitively explained as that the expected additional value discovered from one more visit is less than the search cost. For instance, if the current best option has a value u, and i is the next product to visit according to the order rule, the expected gain from an additional visit is

$$\mathbb{E}[\max\{v_i + Z_i - p_i - u, 0\}] - \underbrace{\mathbb{E}[\max\{Z_i - z_i^*, 0\}]}_{=:s_i},$$

which is strictly negative if $u > v_i + z_i^* - p_i$ and positive otherwise.

Given that all consumers search optimally, the mass of consumers who purchase product i is $D_i^k(\boldsymbol{p}_1, \boldsymbol{p}_2)$, which depends on the product position and product prices. The platform runs a second price auction to sell the sponsored link position (details in Section 5). Sellers simultaneously decide product prices and auction bids to maximize their expected profits net from auction payments.

Though consumer search behavior is more flexible comparing to past sponsored link auction literature, it is worth emphasizing the imposed assumptions and limitations.

First, all consumers search the first block before the second. This assumption is relatively intuitive and is supported by eye-tracking experiments in Joachims et al. [13].

Second, search costs are constant across all consumers and consumers have to pay the search cost before purchasing. When shopping online, some consumers may have target items before the search. Namely, they may learn product characteristics offline (from friends' recommendations or past experiences), and do not need to read the product description and pay the searching cost before the purchase. A recent paper by Chen et al. [6] studies this scenario in which they call it "blind buying". This paper aims to build a tractable model to study the interaction between consumer search, sellers' pricing and bidding in the sponsored link auction.

Lastly, I assume that one seller produces only one type of good. In reality, if a seller produce different products and owns more than one link on the platform, "collusions" in price and bid are likely to happen. The seller can set price and bid to maximize the joint profits from all her products. Studying colluding behavior is out of the scope of the paper. I leave the extensions of blind buying and collusion for future research.

3 Demand Characterization

Given $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, CDK define the *effective value* of product i as $w_i := v_i + \min\{z_i, z_i^*\}$, which is the minimum between i's true value and reservation value. CDK find the eventual purchase condition without blocks: consumers always purchase the product with the highest net effective value $w_i - p_i$ among all $i \in N$, conditional on the net effective value is larger than the outside option value u_0. I adapt CDK's Theorem 1 into the context where consumers search block 1 first and update the outside option value according to the search outcome from block 1 when searching block 2.

Theorem 1 *For a consumer with* $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$*, let* $i^* = \arg\max_{i \in N_1}\{w_i - p_i\}$ *and* $j^* = \arg\max_{i \in N_2}\{w_i - p_i\}$.[12]

- *The consumer purchases product* $i \in N_1$ *if and only if* $i = i^*$*,* $w_i - p_i > u_0$*, and* $v_i + z_i - p_i > w_{j^*} - p_{j^*}$*.*
- *The consumer purchases product* $i \in N_2$ *if and only if* $i = j^*$ *and* $w_i - p_i > u_1$*, where* $u_1 := u_0$ *if* $w_{i^*} - p_{i^*} < u_0$ *and* $u_1 := v_{i^*} + z_{i^*} - p_{i^*}$ *if* $w_{i^*} - p_{i^*} \geq u_0$*.*

Proof Since the search rule in each block follows Weitzman, CDK's eventual purchase theorem applies in each block. Consumers always search block 1 before block 2 and update the current best option when switching blocks, so they compare the net realized value of i^* and the effective value of j^* to make the final purchase decision. ■

Demand for product i is the probability that the purchase conditions in Theorem 1 are satisfied. Theorem 1 says the product with the highest effective value in each block is the *candidate product* to purchase in that block. The consumer compares the outside option value, the **net true value** of the candidate product in block 1, and the **net effective value** of the candidate product in block 2 to make the purchase decision. The asymmetry between

[12] Since both f_i and g_i are continuous, the event $w_i - p_i = w_j - p_j$ for $i \neq j$ happens with zero probability.

how consumers make the purchase decision in block 1 (using realized value) and block 2 (using effective value) is the underlying force making a position in block 1 valuable.

Since consumers select the candidate product in each block based on the *net effective value*, it is useful to set up the distribution of the effective value of product i, $W_i = V_i + \min\{Z_i, z_i^*\}$, to characterize the demands. Denote the distribution of W_i as $H_i(\cdot)$:

$$H_i(w_i) := \int_{\underline{z}_i}^{\overline{z}_i} F_i(w_i - \min\{z_i, z_i^*\}) dG_i(z_i). \tag{2}$$

Notations Before proceeding, I introduce some notations to formulate demand functions. Define $\Gamma_i^k := \max_{j \in N_k \setminus \{i\}}\{W_j - p_j\}$ be the consumer's (random) net effective value of the best alternative to product i in block k, and define $\Gamma_0^k := \max_{j \in N_k}\{W_j - p_j\}$ be the (random) net effective value of the candidate product in block k. The table below summarizes notations used to characterize demand:

random variable	distribution	realization	support
$X_i := \max\{u_0, \Gamma_i^1\}$	$\widetilde{H}_i^1(\cdot)$	x_i	$[\underline{x}_i, \overline{x}_i]$
$Y_i := \max\{u_0, \Gamma_i^2\}$	$\widetilde{H}_i^2(\cdot)$	y_i	$[\underline{y}_i, \overline{y}_i]$
$X_0 := \Gamma_0^1$	$H_*^1(\cdot)$	x_0	$[\underline{x}_0, \overline{x}_0]$
$Y_0 := \Gamma_0^2$	$H_*^2(\cdot)$	y_0	$[\underline{y}_0, \overline{y}_0]$

3.1 Demand in Block 1

When product i belongs to block 1 ($i \in N_1$), for any $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, x_i is the largest effective value in block 1 besides i. By Theorem 1, if $x_i \geq y_0$, a consumer never purchases from block 2, since according to the optimal search rule, the consumer cannot find any product in block 2 better than the candidate product in block 1. Thus, the sufficient condition for the consumer to purchase product i is $w_i - p_i > x_i$, so i is the candidate product in block 1. However, if $y_0 > x_i$, $w_i - p_i > x_i$ alone is no longer sufficient to guarantee the purchase of i, since the consumer may find a product better than i in block 2 if the net realized value of i is not large enough (i.e., $v_i + z_i - p_i < y_0$). The next lemma states a simple condition under which product $i \in N_1$ is purchased.

Lemma 2 *Given* $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$*, the consumer purchases product* i *in block 1 if and and only if*

$$v_i + \min\{z_i, z_i^* + (y_0 - x_i)^+\} - p_i > \max\{x_i, y_0\}, \tag{3}$$

where $(y_0 - x_i)^+ := \max\{y_0 - x_i, 0\}$.

Proof When $x_i \geq y_0$, Ineq. (3) is $w_i - p_i > x_i$, which guarantees the purchase of i. It thus suffices to show when $y_0 > x_i$, i is purchased if and only if

$$v_i + \min\{z_i, z_i^* + y_0 - x_i\} - p_i > y_0. \tag{4}$$

Sufficiency: By Theorem 1, when $y_0 > x_i$, the consumer purchases i if and only if:

1) $w_i - p_i > x_i$, so i is the candidate product in block 1.

2) $v_i + z_i - p_i > y_0$, so the consumer cannot find a product better than i in block 2.

Let's call the two conditions condition 1 and condition 2 respectively.

- Suppose $z_i \leq z_i^* + y_0 - x_i$. By (4), condition 2 is satisfied.
  - If $z_i \geq z_i^*$, $w_i - p_i = v_i + z_i^* - p_i \geq \underbrace{v_i + z_i - p_i - y_0}_{>0 \text{ by } (4)} + x_i > x_i$. So condition 1 is satisfied.
  - If $z_i < z_i^*$, Ineq. (4) implies $w_i - p_i = v_i + z_i - p_0 > y_0 > x_i$. So condition 1 is satisfied.
- Suppose $z_i > z_i^* + y_0 - x_i$. This combined with $y_0 > x_i$ implies $z_i > z_i^*$. Ineq. (4) implies $w_i - p_i = v_i + z_i^* - p_i > x_i$. So condition 1 is satisfied. When $z_i > z_i^* + y_0 - x_i$, $v_i + z_i - p_i > v_i + (z_i^* + y_0 - x_i) - p_i > y_0$, where the second inequality follows from (4). So condition 2 is satisfied.

Necessity: Since $y_0 > x_i$, if $v_i + \min\{z_i, z_i^* + y_0 - x_i\} - p_i < y_0$, either $z_i \leq z_i^* + y_0 - x_i$, so $v_i + z_i - p_i < y_0$ (condition 2 is violated), or $z_i > z_i^* + y_0 - x_i$ (thus, $z_i > z_i^*$ by $y_0 > x_i$), so $w_i - p_i = v_i + z_i^* - p_i < x_i$ (condition 1 is violated). That is, if (4) is violated, the consumer does not purchase product i in block 1. ∎

Given Lemma 2, for any $q \geq 0$, it is useful to define the the distribution of $W_i(q) := V_i + \min\{Z_i, z_i^* + q\}$, which is the *distorted effective value* of product i stems from the block-by-block search behavior.

Remark 1 *One can regard the change of effective value to the distorted effective value of product* i *as if the reservation value of* i *increases from* $v_i + z_i^*$ *to* $v_i + z_i^* + q$*. When there is* *no block, a consumer searches in the descending order of the net reservation values among all products. So an increase of i's reservation value means the consumer searches i earlier if product prices keep the same. Hence, the distorted effective value reflects that search order is distorted to make products in block 1 searched earlier.*

The distribution of $W_i(q)$ can be expressed as:

$$\widehat{H}_i(w_i, q) := \int_{\underline{z}_i}^{\overline{z}_i} F_i(w_i - \min\{z_i, z_i^* + q\}) dG_i(z_i). \tag{5}$$

Lemma 3 below states an obvious stochastic dominance relation of $\widehat{H}_i(\cdot, q)$.

Lemma 3 *For any $q' \geq q \geq 0$, $\widehat{H}_i(\cdot, q')$ first order stochastically dominates (FOSD) $\widehat{H}_i(\cdot, q)$.*

Proof For any realized (v_i, z_i) and any $q' \geq q \geq 0$, $w_i(q') \geq w_i(q)$ by definition. The the difference between $\widehat{H}(\cdot, q')$ and $\widehat{H}(\cdot, q)$ is

$$\widehat{H}_i(w_i, q') - \widehat{H}_i(w_i, q) = \int_{\underline{z}_i}^{\overline{z}_i} \left[F_i(w_i - \min\{z_i, z_i^* + q'\}) - F_i(w_i - \min\{z_i, z_i^* + q\})\right] dG_i(z_i),$$

which is non-positive as F_i is a distribution and is increasing. ■

Let $\boldsymbol{p} := (\boldsymbol{p}_1, \boldsymbol{p}_2)$ be the price vector of all products. Given x_i and y_i, Ineq. (3) holds with probability $1 - \widehat{H}_i\left(\max\{x_i, y_0\} + p_i, (y_0 - x_i)^+\right)$. Thus, the demand for product i in block 1 is:

$$D_i^1(\boldsymbol{p}) \;:=\; \int_{\underline{y}_0}^{\overline{y}_0} \int_{\underline{x}_i}^{\overline{x}_i} \left(1 - \widehat{H}_i\left(\max\{x_i, y_0\} + p_i, (y_0 - x_i)^+\right)\right) d\widetilde{H}_i^1(x_i) dH_*^2(y_0). \tag{6}$$

To compare how the block-by-block search behavior affects product demand, denote $\mathscr{D}_i(\boldsymbol{p})$ as the demand function without blocks, such that consumers are aware of all products when search starts. CDK's Eventual Purchase Theorem says consumers always purchase the item with highest net effective value. Thus,

$$\mathscr{D}_i(\boldsymbol{p}) \;:=\; \int_{\underline{y}_0}^{\overline{y}_0} \int_{\underline{x}_i}^{\overline{x}_i} \left(1 - H_i\left(\max\{x_i, y_0\} + p_i\right)\right) d\widetilde{H}_i^1(x_i) dH_*^2(y_0), \tag{7}$$

which is the probability of $w_i - p_i > \max\{x_i, y_0\}$.

Taking the difference between (6) and (7), a position in block 1 increases product i's demand by:

$$\int_{\underline{y}_0}^{\overline{y}_0} \int_{\underline{x}_i}^{\overline{x}_i} \left(H_i\left(\max\{x_i, y_0\} + p_i\right) - \widehat{H}_i\left(\max\{x_i, y_0\} + p_i, (y_0 - x_i)^+\right)\right) d\widetilde{H}_i^1(x_i) dH_*^2(y_0),$$

which, by Lemma 3 and that $H_i(\cdot) = \widehat{H}_i(\cdot, 0)$, is non-negative and is strictly positive if the event $Y_0 - X_i > 0$ occurs with a strictly positive probability.

3.2 Demand in Block 2

In contrast to how demand in block 1 is characterized, such that net *distorted effective value* matters, the demand in block 2 involves net *true value* of the candidate product in block 1, which is treated as the updated outside option value when consumers search in block 2.

For any $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, I use the notations in Theorem 1 and let $i^* = \arg\max_{i \in N_1}\{w_i - p_i\}$. The consumer finds a *candidate product* in block 1 if $w_{i^*} - p_{i^*} \geq u_0$. For any consumer searching in block 2, the updated outside option value is u_1, which equals u_0 if the consumer does not find a candidate product in block 1 and equals $v_{i^*} + z_{i^*} - p_{i^*}$ otherwise. Theorem 1 says that product i in block 2 is purchased if and only if $w_i - p_i > \max\{u_1, y_i\}$. I denote U_1 as the random variable of u_1, since the updated outside option value is random from block-2 sellers' perspective. To derive demand for products in block 2, it is essential to characterize the distribution of U_1, denoted as $J(\cdot)$.

The probability of that $i \in N_1$ is the candidate product in N_1 and i's net true value is less than u is $\Pr(\{W_i - p_i > X_i\} \cap \{V_i + Z_i - p_i < u\})$, which is equivalent to

$$\Pr\left(W_i - p_i > X_i\right) - \Pr\left(\{W_i - p_i > X_i\} \cap \{V_i + Z_i - p_1 > u\}\right).$$

Given x_i, the former probability is $1 - H_i(x_i + p_i)$ and the latter is $1 - \widehat{H}_i(u + p_1, u - x_i)$, conditional on $u > x_i$.[13] As $u_1 = u_0$ if and only if a consumer does not find a *candidate product* in N_1, we must have $J(u_0) = H_*^1(u_0)$. Thus, the probability of $u_0 < U_1 < u$ can be formulated as:

$$J(u) - H_*^1(u_0) \quad := \quad \sum_{i \in N_1} \int_{u_0}^{u} \left[\widehat{H}_i(u + p_i, u - x_i) - H_i(x_i + p_i)\right] d\widetilde{H}_i^1(x_i). \tag{8}$$

The sum over $i \in N_1$ is taken since every product in block 1 can be selected as the candidate product. Notice that while $\widehat{H}(x, q) \leq H(x)$ for any x and $q \geq 0$, $\widehat{H}(u + p, u - x) \geq H(u + p)$ for any $u \geq x$, so the integrand in (8) is positive.

By Theorem 1, the demand for a product $i \in N_2$ is:

$$D_i^2(\boldsymbol{p}) \quad := \quad \int_{\underline{y}_i}^{\overline{y}_i} \int_{u_0}^{\overline{u}_1} \left(1 - H_i(\max\{u_1, y_i\} + p_i)\right) dJ(u_1) d\widetilde{H}_i^2(y_i), \tag{9}$$

which is the probability that $W_i - p_i$ is larger than $\max\{U_1, Y_i\}$. By definition, given any $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p})_{k=1}^2$, if $w_{i^*} - p_{i^*} \geq u_0$, $u_1 := v_{i^*} + z_{i^*} - p_{i^*} \geq v_{i^*} + \min\{z_{i^*}, z_{i^*}^*\} - p_{i^*} =: x_0$; else,

[13] By the definition of $\widehat{H}$ and Lemma 2, $\Pr\left(\{W_i - p_i > x_i\} \cap \{V_i + Z_i - p_1 > u\}\right) = 1 - \widehat{H}_i(u + p_1, u - x_i)$. The two conditions are exactly the conditions to purchase $i \in N_1$ if $y_0 = u$.

$u_1 = u_0 = x_0$. So $J(\cdot)$ FOSDs $H^1_*(\cdot)$. Notice that for any $u > u_0$, $H^1_*(u) - H^1_*(u_0)$ can be written as (shown in Appendix A):

$$H^1_*(u) - H^1_*(u_0) = \sum_{i \in N_1} \left[\int_{u_0}^{u} H_i(u + p_i) - H_i(x_i + p_i) \right] d\widetilde{H}^1_i(x_i). \tag{10}$$

Comparing (8) and (10), for any $u > u_0$, the difference between H^1_* and J is

$$\begin{aligned} \mathcal{K}(u) &:= H^1_*(u) - J(u), \\ &= \sum_{i \in N_1} \int_{u_0}^{u} \left[H_i(u + p_i) - \widehat{H}_i(u + p_i, u - x_i) \right] d\widetilde{H}^1_i(x_i), \end{aligned} \tag{11}$$

for any $u \geq u_0$. $\mathcal{K}$ has an intuitive meaning. Given i as the candidate product in N_1, $\Pr(\{X_i < W_i - p_i < u\} \cap \{V_i + Z_i - p_i > u\})$ is the probability of $X_0 < u$ and $U_1 > u$. And $\mathcal{K}$ is the probability of that event, since the integrand in (11) is

$$\begin{aligned} & (1 - \widehat{H}_i(u + p_i, u - X_i)) - (1 - H_i(u + p_i)) \\ = \ & \Pr\left(\{W_i - p_i > X_i\} \cap \{V_i + Z_i - p_i > u\}\right) - \Pr\left(W_i - p_i > u\right) \\ = \ & \Pr(\{X_i < W_i - p_i < u\} \cap \{V_i + Z_i - p_i > u\}). \end{aligned}$$

By Lemma 3, $H_i(u + p_i) > \widehat{H}_i(u + p_i, u - x_i)$, so (11) is always positive and formally shows the stochastic dominance relationship.

When there is no block, following CDK, the demand for product $i \in N_2$ is:

$$\mathscr{D}_i(\boldsymbol{p}) = \int_{\underline{y}_i}^{\overline{y}_i} \int_{\underline{x}_0}^{\overline{x}_0} (1 - H_i(\max\{x_0, y_i\} + p_i))\, dH^1_*(x_0) d\widetilde{H}^2_i(y_i). \tag{12}$$

Comparing (12) and (9), the demand for product $i \in N_2$ decreases if consumer search block-by-block, as J FOSDs H^1_* and $1 - H_i(\max\{u_1, y_i\} + p_i)$ is decreasing in u_1.

The lemma below concludes the comparison of product i's demand when consumers search block-by-block ($D^k_i(\boldsymbol{p})$) and i's demand without blocks ($\mathscr{D}_i(\boldsymbol{p})$).

Lemma 4 *Given a set of products N and any $\boldsymbol{p} \in \mathbb{R}^n_{++}$, for any position (N_1, N_2), if $i \in N_1$, then $D^1_i(\boldsymbol{p}) \geq \mathscr{D}_i(\boldsymbol{p})$, while if $i \in N_2$, then $D^2_i(\boldsymbol{p}) \leq \mathscr{D}_i(\boldsymbol{p})$.*

Lemma 4 states that the demand for any product increases if its position is in the first block, and thus sellers have incentives to win the sponsored link position. The intuition behind Lemma 4 is that: when the search order is distorted, it is possible that product i in block 1 is searched earlier comparing to the case without blocks. Thus, even if a product j in block 2 has a net effective value higher than that of i, and therefore has a higher priority to purchase than i without block, a consumer may end up purchasing i instead of j if the realized match value of product i is large enough. This is because j is no longer considered when the consumer searches in the second block if $i \in N_1$ is found to be good enough.[14] Thus, the demand for product $i \in N_1$ increases, and the demand for product $j \in N_2$ decreases, compared to the case without blocks.

4 Pricing Equilibrium

This section establishes existence and uniqueness of the pure strategy pricing equilibrium under any product position. The solution concept is Bertrand-Nash price competition. Given equilibrium prices, sellers then decide their auction bids by comparing equilibrium profits under different positions resulted from the sponsored link auction. Optimal bidding strategy and its effect on pricing strategy are studied in Section 5.

Given any position (N_1, N_2), each seller i chooses the product price p_i to maximize the profit, $(p_i - c_i)\, D_i^k(\boldsymbol{p})$. The following three assumptions are needed to establish the existence and uniqueness of the pricing equilibrium.

Assumption 1 *For all i, $f_i(\cdot)$ and $g_i(\cdot)$ are log-concave.*

Assumption 2 *For all i, the support of $f_i(\cdot)$ has no upper bound (i.e., $\overline{v}_i = \infty$). Moreover, the variance of V_i is larger than a constant $\sigma < \infty$, and either F_i has no lower bound (i.e., $\underline{v}_i = -\infty$) or $f_i(\underline{v}_i) = 0$ for all i.*

Assumption 3 *For all i, either $g_i'(\overline{z}_i) \leq 0$, or G_i has no upper bound (i.e., $\overline{z}_i = \infty$).*

Theorem 2 *Under Assumption 1, 2 and 3, for any $k \in \{1, 2\}$ and any $i \in N_k$, $D_i^k(\boldsymbol{p})$ is log-concave in p_i and $\log D_i^k(\boldsymbol{p})$ has strictly increasing differences in p_i and p_j for any $j \neq i$. And there is a unique pure strategy pricing equilibrium for all sellers.*

Theorem 2 is proved in Appendix B. Log-concavity is a standard assumption in literature of

[14] This is the case where $v_i + \min\{z_i, z_i^*\} - p_i < v_j + \min\{z_j, z_j^*\} - p_j$ but $v_i + z_i - p_i > v_j + z_j^* - p_j$.

pricing equilibrium.[15] The necessary first-order condition (FOC) for pricing equilibrium is:

$$\frac{1}{p_i - c_i} = -\frac{dD_i^k(\boldsymbol{p})/dp_i}{D_i^k(\boldsymbol{p})}, \tag{13}$$

which is also sufficient given Theorem 2. Assumption 2 guarantees any product has a strictly positive possibility to be purchased at any price, and eliminates the equilibrium with a zero demand. The intuition of Theorem 2 is similar to Theorem 2 in CDK and Theorem 1 in Quint (2014) [18]. Assumption 1 implies $1-\widehat{H}_i(\cdot, q)$ is log-concave for all i and $q > 0$. Assumption 1 and 2 together implies H_i is also log-concave (See the right panel of Figure 1 and CDK's Proposition 2). All three assumptions together implies J is log-concave. Since log-concavity in preserved under marginalization (Prékopa-Leindler's inequality), by the demand functions in Section 3, $D_i^k(\boldsymbol{p})$ is log-concave for any $k \in \{1, 2\}$ and $i \in N_k$. Log-concavity of $D_i^k(\boldsymbol{p})$ guarantees the right-hand side of (13) being monotonically increasing in p_i, while the left-hand side is strictly decreasing in p_i. So an equilibrium exists. Strictly increasing difference of $\log D_i^k(\boldsymbol{p})$ in p_i and p_j guarantees the best respond is unique.

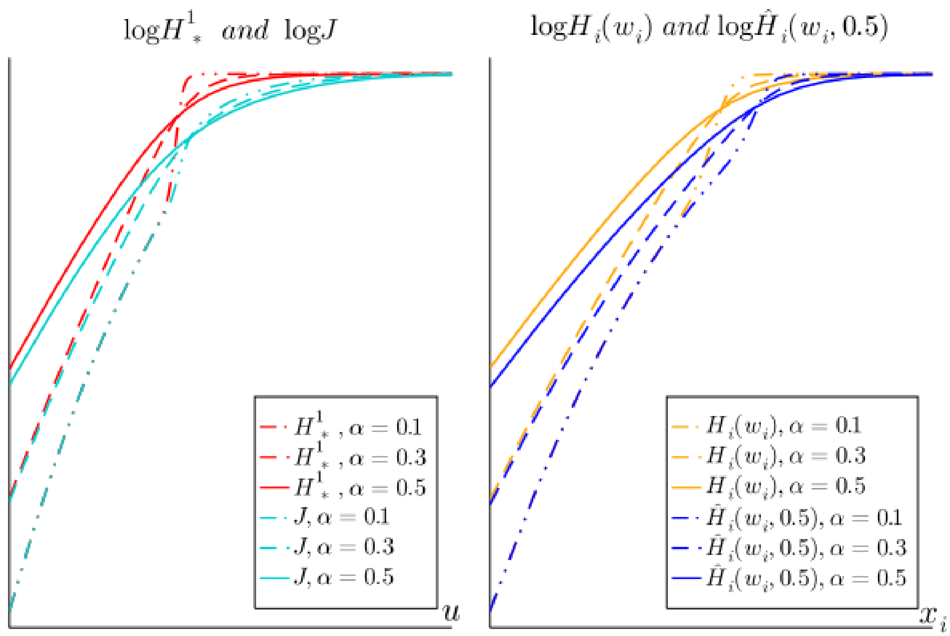


Figure 1: $\log H_*^1(u)$ and $\log J(u)$ (left), and $\log H_i(w_i)$ and $\log \widehat{H}_i(w_i, 0.5)$ (right) with different variances of V_i. In both graphs, $F_i(v_i) = 1/(1 + e^{-v_i/\alpha})$, $G_i(z_i)$ is standard normal and $s_i = 1$ (z_i^* is approximately -0.9) for all $i \in N$. In the left panel, $p_1 = 1$ and $n_1 = 3$.

[15] A comprehensive list of log concave density is given by Quint (2014) [18] and Bagnoli and Bergstrom (2005) [3]. The latter shows a distribution function is log-concave if it has a differentiable and log-concave density.

To prove Theorem 2, I take advantage of existing results from Quint (2014). With the discrete choice formulation of the demand (Theorem 1), Quint (2014)'s Theorem 1 implies it suffices to show: 1) for any $i \in N_1$ and $j \neq i$, $\Pr(\max\{X_i, Y_0\} - \min\{Z_i, z_i^* + (Y_0 - X_i)^+\} < t)$ is log-supermodular in t and p_j; and 2) $J(u)$ is log-supermodular in u and p_i for any $i \in N_1$.[16] Both are proved in Appendix B given Assumptions 1, 2 and 3. In particular, by (11), $J(u) = H_*^1(u) - \mathcal{K}(u)$, and $H_*^1(u) = \prod_{i \in N_1} H_i(u + p_i)$ is log-concave provided H_i is log-concave. I show when the variance of V_i is sufficiently large and the density of Z_i is weakly decreasing at the upper bound (Asumption 3), $\mathcal{K}(u)$ is spread out and its effect on the shape of $\log J(u)$ vanishes (See the left panel of Figure 1). Assumption 3 guarantees there is no spike in J at the upper bound when the variance of V_i is large, and is relatively weak as it holds under many log-concave density functions.[17]

5 Sponsored Link Auctions

Since *status quo* position (N_1^0, N_2^0) is exogenously fixed, with one spot in block 1 been auctioned, only two forms of position exist: 1) If seller $i \in N_1^0$ wins the sponsored link auction, position stays the same as the *status quo*, i.e., $(N_1, N_2) = (N_1^0, N_2^0)$; 2) If seller $i \in N_2^0$ wins the sponsored link auction, position becomes $(N_1, N_2) = (N_1^0 + i - l(i), N_2^0 - i + l(i))$.

By Lemma 4, if the product position switches from (N_1^0, N_2^0) (*status quo* position) to $(N_1^0 + i - l(i), N_2^0 - i + l(i))$ for any $i \in N_2^0$, the demand for product i increases and that for $l(i)$ decreases. Thus, the sponsored link position has a strictly positive value for sellers in N_2^0. Seller $l(i) \in N_1^0$ thus bids in order to keep the position in block 1. Other sellers in N_1^0 also have incentives to submit a positive bid if i is a stronger competitor than $l(i)$, and i's entry decreases the profits of some sellers in $N_1^0 \setminus \{l(i)\}$.

I treat the sponsored link auction game as a complete information game, since given position (N_1, N_2) and price $(p_i)_{i=1}^n$, the demand for any product is completely determined by the priors $(F_i)_{i=1}^n$ and $(G_i)_{i=1}^n$, known to all sellers. The complete information assumption is also used by Varian (2007) [20] and Edelman, Ostrovsky, and Schwarz (2007) [11]. Since sponsored link auctions occur repeatedly and all sellers can update their bids frequently,

[16] Quint defines a function is log-supermodular if its log is supermodular. For simplicity, I abuse the notation by defining J as a single argument function, but p_i affects J for any $i \in N_1$.

[17] For example, uniform, normal, logistic, and exponential distributions.

they can easily learn the demands and competitors' bids by experimenting on their own bids. Edelman and Schwarz (2010) [10] provide a detailed rationale to assume complete information.

The sponsored link auction is modelled as the second price auction.[18] The platform can choose between two different payment rules, fixed payment and per-transaction payment. The winner pays the second highest bid only once under the fixed payment and pays the second highest bid whenever his/her product is purchased under the per-transaction payment. Since sellers' profits are uniquely determined under any position outcome by Theorem 2, a seller's gain from winning the sponsored auction is the difference between the profits under the losing position and that under the winning position. Same as Börgers et al. [4], I study the pure strategy Nash equilibrium of the bidding game. The bidding equilibrium exists if no seller has an incentive to revise the bid and switch the position unilaterally under the equilibrium.

For any i, let b_i be the equilibrium bid submitted by seller i, and the vector of bids is $\boldsymbol{b} := \{b_1, b_2, \ldots, b_n\}$. Denote $\boldsymbol{b}_{-i} = \{b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n\}$ as the vector of bids besides i, and $\boldsymbol{b}_{-i}^{(1)} = \max \boldsymbol{b}_{-i}$ be the maximum bid in $\boldsymbol{b}_{-i}$.

5.1 Fixed payment

Under the fixed payment, if $b_i = \max \boldsymbol{b}$, the winner i's payoff is $(p_i - c_i)D_i^1(\boldsymbol{p}) - \boldsymbol{b}_{-i}^{(1)}$.[19] Since $\boldsymbol{b}_{-i}$ and p_i are in additive separable terms in the payoff function, bidding and pricing strategies do not interact since the FOCs in (13) are not affected. Denote π_i as the equilibrium profit of seller $i \in N$ under *status quo* position (N_1^0, N_2^0), and π_i^j as the equilibrium profit of seller i if j wins the auction. So π_i^j is the profit of i under the position $(N_1^0 + j - l(j), N_2^0 - j + l(j))$ if $j \in N_2^0$, and $\pi_i^j = \pi_i$ if $j \in N_1^0$.

For any seller $i \in N$, let b_i be the change of i's profit from the *status quo* position (N_1^0, N_2^0) to the position in which i wins the sponsored link auction, under the pricing equilibrium. That is,

$$b_i := \pi_i^i - \pi_i,$$

which is zero if $i \in N_1^0$. Let b_i^j be the change of i's profit from $j \in N_2^0$'s winning position, $(N_1^0 + j - l(j), N_2^0 - j + l(j))$, to i's winning position, which is (N_1^0, N_2^0) if $i \in N_1^0$ and is

[18] Under complete information, the second price auction is equivalent to the ascending bid auction.

[19] Fixed payment here is equivalent to pay-per-impression in marketing literature.

$(N_1^0 + i - l(i), N_2^0 - i + l(i))$ if $i \in N_2^0$. That is,

$$b_i^j := \pi_i^i - \pi_i^j.$$

If $i \in N_1^0$ and $j \in N_2^0$, b_i^j is i's willingness to pay to deter j entering block 1.

Theorem 3 *Under the fixed payment, conditional on all sellers set the equilibrium prices, there exists a complete information Nash equilibrium in which no seller benefits from revising the bid unilaterally.*

Proof To prove the theorem, I provide a tatonnement process that reaches an equilibrium:

1. Seller $j \in N_2^0$ submits bid b_j. Fix any $j' \in \arg\max_{j \in N_2^0} b_j$.
2. Seller $i \in N_1^0$ submits bid $b_i^{j'}$.
   - If $\max_i b_i^{j'} \geq b_{j'}$, the auction ends. The stable position is (N_1^0, N_2^0).
   - Else, proceed to step 3:
3. Seller $j \in N_2^0 \setminus \{j'\}$ submits $b_j^{j'}$.
   - If $b_{j'} \geq b_j^{j'}$ for all $j \in N_2^0$, the auction ends. The stable position is $(N_1^0 + j' - l(j'), N_2^0 - j' + l(j'))$.
   - Else, let $j'' := \arg\max_{j \in N_2^0 \setminus \{j'\}} b_j^{j'}$ and repeat step 2 and 3 by replacing j' with j''.

The above process eventually reaches an equilibrium since it stops if and only if the winner's bid exceeds all other sellers' willingness to pay for the sponsored link position under the current position. The equilibrium exits because whenever $b_j^{j'} > b_{j'}$ in step 3, the current highest bid strictly increases (from $b_{j'}$ to $b_{j''}^{j'}$). So it is impossible to have an infinite loop in steps 2-3, as all sellers have finite profits. ■

5.2 Per-transaction payment

Under the per-transaction payment, the winner pays to the platform when a consumer purchases the winner's product, and the winner i's payoff is $(p_i - c_i - \boldsymbol{b}_{-i}^{(1)})D_i^1(p_i, \boldsymbol{p}_{-i})$, where $\boldsymbol{b}_{-i}^{(1)}$ is the runner-up's bid. Thus, if seller i wins the sponsored link auction, it is equivalent to regard product i's marginal cost as $c_i + \boldsymbol{b}_{-i}^{(1)}$, with position changed according to the auction rule. Hence, Theorem 2 applies, and the prices of all products are determined uniquely given the bid payment $\boldsymbol{b}_{-i}^{(1)}$.

It turns out that with per-transaction payment, there is no difference between using first or second price auction from the winner's point of view, since under the bidding equilibrium, the highest bid and the second highest bid coincides.[20] This is because the winner's optimal price is increasing in the marginal cost, which "consists of" the runner-up's bid. In the complete information game, non-winners benefit from increasing their bids to increase the winner's marginal cost.

Lemma 5 *The equilibrium demand and profit of seller i increase if the marginal cost of i decreases, or if the marginal cost of j increases for any $j \neq i$.*

Lemma 5 follows from Quint's Theorem 4, which applies because of the log-supermodular demand structure.[21] Thus, all non-winner sellers benefit if the runner-up's bid becomes closer to the winner's bid, regardless who is the runner-up. This process continues until the runner-up's bid is just below the winner's bid. Thus, in the rest of the section, I take the auction payment as if it is a first price auction. The only difference is that the runner-up's bid must match the winner's bid, and the tie breaking rule is that the runner-up never wins.[22] This avoids empty best response of the runner-up.

Similar to Section 5.1, I define $\pi_i^j(b)$ as the equilibrium profit of seller i conditional on j wins the auction with bid b. In particular, I define the winner's profit as that net from auction payments to simplify the notation. That is, if seller i submits the highest bid b, the equilibrium profit is $\pi_i^i(b) := (p_i - c_i - b)D_i^1(p_i, \boldsymbol{p}_{-i})$. The winner always pays his/her own bid since the auction is treated as first price by the reason discussed above.

To characterize the auction equilibrium, I consider the hypothetical scenario where the auction winner r, the winning bid b and the runner-up j are given ex ante. Under the bidding equilibrium that r wins with bid b, two conditions need to be satisfied. First, no seller finds it profitable to beat r:

$$\pi_i^i(b) - \pi_i^r(b) \leq 0, \tag{14}$$

[20]More precisely, the runner-up's bid is arbitrarily close to the winner's bid.

[21]I do not need Quint's Assumption 3 since each product only has one component in my model.

[22]Since the auction takes place repeatedly, the runner-up can submit and retrieve the bid until the tie is broken such that he/she does not win.

for all $i \in N \setminus \{r\}$. Lemma 5 states that any seller's equilibrium profit is decreasing in the marginal cost and is increasing in a competitor's marginal cost. Hence the left-hand side of (14) is decreasing in b, which pins down the lower bound of b. Second, it is not profitable for winner r to withdraw the bid and let the runner-up j wins:

$$\pi_r^r(b) - \pi_r^j(b) \geq 0, \tag{15}$$

for a given $j \in N \setminus \{r\}$, which characterizes the upper bound of b as (15) is decreasing in b. The bidding equilibrium exists if there exist r and j such that the upper bound of b characterized by (15) is larger than the lower bound of b characterized by (14).

Theorem 4 *Under the per-transaction payment, conditional on all sellers set the equilibrium prices, there exists a complete information Nash equilibrium in which no seller benefits from revising the bid unilaterally.*

Proof The bidding equilibrium exists if there exits the winner r and the runner-up j, and the winning bid b, such that (14) and (15) are satisfied. The proof is constructive to find r, j and b. First, there exists $r \in N_1^0$ and $j \in N_2^0$ such that $r = l(j)$ and (15) holds with sufficiently small $b > 0$. This is because r's position is moved from block 1 into block 2 if j wins. And in the extreme case where $b \to 0$, seller r can get the sponsored link for free and is strictly better off to win the auction. Now, with (15) holds for some r, j and b, the equilibrium is constructed by the following process:

1. If (14) also holds for r and b then the equilibrium exists in which r wins with bid b.

2. Else, violation of (14) means there exists another j' such that

$$\pi_{j'}^{j'}(b) - \pi_{j'}^r(b) > 0, \tag{16}$$

   which is (15) under strict inequality. Since the left-hand side of (16) is decreasing in b, there exists $b' > b$ such that (16) holds with equality.

3. Since $\pi_{j'}^{j'}(b') - \pi_{j'}^i(b') = 0$. Let j' be the current winner who submit a bid b' and r be the runner-up: replace j by r, r by j' and b by b'. So (15) holds with equality. Then go back to step 1.

An equilibrium is constructed following the 3-step procedure above. Every time step 3 is activated, the current standing bid increases. Whenever the process stops, we find r, j and b such that (14) and (15) are satisfied, so an bidding equilibrium exists.

The only case that an equilibrium does not exist is that the three-step process above forms a loop that never ends. That is, there is a sequence of sellers $(i_1, i_2, \ldots, i_m)$ and a corresponding sequence of bids $(b_1, b_2, \ldots, b_m)$ such that $\pi_{i_s}^{i_s}(b_{s-1}) > \pi_{i_s}^{i_{s-1}}(b_{s-1})$ for any $2 \leq s \leq m$, and $\pi_{i_1}^{i_1}(b_m) > \pi_{i_1}^{i_m}(b_m)$, so there is a cyclic relation in (16).

However, since b_s are adjusted in the way that $\pi_{i_s}^{i_s}(b_s) = \pi_{i_s}^{i_{s-1}}(b_s)$ for any $s \in \{2, \ldots, m\}$, and $\pi_{i_1}^{i_1}(b_1) = \pi_{i_1}^{i_m}(b_1)$, monotonicity of the left-hand side of (16) in b implies $b_s > b_{s-1}$ for $s \in \{2, \ldots, m\}$ and $b_1 > b_m$. A contradiction. ■

One special case is where sellers are symmetric (Section 6), such that the cycle holds whenever the bid is lower than the equilibrium bid. But once the bid reaches the equilibrium bid, all strict inequalities in the cycle become equality, so (14) is not violated.

6 Comparative Statics

This section shows a set of comparative static results when sellers are symmetric. To be specific, for any $i \in N$, let $F_i = F$, $G_i = G$, $c_i = c$, $s_i = s$. I compare the auction revenue, consumer surplus, and seller profits between the fixed payment and the per-transaction payment rule. Assumption 1, 2 and 3 are kept in this section.

With Theorem 2, equilibrium prices are pinned down by FOCs. Under the symmetric environment, it is without loss of generality to fix the winner as $r \in N$. If the bid payment is fixed, by Theorem 2, all sellers in block 1 (include the winner r) set the price at p_1^f and all sellers in block 2 set the price at p_2^f, such that p_1^f and p_2^f satisfy:

$$\frac{1}{p_k^f - c} = -\left.\frac{dD_i^k(p, \boldsymbol{p}_{-i}^f)/dp}{D_i^k(p, \boldsymbol{p}_{-i}^f)}\right|_{p=p_k^f},$$

for $k \in \{1, 2\}$, where $D_i^k(p, \boldsymbol{p}_{-i}^f)$ is the demand for $i \in N_k$ conditional on i sets a price p and all other sellers set the equilibrium price $\boldsymbol{p}_{-i}^f$ under the fixed payment. If the bid payment is based on the number of transactions and the winner's bid is b, the equilibrium is that the winner sets a price p_r, all sellers in $N_1 \setminus \{r\}$ set price at p_1^t, and all sellers in N_2 set price at

p_2^t. And p_r, p_1^t and p_2^t satisfy

$$\frac{1}{p_r - c - b} = -\frac{dD_r^k(p, \boldsymbol{p}_{-i}^t)/dp}{D_r^k(p, \boldsymbol{p}_{-i}^t)}\bigg|_{p=p_r}, \qquad \frac{1}{p_k^t - c} = -\frac{dD_i^k(p, \boldsymbol{p}_{-i}^t)/dp}{D_i^k(p, \boldsymbol{p}_{-i}^t)}\bigg|_{p=p_k^t},$$

for $k \in \{1, 2\}$, where $D_i^k(p, \boldsymbol{p}_{-i}^t)$ is the demand for $i \in N_k$ conditional on i set price at p and all other sellers set the equilibrium price $\boldsymbol{p}_{-i}^t$ under the per-transaction payment. The proof of the uniqueness of the equilibrium is provided in Appendix C.

For notation simplicity, I denote D_f^1 and D_f^2 as the equilibrium demands for a product in block 1 and block 2 under the fixed payment. And define $D_r(b)$, $D_t^1(b)$ and $D_t^2(b)$ as the equilibrium demand for the winner's product, a product in block 1 and block 2 under the per-transaction payment when the winner's bid is b.

With the notations from Section 5.1, we cans show

$$b_i = \begin{cases} 0 & \text{if } i \in N_1^0 \\ (p_1^f - c)D_f^1 - (p_2^f - c)D_f^2 & \text{if } i \in N_2^0 \end{cases},$$

$$b_i^j = \begin{cases} (p_1^f - c)D_f^1 - (p_2^f - c)D_f^2 & \text{if } j \in N_0^2 \text{ and } i = l(j) \\ b_i & \text{else} \end{cases}.$$

Thus, the auction revenue under fixed payment is $(p_1^f - c)D_f^1 - (p_2^f - c)D_f^2$.

I use the notations in Section 5.2 to find the bidding equilibrium with per-transaction payment. With the symmetric assumption, the winner r's net profit is $\pi_r^r(b) = (p_r^t - c - b)D_r(b)$, and

$$\pi_i^j(b) = \begin{cases} (p_1^t - c)D_t^1(b) & \text{if } i \in N_1^0 \text{ and } i \neq l(j) \\ (p_2^t - c)D_t^2(b) & \text{else} \end{cases}.$$

Because of the "for all" quantifier, (14) becomes:

$$(p_r^t - c - b)D_r(b) \leq (p_2^t - c)D_t^2(b) \quad \textit{and} \quad (p_r^t - c - b)D_r(b) \leq (p_1^t - c)D_t^1(b),$$

And because of the "there exists" quantifier, (15) becomes

$$(p_r^t - c - b)D_r(b) \geq (p_2^t - c)D_t^2(b) \quad \textit{or} \quad (p_r^t - c - b)D_r(b) \geq (p_1^t - c)D_t^1(b).$$

The two conditions in (14) and (15) together imply, under the equilibrium,

$$(p_r^t - c - b)D_r(b) = (p_2^t - c)D_t^2(b), \tag{17}$$

and

$$(p_r^t - c - b)D_r(b) \leq (p_1^t - c)D_t^1(b). \tag{18}$$

Condition (17) thus determines the equilibrium bid and the auction revenue is

$$bD_r(b) = (p_r^t - c)D_r(b) - (p_2^t - c)D_t^2(b). \tag{19}$$

Theorem 5 *In the symmetric environment, both consumer surplus and auction revenue are higher under the fixed payment than that under the per-transaction payment; profits of non-winners are higher under the per-transaction payment than that under the fixed payment.*

Proof By Monotone Selection Theorem of Milgrom and Shannon [15], the product price of the winner r increases when the payment rule switches from the fixed to the per-transaction ($p_r^t > p_1^f$). Since prices are complements in the supermodular game, prices of all products increase. Lemma 5 implies an increase of winner r's marginal cost increases the equilibrium profits of non-winners.

The winner r's profit, $(p_r^t - c)D_r(b)$, must be weakly smaller. This is because non-winners' FOCs are the same under the fixed payment and per-transaction payment. If $(p_r^t - c)D_r(b) > (p_1^f - c)D_1(b)$, winner's profit under the fixed payment can be increased if the winner switches the price from p_1^f to p_r^t. Contradict p_1^f is the best response of the winner under the fixed payment.

These together imply $(p_r^t - c)D_r(b) < (p_1^f - c)D_f^1$ and $(p_2^t - c)D_t^2(b) > (p_2^f - c)D_f^2$ for any $b > 0$. By (19), the total bid payment of winner r decreases. The consumer surplus decreases as the prices of all products increase, i.e., $p_r^t > p_1^f$, $p_1^t > p_1^f$ and $p_2^t > p_2^f$. ■

The intuition of Theorem 5 is comparable to the distortion effect of proportional income/consumption tax. The trade-offs between using the proportional tax and the lump-sum tax are efficiency (higher tax income) and redistribution (tax more on high income).[23] While in a sponsored link auction, the trade-off between using the fixed-payment and using the per-transaction payment is how to split total surplus among the platform, buyers, and sellers: fixed payment leads to higher auction revenue and consumer surplus, and per-transaction payment leads to higher seller surplus. In reality, an online shopping platform may care both consumer surplus and seller surplus for a long-run objective. Amazon also collects revenue from Amazon Prime membership subscription, which relates to customers' shopping experience, or consumer surplus. The per-click payment used by Amazon is close to a mixture

[23] See Chapter 6 in Kaplow (2008) [14].

between the fixed payment and the per-transaction payment since every click has a probability of converting to a transaction. Theorem 5 thus implies the per-click payment used by Amazon splits total surplus from the online transaction between consumers and sellers in a relatively equal way. While the per-transaction payment used by eBay is a seller-friendly rule.

A recent paper by Ostrovsky (2021) [17] studies choice screen auctions with different payment methods. Though the underlying mechanisms are different, he finds a similar result on consumer surplus. The choice screen auction is operated on Android platform smartphones, and search engine providers compete to place their apps on the screen at the first time consumers use the smartphone to pick a default search engine. Search engine providers are allowed to choose the popularity (demand) of the search engine and the revenue-per-use (product price), which are negatively correlated. Ostrovsky (2021) shows that "per-install" (per-transaction) payment distorts the search engine toward extracting as much revenue from its users (higher price and lower consumer surplus) comparing to "per-appearance" (fixed) payment, as the winner has the incentive to decrease its number of installs (transactions).

7 Consumer Optimal Positioning

If the platform aims to improve the users' experiences for a long-term goal (e.g., attract more users and more membership subscriptions like Amazon Prime), or if a social planner regulates the platform, consumer surplus maximization may become a part of the goals the platform target. In this section, I assume all sellers commit to product prices before the position is allocated.

To maximize consumer surplus, a naive platform may intend to allocate products with higher expected net true value into block 1. But such a practise may fail to be optimum. This is because under the search environment, the match value z_i is unknown a priori and consumers in general do not purchase the product with the highest net true value (i.e., existence of the search friction). As search is costly, consumers' payoffs increase from a lower search cost paid during the search if the purchase decision remains the same. The question is how the platform should allocate products to minimize the search frictions, which requires balancing consumers' payoff from the final purchasing and the cost paid during search. This section provides a consumer optimal position rule that answers the question.

As consumers pay search cost whenever they visit (click) a product selling page. I need to formalize the conditions under which i's link is click to calculate the total search cost.

Lemma 6 *Given* $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$, *product* $i \in N_k$ *for* $k \in \{1,2\}$ *is clicked if and only if both conditions below are satisfied.*

1. $v_i + z_i^* - p_i > u_{k-1}$,
2. $\forall j \in N_k \setminus \{i\}$, $v_j + \min\{z_j, z_j^*\} - p_j < v_i + z_i^* - p_i$.

Proof The proof directly follows from the optimal search rule in Lemma 1. The first condition states the consumer never click a link with a reservation value less than the outside option value, u_0 if searching in block 1 and u_1 if searching in block 2. The second condition is: $\forall j \in N_1 \setminus \{i\}$ such that $v_j + z_j^* - p_j > v_i + z_i^* - p_i$, it has to be $v_j + z_j - p_j < v_i + z_i^* - p_i$. That is, for any product j that is searched before i according to the optimal search rule, the net realized value of j must be less than the net reservation value of i. So the consumer continues searching to product i. ■

For any position (N_1, N_2), I use the definition of X_i, X_0, Y_i, Y_0 and U_1 as that in Section 3. Since X_i, X_0, Y_i, Y_0 and U_1 purely depend on V_i, Z_i and product position, all expectations in this section are taken with respect to V_i and Z_i for all $i \in N$. Under the optimal search rule, consumer surplus is a function $W : (N_1, N_2) \to \mathbb{R}$ such that

$$\begin{aligned} W(N_1, N_2) \; &:= \; \mathbb{E}\left[\mathbb{1}_{\{\max\{X_0, Y_0\} < u_0\}}\right] u_0 \\ &+ \; \sum_{i \in N_1} \mathbb{E}\left[\mathbb{1}_{\{X_i < W_i - p_i\} \cap \{Y_0 < V_i + Z_i - p_i\}\}}(V_i + Z_i - p_i) - \mathbb{1}_{\{X_i < V_i + z_i^* - p_i\}} s_i\right] \\ &+ \; \sum_{i \in N_2} \mathbb{E}\left[\mathbb{1}_{\{\max\{U_1, Y_i\} < W_i - p_i\}}(V_i + Z_i - p_i) - \mathbb{1}_{\{\max\{U_1, Y_i\} < V_i + z_i^* - p_i\}} s_i\right]. \end{aligned} \tag{20}$$

The first term in (20) is the payoff of consumers who pick the outside option eventually. In the two summations (line 2 and 3), the first terms are payoffs of consumers purchasing product i in block 1 and 2 respectively, where the indicators incorporate the purchase conditions from Theorem 1. Lemma 6 says that product $i \in N_k$ is visited (clicked) if and only if i's net reservation value is larger than the highest effective value in N_k. This forms the second terms in the two summations.

Given sizes of the two blocks n_1 and n_2, and sellers' committed prices $(p_i)_{i=1}^n$, the planner solves the following consumer surplus maximization problem:

$$\begin{aligned} \max_{N_1,N_2} \quad & W(N_1,N_2) \\ \text{s.t.} \quad & N_1 \cup N_2 = N, N_1 \cap N_2 = \emptyset \\ & |N_1| = n_1, |N_2| = n_2. \end{aligned} \tag{21}$$

Theorem 6 *Given F_i, G_i for all i, and the size of the two blocks n_1, n_2, and if each seller i commits to a price p_i, the problem (21) is equivalent to minimizing*

$$W^L(N_1,N_2) := \mathbb{E}\left[(Y_0 - X_0)^+\right] - \mathbb{E}\left[(Y_0 - U_1)^+\right], \tag{22}$$

subject to the constraints in (21).

Theorem 6 is proved in Appendix D, which shows

$$W(N_1,N_2) = \overline{W} - W^L(N_1,N_2),$$

where $\overline{W} = \mathbb{E}[u_0, X_0, Y_0]$ is the expectation of the highest net effective value among all products (including the outside option) and is uncorrelated with product position. $\overline{W}$ is exactly the consumer surplus without blocks.[24] Notice that W^L is non-negative given U_1 FOSDs X_0, so the consumer surplus is indeed lower comparing to the case without blocks (i.e., $W \leq \overline{W}$). W^L can thus be regarded as the welfare loss from the block-by-block search behavior.

To get implications of Theorem 6, rewrite W^L as:

$$W^L(N_1,N_2) = \mathbb{E}\left[\mathbb{1}_{\{U_1 \geq Y_0\}} (Y_0 - X_0)^+\right] + \mathbb{E}\left[\mathbb{1}_{\{U_1 < Y_0\}} (U_1 - X_0)\right]. \tag{23}$$

Theorem 6 thus tells us that consumer surplus is maximized under two criteria. First, $\mathbb{E}\left[\mathbb{1}_{\{U_1 \geq Y_0\}} (Y_0 - X_0)^+\right] = \mathbb{E}\left[\mathbb{1}_{\{U_1 \geq Y_0 \geq X_0\}} (Y_0 - X_0)\right]$ should be small. That is, conditional on consumers purchase from block 1, or $U_1 \geq Y_0$, products with high (low) net effective values should be placed at block 1 (block 2). Second, $\mathbb{E}\left[\mathbb{1}_{\{U_1 < Y_0\}} (U_1 - X_0)\right]$ should also be small. That is, conditional on consumers purchasing from block 2 ($U_1 < Y_0$), candidate product in block 1 should have a low search cost. This is because $U_1 - X_0 := (Z_{i^*} - z^*_{i^*})^+$, conditional on i^* is the candidate product in block 1. By (1), $\mathbb{E}\left[(Z_{i^*} - z^*_{i^*})^+\right] = s_{i^*}$.

There are two extreme cases where the loss from block-by-block search, $W^L(N_1,N_2)$, achieves zero under the optimal positioning. The first case is $w_{i^*} - p_{i^*} \geq w_{j^*} - p_{j^*}$ for any

[24] See Corollary 1 of Choi, Dai and Kim (2018) [7]

candidate product i^* in block 1 and any candidate product j^* in block 2 under any realized $(\boldsymbol{v}_k, \boldsymbol{z}_k, \boldsymbol{p}_k)_{k=1}^2$. That is, $\Pr(X_0 \geq Y_0) = 1$: any product in block 1 has a net effective value larger than that in block 2. Under this case, the block-by-block search aligns perfectly the optimal search without blocks, since products in the second block are never visited. $W^L = 0$ since both terms in (22) are zeros.

The second case is $\Pr(U_1 = X_0) = 1$, which is equivalent to $\Pr(Z_{i^*} = z^*_{i^*}) = 1$. Under this case, the candidate product in block 1 has zero uncertainty in match value and incurs a zero search cost to be visited. The distorted block-by-block search has no impact on consumer surplus, since no search cost is paid in the first block. $W^L = 0$ since the two terms in (22) are identical. The two cases mentioned above have a strong requirements on products' priors (non-overlapping effective value supports and zero search cost), and are not like to happen in general. A consumer optimal platform needs to consider both criteria when positioning products.

Notice that the algorithm provided by Theorem 6 requires all sellers to commit to their prices before the position is allocated. If sellers are not able to make the commitment, a social surplus maximization platform needs complete a much more complicated task by enumerating and comparing $W = \overline{W} - W^L$ under all possible positions with the corresponding equilibrium product prices.

8 Conclusion

Internet changes every aspect of economics as a low-cost communication method. Since the pandemic, more and more people have shifted from in-store shopping to online shopping to keep social distancing. Compared to advertising products in-store, advertising a product online is much more straightforward: a seller can buy a position at the top of the product webpage. Online shopping platforms use sponsored link auctions to determine who gets the limited number of positions and how much to pay. A typical online shopping activity is an interaction between three parties: consumers, sellers, and the shopping platform, with each party a different objective: consumers maximize the consumption payoffs from search, sellers maximize the selling profits, and the platform either maximizes the sponsored link auction revenue or coordinates on the surplus split between the three parties as a social planner.

By assuming consumers follow a block-by-block searching behavior, this paper formulates consumers' discrete choice shopping problem and the demand functions. The paper also establishes pricing and bidding equilibrium, and finds how pricing strategy, bidding strategy, and product position interact in a complete information second price auction. A comparison between the fixed payment and per-transaction payment in the symmetric environment shows that consumer surplus and auction revenue are higher under fixed payment. By contrast, sellers' profits are higher under per-transaction payment. The paper also finds a simple position rule to maximize consumer surplus if sellers commit to price before the position is allocated. The demand functions in my model have standard properties, and the discrete choice formulation may be applied to empirical researches.

Many interesting questions remain to be answered. A natural question is how results might change if more than one position in the first block is auctioned. With more than one position, the pricing equilibrium exists, but the allocative externality complicates bidding equilibrium as the winner of one position needs to make the bidding decision based on the identities of other winners. And this problem can be potentially removed under the symmetric environment. The other is to allow "blind buying" in Chen et al. [6] such that consumer need not learn the match value before purchase.

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A Derivation of (10)

The probability of $x_0 \in [u_0, s]$ is:

$$
\begin{aligned}
H_*^1(s) - H_*^1(u_0) &= \prod_{i \in N_1} H_i(s+p_i) - \prod_{i \in N_1} H_i(u_0+p_i) \\
&= \int_{u_0}^{s} d \prod_{i \in N_1} H_i(w+p_i) \\
&= \sum_{i \in N_1} \int_{u_0}^{s} \prod_{j \in N_1 \setminus \{i\}} H_j(w+p_j) dH_i(w+p_i), \\
&= \sum_{i \in N_1} \left[H_*^1(s) - H_*^1(u_0) - \int_{u_0}^{s} H_i(w+p_i) d \prod_{j \in N_1 \setminus \{i\}} H_j(w+p_j) \right] \qquad (24) \\
&\underset{(x_i = w)}{=} \sum_{i \in N_1} \left[\int_{u_0}^{s} H_i(s+p_i) - H_i(x_i+p_i) d\widetilde{H}_i^1(x_i) \right],
\end{aligned}
$$

where the third equality follows from factorial decomposition, the fourth equality comes from integration by parts and the last line is follow by the definition of $\widetilde{H}_i^1$. Namely, $\widetilde{H}_i^1(x_i)$ have a mass at $x_i = u_0$: for any function f, $\int_{u_0}^{u} f(x) d\widetilde{H}_i^1(x) = \int_{u_0}^{u} f(x) d \prod_{j \in N_1 \setminus \{i\}} H_j(x+p_j)) + f(u_0)\widetilde{H}_i^1(u_0)$.

B Proof of Theorem 2

Under Assumption 1, Theorem 1 and 3 in Bagnoli and Bergstrom (2005) [3] imply for all i, $F_i(\cdot)$, $G_i(\cdot)$, $1 - F_i(\cdot)$ and $1 - G_i(\cdot)$ are log-concave. With Assumption 1 and 2, Proposition 2 in CDK shows both $1 - H_i(\cdot)$ and $H_i(\cdot)$ are log-concave. Since for $k \in \{1, 2\}$, $i \in N_k$ and $x \geq u_0$,

$$
\widetilde{H}_i^k(x) = \prod_{j \in N_k \setminus \{i\}} H_j(x+p_j), \qquad (25)
$$

and log-concave is preserved under product operations, both $\widetilde{H}_i^k(\cdot)$ and $H_*^k(\cdot)$ are log-concave. That is, the distribution of x_i, y_i, x_0 and y_0 are log-concave.

Demand in block 1 By Lemma 2, $D_i^1(\boldsymbol{p})$ is the probability of

$$
V_i + \min\{Z_i, z_i^* + (Y_0 - X_i)^+\} - p_i > \max\{X_i, Y_0\}.
$$

If two independent random variables have distributions with increasing hazard rates, so does their sum. Since V_i, Z_i, X_i and Y_0 are independent from each other, a sufficient condition of log-concavity of $D_i^1(\boldsymbol{p})$ in p_i is that $\Pr(\max\{X_i, Y_0\} - \min\{Z_i, z_i^* - (Y_0 - X_i)^+\} < t)$ is log-concave in t. Under assumption 2, for any $t > u_0$, the probability can be transferred to:

$$\begin{aligned}
& \Pr\left(\max\{X_i, Y_0\} - \min\{Z_i, z_i^* + (Y_0 - X_i)^+\} < t\right) \\
= & \Pr\left(\max\{X_i - z_i^*, X_i - Z_i, Y_0 - Z_i\} < t\right), \\
= & \Pr\left(\{X_i < t + \min\{Z_i, z_i^*\}\} \cap \{Y_0 < t + Z_i\}\right), \\
= & \int_{\underline{z}_i}^{\overline{z}_i} \widetilde{H}_i^1\left(t + \min\{z_i, z_i^*\}\right) H_*^2\left(t + z_i\right) dG_i(z_i), \\
= & \int_{\underline{z}_i}^{\overline{z}_i} \prod_{j \in N_1 \setminus \{i\}} H_j\left(t + \min\{z_i, z_i^*\} + p_j\right) \prod_{j \in N_2} H_j\left(t + z_i + p_j\right) dG_i(z_i), \qquad (26)
\end{aligned}$$

where the first equality follows from exploring the cases that $X_i \geq Y_0$ and $X_i < Y_0$ separately, and the last equality comes from the definition of $\widetilde{H}_i^1$ and H_*^2. Since $\min\{z_i, z_i^*\}$ is a concave function in z_i, and log-concavity is preserved under multiplication, the integrand of (26) is log-concave in t, z_i and p_j. By Prékopa-Leinder inequality, log-concavity is preserved under integration. This, coupled with g_i is log-concave (Assumption 1), implies (26) is log-concave and log-supermodular in t and p_j for any $j \neq i$, as both t and p_j are additive seperatble terms in H_j in (26).[25] Quint (2014) [18]'s Theorem 1 applies and $D_i^1(\boldsymbol{p})$ is log-supermodular in p_i and p_j for any $i \neq j$. [26]

Demand in block 2 $D_i^2(\boldsymbol{p})$ is the probability of

$$W_i - p_i > \max\{U_1, Y_i\}$$

I show $J(\cdot)$ is log-concave under Assumption 1, 2 and 3, the result then follows by the same arguments as that for block 1 demand. By (11),

$$J(u) = H_*^1(u) - \mathcal{K}(u).$$

Since $H_*^1(u) = \prod_{i \in N_1} H_i(u + p_i)$ is log-concave in both u and p_i, it suffices to prove the sign of $(\log(H_*^1(u) - \mathcal{K}(u)))''$ is the same as the sign of $(\log H_*^1(u))''$ when the variance of V_i is large enough.

[25]By Quint's definition, a function is log-supermodular if its log is supermodular.

[26]In specific, one can replace $G_2(t)$ in the proof of Quint's Theorem 1 by (26).

Lemma 7 *For any $u \geq u_0$, $\mathcal{K}(u)$ defined in (11) can be written as:*

$$\mathcal{K}(u) = \sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i)(1 - G_i(z_i^* + u - x_i))dF_i(x_i + p_i - z_i^*). \tag{27}$$

And the derivative is

$$\begin{aligned} \frac{d\mathcal{K}(u)}{du} &= \sum_{i \in N_1} \widetilde{H}_i^1(u)\,(1 - G_i(z_i^*))\, f_i(u + p_i - z_i^*) \\ &\quad - \sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i) g_i(z_i^* + u - x_i) dF_i(x_i + p_i - z_i^*). \end{aligned} \tag{28}$$

Proof By (11), and the definition of $\widetilde{H}_i^1$,

$$\begin{aligned} \mathcal{K}(u) = \sum_{i \in N_1} \left[\int_{u_0}^{u} H_i(u + p_i) - \widehat{H}_i(u + p_i, u - x_i)\right] d \prod_{j \in N_1 \setminus \{i\}} H_j(x_i + p_j) \\ + \sum_{i \in N_1} \left[H_i(u_0 + p_i) - \widehat{H}_i(u + p_i, u - u_0)\right] \widetilde{H}_i^1(u_0), \end{aligned}$$

which by integration by parts and $\widetilde{H}_i^1(x_i) = \prod_{j \in N_1 \setminus \{i\}} H_j(x_i + p_j)$ on $[u_0, u]$, equals

$$\begin{aligned} \mathcal{K}(u) &= -\sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i) d_x \left[\widehat{H}_i(u + p_i, u - x_i) - H_i(u + p_i)\right], \\ &= -\sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i) d_x \widehat{H}_i(u + p_i, u - x_i), \end{aligned}$$

where $d_x \widehat{H}_i(u + p_i, u - x_i)$ is the derivative with respect to x_i, which is

$$\begin{aligned} \frac{d}{dx_i} \widehat{H}_i(u + p_i, u - x_i) &= \frac{d}{dx_i} \left[(1 - G_i(z_i^* + u - x_i))F_i(x_i + p_i - z_i^*)\right] \\ &\quad + \frac{d}{dx_i} \int_{\underline{z}_i}^{z_i^* + u - x_i} F_i(u + p_i - z_i) dG_i(z_i), \\ &= (1 - G_i(z_i^* + u - x_i)) f_i(x_i + p_i - z_i^*), \end{aligned}$$

if $z_i^* + u - x_i \leq \overline{z}_i$, and equals zero otherwise. Thus

$$\mathcal{K}(u) = \sum_{i \in N_1} \int_{u_0}^{u} \widetilde{H}_i^1(x_i)(1 - G_i(z_i^* + u - x_i))dF_i(x_i + p_i - z_i^*).$$

The derivative, by Leibniz rule, is exactly (28). ■

Let $V_i^\sigma = \sigma V_i$ with distribution $F_i^\sigma(v) = F_i(v/\sigma)$, and $W_i^\sigma = V_i^\sigma + \min\{Z_i, z_i^*\}$. Denote the distribution of $\max_{j \in N_1 \setminus \{i\}} W_j^\sigma$ as $(\widetilde{H}_i^1)^\sigma$. By Lemma 7, $\mathcal{K}$ with V_i^σ can be written as:

$$\mathcal{K}^\sigma(u) = \sum_i \int_{u_0}^{u} (\widetilde{H}_i^1)^\sigma(x_i)\,(1 - G_i(z_i^* + u - x_i))\, dF_i^\sigma(x_i + p_i - z_i^*).$$

Denote $b_i := F_i^\sigma(u + p_i - z_i^*)$, $a_i := F_i^\sigma(u_0 + p_i - z_i^*)$, and $r_i := F_i^\sigma(x_i + p_i - z_i^*)$, by change of variable:

$$\mathcal{K}^\sigma(u) = \sum_i \int_{a_i}^{b_i} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(r_i) + z_i^* - p_i) \left(1 - G_i(z_i^* + (F_i^\sigma)^{-1}(b_i) - (F_i^\sigma)^{-1}(r_i))\right) dr_i.$$

With $F_i^\sigma(v_i) = F_i(v_i/\sigma)$, we have $(F_i^\sigma)^{-1}(r) = \sigma F_i^{-1}(r)$, $f_i^\sigma((F_i^\sigma)^{-1}(r)) = f_i(F_i^{-1}(r))/\sigma$ and $(f_i^\sigma)'(F_i^{-1}(r)) = f_i(F_i^{-1}(r))/\sigma^2$. Thus,

$$\mathcal{K}^\sigma(u) = \sum_i \int_{a_i}^{b_i} (\widetilde{H}_i^1)^\sigma(\sigma F_i^{-1}(r_i) + z_i^* - p_i) \left(1 - G_i(z_i^* + \sigma(F_i^{-1}(b_i) - F_i^{-1}(r_i))\right) dr_i,$$

Since $(\widetilde{H}_i^1)^\sigma$ is bounded between zero and one, given $F_i^{-1}(b_i) > F_i^{-1}(r_i)$, $\mathcal{K}^\sigma$ converges to zero when σ explodes as $\left(1 - G_i(z_i^* + \sigma(F_i^{-1}(b_i) - F_i^{-1}(r_i))\right)$ goes to zero.

The derivative of $\mathcal{K}^\sigma(u)$ is

$$\begin{aligned} \frac{d\mathcal{K}^\sigma(u)}{du} &= \sum_{i \in N_1} (\widetilde{H}_i^1)^\sigma(u)\,(1 - G_i(z_i^*))\, f_i^\sigma(u + p_i - z_i^*) \\ &\quad - \sum_{i \in N_1} \int_{u_0}^{u} (\widetilde{H}_i^1)^\sigma(x_i) g_i(z_i^* + u - x_i) dF_i^\sigma(x_i + p_i - z_i^*). \end{aligned} \tag{29}$$

Rewrite the above expression in terms of a_i, b_i and r_i:

$$\sum_{i \in N_1} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i) + z_i^* - p_i)\,(1 - G_i(z_i^*))\, f_i^\sigma\left((F_i^\sigma)^{-1}(b_i)\right)$$
$$- \sum_{i \in N_1} \int_{a_i}^{b_i} (\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(r_i) + z_i^* - p_i) g_i(z_i^* + (F_i^\sigma)^{-1}(b_i) - (F_i^\sigma)^{-1}(r_i)) dr_i,$$

where the first term converges to zero when σ explodes since $f_i^\sigma((F_i^\sigma)^{-1}(b_i)) = f_i(F_i^{-1}(b_i))/\sigma$ and $(\widetilde{H}_i^1)^\sigma$ is bounded between zero and one. The second term is negative as all terms in its integrand has non-negative value. That is, the first order derivative of $\mathcal{K}^\sigma$ is non-positive when σ is large enough.

Similarly, we can express the second order derivative of $\mathcal{K}^\sigma$ as:

$$\begin{aligned} \frac{d^2}{du^2}\mathcal{K}^\sigma(u) &= \sum_{i \in N_1} \left((\widetilde{H}_i^1)^\sigma(u)\right)' (1 - G_i(z_i^*)) f_i^\sigma(u + p_i - z_i^*) \\ &\quad + \sum_{i \in N_1} (\widetilde{H}_i^1)^\sigma(u)\,(1 - G_i(z_i^*))\,(f_i^\sigma)'(u + p_i - z_i^*) \\ &\quad - \sum_{i \in N_1} (\widetilde{H}_i^1)^\sigma(u) g_i(z_i^*) f_i^\sigma(u + p_i - z_i^*) \\ &\quad - \sum_{i \in N_1} \int_{u_0}^{u} (\widetilde{H}_i^1)^\sigma(x_i) g_i'(z_i^* + u - x_i) dF_i^\sigma(x_i + p_i - z_i^*), \end{aligned}$$

which, in terms of a_i, b_i and r_i is:

$$\begin{aligned}\frac{d^2}{du^2}\mathcal{K}^\sigma(u) &= \sum_{i\in N_1}\Big((\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i)+z_i^*-p_i)\Big)'(1-G_i(z_i^*))f_i^\sigma((F_i^\sigma)^{-1}(b_i)) \\ &\quad+\sum_{i\in N_1}(\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i)+z_i^*-p_i)\,(1-G_i(z_i^*))\,(f_i^\sigma)'((F_i^\sigma)^{-1}(b_i)) \\ &\quad-\sum_{i\in N_1}(\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i)+z_i^*-p_i)g_i(z_i^*)f_i^\sigma((F_i^\sigma)^{-1}(b_i)) \\ &\quad-\sum_{i\in N_1}\int_{u_0}^{u}(\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(r_i)+z_i^*-p_i)g_i'(z_i^*+(F_i^\sigma)^{-1}(b_i)-(F_i^\sigma)^{-1}(r_i))dr_i,\end{aligned}$$

where the second and third term converges to zero when σ is large enough as $f_i^\sigma((F_i^\sigma)^{-1}(b)) = f_i(F_i^{-1}(b))/\sigma$ and $(f_i^\sigma)'(F_i^{-1}(b)) = f_i(F_i^{-1}(b))/\sigma^2$ and $(\widetilde{H}_i^1)^\sigma$ is bounded between zero and one. The first term is zero as $H_i^\sigma(w_i+p_i)$ converges to either zero or $F_i^\sigma(w_i - z_i^* + p_i)$ (shown by CDK in the supplementary material). So $\Big((\widetilde{H}_i^1)^\sigma((F_i^\sigma)^{-1}(b_i)+z_i^*-p_i)\Big)'$ is either converges to zero or

$$\sum_{j\in N_1\setminus\{i\}}\prod_{k\in N_1\setminus\{i,j\}}F_k^\sigma((F_i^\sigma)^{-1}(b_i)+z_i^*-p_i-z_k^*+p_k)f_j^\sigma((F_i^\sigma)^{-1}(b_i)+z_i^*-p_i-z_j^*+p_j),$$

which converges to zero since for any $q\in\mathbb{R}$,

$$F_k^\sigma((F_i^\sigma)^{-1}(b_i)+q) = F_k\left(\frac{\sigma F_i^{-1}(b_i)+x}{\sigma}\right)\underset{\sigma\to\infty}{\longrightarrow} F_k(F_i^{-1}(b_i)),$$

which is finite, and

$$f_j^\sigma((F_i^\sigma)^{-1}(b_i)+q) = \frac{1}{\sigma}f_i\left(\frac{\sigma F_i^{-1}(b_i)+x}{\sigma}\right)\underset{\sigma\to\infty}{\longrightarrow} 0.$$

The last term is non-negative as $g_i'(z_i^*+(F_i^\sigma)^{-1}(b_i)-(F_i^\sigma)^{-1}(r_i))$ converges to $g_i'(\overline{z}_i)$ when σ explodes by Assumption 3 (either $g_i'(\overline{z}_i)\le 0$ or $\overline{z}_i=\infty$).

Putting all things together, we have $\mathcal{K}^\sigma(u)$ converges to zero, $\mathcal{K}^\sigma(u)'$ is non-positive and $\mathcal{K}^\sigma(u)''$ is non-negative when σ is large enough. Thus,

$$\begin{aligned}\lim_{\sigma\to\infty}(\log J(u))'' &= \lim_{\sigma\to\infty}\frac{((H_*^1)^\sigma(u)-\mathcal{K}^\sigma(u))''((H_*^1)^\sigma(u)-\mathcal{K}^\sigma(u))-((H_*^1)^\sigma(u)'-\mathcal{K}^\sigma(u)')^2}{((H_*^1)^\sigma(u)-\mathcal{K}^\sigma(u))^2}, \\ &\le \lim_{\sigma\to\infty}\frac{(H_*^1)^\sigma(u)''(H_*^1)^\sigma(u)-((H_*^1)^\sigma(u)')^2}{((H_*^1)^\sigma(u))^2}, \\ &= \lim_{\sigma\to\infty}(\log H_*^1(u))'',\end{aligned}$$

which is negative. Thus,

$$\begin{aligned}\Pr(\max\{u_1, Y_i\} < t) &= J(t)\widetilde{H}_i^2(t), \\ &= J(t) \prod_{j \in N_2 \setminus \{i\}} H_j(t + p_j)\end{aligned}$$

is a product of two log-concave functions, and is thus log-concave. Log-supermodular of demand follows the same arguments before.

C Pricing Equilibrium with Symmetric Sellers

Under the fixed payment, the price equilibrium in Section 6 is that all sellers in block 1 set an identical price at p_1^f and all sellers in block 2 set an identical price at p_2^f.

Suppose not. There exists seller $j \in N_1$ such that $p_j \neq p_1^f$ and all other sellers following the equilibrium price. Let $\boldsymbol{p}$ be the corresponding vector of prices, then Theorem 2 implies

$$\frac{1}{p_1^f - c} = -\frac{dD_i^1(p, \boldsymbol{p}_{-i})/dp}{D_i^1(p, \boldsymbol{p}_{-i})}\bigg|_{p=p_1^f}, \tag{30}$$

for any $i \in N_1 \setminus \{j\}$ and

$$\frac{1}{p_j - c} = -\frac{dD_j^1(p, \boldsymbol{p}_{-j})/dp}{D_j^1(p, \boldsymbol{p}_{-j})}\bigg|_{p=p_j}. \tag{31}$$

However, (30) and (31) cannot hold simultaneously. This is because by Theorem 2, for any $i \neq j$ and $k \in \{1, 2\}$, $\partial^2 \log D_i^k(\boldsymbol{p})/\partial p_i \partial p_j > 0$. If $p_j > p_1^f$ and $p_i = p_1^f$ for any $i \in N_1 \setminus \{j\}$, then

$$-\frac{dD_i^1(p, \boldsymbol{p}_{-i})/dp}{D_i^1(p, \boldsymbol{p}_{-i})} < -\frac{dD_j^1(p, \boldsymbol{p}_{-j})/dp}{D_j^1(p, \boldsymbol{p}_{-j})},$$

for any p, since $\boldsymbol{p}_{-i}$ contains $p_j > p_1^f$, while $\boldsymbol{p}_{-j}$ only contains p_1^f. Since the right-hand sides of both FOCs are increasing in p and $1/(p - c)$ is decreasing in p, the solution of the FOCs implies $p_j < p_1^f$. A contradiction.

Similarly, $p_j > p_1^f$ can never hold. So all sellers in block 1 charge the same price. Sellers in block 2 charge the same price by the same reason. The uniqueness of the pricing equilibrium under per-transaction follows from the same arguments and is thus omitted.

D Proof of Theorem 6

Theorem 6 is proved by separating consumer surplus in (20) into two parts. One is independent of product position and the other depends on product position. The latter one turns out to be (22).

Step 1: reformulate the first sum in (20): I use $\wedge$ to denote the intersection of sets and $\vee$ to denote the union of sets. By definition of z_i^* in (1), $s_i = \mathbb{E}[(Z_i - z_i^*)^+]$. The first sum in (20) can be rewritten as:

$$\begin{aligned}
& \sum_{i \in N_1} \mathbb{E}\left[\mathbb{1}_{\{\{X_i < W_i - p_i\} \wedge \{Y_0 < V_i + Z_i - p_i\}\}}(V_i + Z_i - p_i) - \mathbb{1}_{\{X_i < V_i + z_i^* - p_i\}}\left(Z_i - z_i^*\right)^+\right] \\
= & \sum_{i \in N_1} \mathbb{E}\left[\left(\mathbb{1}_{\{\{\max\{X_i, Y_0\} < W_i - p_i\}} + \mathbb{1}_{\{X_i < W_i - p_i \le Y_0 < V_i + Z_i - p_i\}}\right)(V_i + Z_i - p_i)\right. \\
& \left. - \mathbb{1}_{\{X_i < W_i - p_i\}}\left(Z_i - z_i^*\right)^+\right], \\
= & \sum_{i \in N_1} \mathbb{E}\left[\left(\mathbb{1}_{\{\{\max\{X_i, Y_0\} < W_i - p_i\}} + \mathbb{1}_{\{X_i < W_i - p_i \le Y_0 < V_i + Z_i - p_i\}}\right)(V_i + Z_i - p_i)\right. \\
& \left. - \left(\mathbb{1}_{\{\max\{X_i, Y_0\} < W_i - p_i\}} + \mathbb{1}_{\{\{X_i < W_i - p_i\} \wedge \{Y_0 \ge W_i - p_i\}\}}\right)\left(Z_i - z_i^*\right)^+\right], \\
= & \sum_{i \in N_1} \mathbb{E}\left[\mathbb{1}_{\{\{\max\{X_i, Y_0\} < W_i - p_i\}}(W_i - p_i)\right] + \sum_{i \in N_1} \mathbb{E}\left[\mathbb{1}_{\{X_i < W_i - p_i \le Y_0 < V_i + Z_i - p_i\}}(V_i + Z_i - p_i)\right] \\
& - \sum_{i \in N_1} \mathbb{E}\left[\mathbb{1}_{\{X_i < W_i - p_i \le Y_0\}}\left(Z_i - z_i^*\right)^+\right], \qquad (32)
\end{aligned}$$

where the first equality comes from:

$$\begin{aligned}
& \{X_i < W_i - p_i\} \wedge \{Y_0 < V_i + Z_i - p_i\} \\
= & \{\max\{X_i, Y_0\} < W_i - p_i\} \vee \{X_i < W_i - p_i \le Y_0 < V_i + Z_i - p_i\},
\end{aligned}$$

and $\mathbb{1}_{\{X_i < V_i + z_i^* - p_i\}}\left(Z_i - z_i^*\right)^+ = \mathbb{1}_{\{X_i < W_i - p_i\}}\left(Z_i - z_i^*\right)^+$.[27]

Step 2: reformulate the second sum in (20): Similarly, by $s_i = \mathbb{E}[\max\{0, z_i - z_i^*\}]$, the

[27] If $z_i \le z_i^*$, both terms equal zero. If $z_i > z_i^*$, $w_i = v_i + z_i^*$, the indicators are the same. The logic is borrowed from CDK (2018).

second sum in (20) can be rewritten as:

$$\begin{aligned}
&\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(V_i+Z_i-p_i)-\mathbb{1}_{\{\max\{U_1,Y_i\}<V_i+z_i^*-p_i\}}\left(Z_i-z_i^*\right)^+\right]\\
=&\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(W_i-p_i)\right]\\
=&\sum_{i\in N_2}\mathbb{E}\left[\left(\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\}}-\mathbb{1}_{\{\{X_0\le W_i-p_i\le U_1\}\wedge\{Y_i<W_i-p_i\}\}}\right)(W_i-p_i)\right], \qquad (33)
\end{aligned}$$

where the first equality uses the fact that

$$\mathbb{1}_{\{\max\{U_1,Y_i\}<V_i+z_i^*-p_i\}}\left(Z_i-z_i^*\right)^+=\mathbb{1}_{\{\{\max\{U_1,Y_i\}<W_i-p_i\}}\left(Z_i-z_i^*\right)^+,$$

and the second equality uses the fact that $u_1\ge x_0$ for any realization of $(\boldsymbol{v}_k,\boldsymbol{z}_k)_{k=1}^2$ and $\boldsymbol{p}$.

Step 3: rearrangement: Since (20) is equal to $\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_0\}<u_0\}}\right]u_0$ plus (32) plus (33), consumer surplus can be expressed as $W=\overline{W}-W^L$, where

$$\begin{aligned}
W^L\ :=&\ -\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0<V_i+Z_i-p_i\}}(V_i+Z_i-p_i)\right]\\
&+\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}\left(Z_i-z_i^*\right)^+\right]+\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\le U_1\}}(W_i-p_i)\right],\\
\overline{W}\ :=&\ \mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_0\}<u_0\}}\right]u_0+\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{\{\max\{X_i,Y_0\}<W_i-p_i\}}(W_i-p_i)\right],\\
&+\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\}}(W_i-p_i)\right]\\
=&\ \mathbb{E}\left[\max\{u_0,X_0,Y_0\}\right].
\end{aligned}$$

We can further simplify W^L into an expression of X_0, Y_0 and U_1:

$$\begin{aligned}
W^L\ =&\ -\sum_{i\in N_1}\mathbb{E}\left[\left(\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}-\mathbb{1}_{\{X_i<W_i-p_i\le V_i+Z_i-p_i\le Y_0\}}\right)(V_i+Z_i-p_i)\right]\\
&+\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}\left(Z_i-z_i^*\right)^+\right]+\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\le U_1\}}(W_i-p_i)\right],\\
=&\ \sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le V_i+Z_i-p_i\le Y_0\}}(V_i+Z_i-p_i)\right]-\sum_{i\in N_1}\mathbb{E}\left[\mathbb{1}_{\{X_i<W_i-p_i\le Y_0\}}(W_i-p_i)\right]\\
&-\left(\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{U_1,Y_i\}<W_i-p_i\}}(W_i-p_i)\right]-\sum_{i\in N_2}\mathbb{E}\left[\mathbb{1}_{\{\max\{X_0,Y_i\}<W_i-p_i\}}(W_i-p_i)\right]\right),\\
=&\ \mathbb{E}[\mathbb{1}_{\{U_1\le Y_0\}}U_1]-\mathbb{E}[\mathbb{1}_{\{X_0\le Y_0\}}X_0]-\left(\mathbb{E}[\mathbb{1}_{\{U_1\le Y_0\}}Y_0]-\mathbb{E}[\mathbb{1}_{\{X_0\le Y_0\}}Y_0]\right),\\
=&\ \mathbb{E}[\mathbb{1}_{\{X_0\le Y_0\}}(Y_0-X_0)]-\mathbb{E}[\mathbb{1}_{\{U_1\le Y_0\}}(Y_0-U_1)],\\
=&\ \mathbb{E}\left[(Y_0-X_0)^+\right]-\mathbb{E}\left[(Y_0-U_1)^+\right], \qquad (34)
\end{aligned}$$

where the first equality comes from $W_i - p_i = V_i + Z_i - p_i - (Z_i - z_i^*)^+$ and the fact that

$$\{\max\{X_0, Y_i\} < W_i - p_i\} = \{\max\{U_1, Y_i\} < W_i - p_i\} \wedge \{\max\{X_0, Y_i\} < W_i - p_i \leq U_1\},$$

as $u_1 > x_0$ under any realization, and the third equality follows from interchanging sum and expectation in a linear operator and the definitions of X_0, Y_0 and U_1.