LORENZ MAP, INEQUALITY ORDERING AND CURVES BASED ON MULTIDIMENSIONAL REARRANGEMENTS

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Abstract. We propose a multivariate extension of the Lorenz curve based on multivariate rearrangements of optimal transport theory. We define a vector Lorenz map as the integral of the vector quantile map associated to a multivariate resource allocation. Each component of the Lorenz map is the cumulative share of each resource, as in the traditional univariate case. The pointwise ordering of such Lorenz maps defines a new multivariate majorization order. We define a multi-attribute Gini index and complete ordering based on the Lorenz map. We propose the level sets of an Inverse Lorenz Function as a practical tool to visualize and compare inequality in two dimensions, and apply it to income-wealth inequality in the United States between 1989 and 2019.

Keywords: Multidimensional inequality, Lorenz curve, Gini index, vector quantiles, optimal transport, majorization

Introduction

The Lorenz curve, first proposed in Lorenz [1905], is a compelling visual and simple quantification tool for the analysis of dispersion in univariate distributions. It allows easy visualization of dispersion from the curvature of a convex curve and its distance from the diagonal. It also allows easy computation, reading off the curve, as it were, of the share of a resource held by the top or bottom of the allocation distribution for that resource. These two features of the Lorenz curve account for much of its enduring appeal among practitioners, policy analysts and policy makers. This appeal is further enhanced by the relation between majorization and the pointwise ordering of Lorenz curves, which provides a way to visualize inequality comparisons between

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populations and within a given population between time periods. Comprehensive accounts are given in Marshall et al. [2011] and Arnold and Sarabia [2018].

The appealing properties of the Lorenz curve are well captured by formulation given in Gastwirth [1971]. In that formulation, the Lorenz curve is the graph of the Lorenz map, and the latter is the cumulative share of individuals below a given rank in the distribution, i.e., the normalized integral of the quantile function. The relation to majorization and the convex order follows immediately, as shown in Section C of Marshall et al. [2011]. As pointed out by Arnold [2008], this makes the Lorenz ordering an uncontroversial partial inequality ordering of univariate distributions, and most open questions concern the higher dimensional case.

Dispersion in multivariate distributions is not adequately described by the Lorenz curve of each marginal, and a genuinely multidimensional approach is needed. Even for utilitarian welfare inequality, Atkinson and Bourguignon [1982] motivate the need for the multi-dimensional approach initiated by Fisher [1956]. More generally, the literature on multi-dimensional inequality of outcomes and its measurement is vast, as evidenced by many recent surveys, see for instance Decancq and Lugo [2012], Aaberge and Brandolini [2014], Andreoli and Zoli [2020]. We only discuss it insofar as it relates to the Lorenz curve.

Multivariate extensions have been proposed for the Lorenz curve, most notably Taguchi (1972a,1972b), Arnold [1983], and Koshevoy and Mosler (1996,1999). They are reviewed in Marshall et al. [2011] and Sarabia and Jorda [2014] and discussed in more details in the next subsection. We contribute to this literature with a vector version of the Gastwirth [1971] formulation of the Lorenz map, based on a multivariate reordering of the data, with optimal transport rearrangement theory. With this reordering, we emulate the features of the Lorenz curve that most contributed to its success: we derive a tool to compute the share of each resource held by the subset of the population below any given rank, and we provide tools to visualize and compare inequality in multivariate distributions.

The solution of quadratic optimal transport problems, see Villani (2003, 2009), can be seen as rearrangements of random vectors. For a given random vector $X \in \mathbb{R}^d$, interpreted as the resource allocation in the population, there is a uniform random vector $U \in [0,1]^d$ and a gradient $\nabla\psi_X$ of a convex function such that $X = \nabla\psi_X(U)$ almost surely. The vector U can be interpreted as a vector of multivariate ranks of individuals in the population, and the Lorenz map at $r \in [0,1]^d$ can be defined as the cumulative vector share of resources held by individuals with vector ranks below r, componentwise. Equivalently, we define the Lorenz map as the integral of the vector quantile of Chernozhukov et al. [2017]. Correspondingly, we define a Lorenz ordering from the pointwise dominance of Lorenz maps. Since the latter are integrated quantiles, it is a multivariate majorization ordering, based on optimal transport reorderings. The Lorenz map and Lorenz ordering defined here reduce to the traditional Lorenz map and ordering in the univariate case. We construct a Gini index based on the distance to the Lorenz map of perfect equality, which preserves the Lorenz ordering and also reduces to the traditional Gini in the univariate case. Our multivariate Lorenz map also shares with the traditional Lorenz map the property that it characterizes the distribution of the random vector it is constructed from.

For our vizualization device, we define an *Inverse Lorenz Function* at a given vector of resource shares as the share of the population that cumulatively holds those shares. It is characterized by the cumulative distribution function of the image of a uniform random vector by the Lorenz map. Hence, it is a cdf by construction, like the univariate inverse Lorenz function. In two dimensions, the α-level sets of this cdf, which we call α-Lorenz curves, are non crossing downward sloping curves that shift to the south-west when inequality increases, as defined by the Lorenz ordering. Because this compelling visual tool for the comparison of multi-attribute inequality is one of the major objectives we pursue with this work, we report all our results in the bivariate case, and propose an illustration to the analysis of income-wealth inequality in the United States between 1989 and 2019.

Closely related work. The original definition of the Lorenz curve is implicit: The Lorenz curve L_Y of allocation Y is defined by $F_Y(y) = q$ and $L_Y(q) := \mathbb{E}[Y \; 1\{Y \leq y\}]$. Arnold [1983] extends the implicit formulation to multiple univariate ranks $F_1(x_1) = p_1$ and $F_2(x_2) = p_2$, and defines a scalar valued Lorenz map $L_X(p_1, p_2)$. Koshevoy and Mosler [1996] keep $F_X(x) = p$ for random vector X, but extend univariate shares to vector shares. The Lorenz zonoid of Koshevoy and Mosler [1996] relates any function $\phi : \mathbb{R}^d \to [0,1]$ to the vector $(\mathbb{E}[\phi(X)], \mathbb{E}[X_1 \phi(X)], \dots, \mathbb{E}[X_d \phi(X)])$. Hence, to a proportion of the population $\mathbb{E}[\phi(X)]$, it associates their share of each of the resources. The Lorenz zonoid is difficult to compute in general. The choice $\phi_r(\cdot) := 1\{\nabla\psi^*(\cdot) \leq r\}$, for each rank $r \in [0,1]^d$, where $\nabla\psi^*$ is the vector rank defined in Section 1 provides an easily computable selection of the Lorenz zonoid, that is closely related to our proposal. Koshevoy and Mosler [1999] propose an Inverse Lorenz Function that can be interpreted as the maximum share of the population that holds a given vector share of resources, but it is the solution of a complex optimization problem. Grothe et al. [2021] define an Inverse Lorenz Function as the probability that for each i, a share z_i of resource i is collectively held by individuals whose rank in the distribution of resource i is below $F_i(X_i)$, which they characterize with a clever coupling argument. Multivariate Lorenz dominance is defined and discussed in Koshevoy [1995], Koshevoy and Mosler [2007], Banerjee [2010]. See Arnold and Sarabia [2018] for a comprehensive account. Gajdos and Weymark [2005] discuss multivariate versions of the Gini index based exclusively on functionals of individually reordered marginals, as opposed to the multivariate reorderings proposed here. Koshevoy and Mosler [1997], Banerjee [2016], Grothe et al. [2021], Mosler [2013], Sarabia and Jorda [2020] also propose multivariate Gini indices based on multivariate Lorenz curve proposals. Finally, optimal transport rearrangements are used to define multivariate comonotonicity and quantiles in Ekeland et al. [2012], Galichon and Henry [2012] and Puccetti and Scarsini [2010].

Notation. Let λ be the uniform distribution on $[0,1]^d$. The indicator function is denoted $1\{\cdot\}$. We call $\mathbb{R}^d_+$ the set $\{z \in \mathbb{R}^d : z \geq 0\}$, where $\geq$ denotes the componentwise inequality. We use Y to denote a random variable, and $X = (X_1, X_2)$ a random vector on $\mathbb{R}^d$. Let P_X stand for the distribution of the random vector X. Following Villani [2003], we let $g_{\#}\nu$ denote the *image measure* (or *push-forward*) of a measure ν by a measurable map $g : \mathbb{R}^d \to \mathbb{R}^d$. Explicitly, for any Borel set A, $g_{\#}\nu(A) := \nu(g^{-1}(A))$. The generalized inverse of a function f is denoted f^-. The symbol ∇ denotes the gradient, and D the Jacobian. The convex conjugate of a convex lower semicontinuous function ψ is denoted ψ^*. We use the standard convention of calling weak monotonicity non-decreasing or non-increasing, as the case may be.

1. Vector Lorenz Maps and Curves

We propose a method to compare, measure and visualize inequality of allocations of two resources. A resource allocation is modeled as a random vector $X = (X_1, X_2)$ with support $\mathcal{X} \subseteq \mathbb{R}^2_+$. The distribution P_X of X is assumed to have unit mean, to avoid issues of normalization or change of units of measurement. Most of the definitions and properties we present here are valid in higher dimensions. However, the crucial visualization aspect is lost, so we don't pursue a greater level of generality.

1.1. **Lorenz map.**

The Gastwirth [1971] formulation of the Lorenz curve for a scalar allocation relies on the quantile function. Let Y be a scalar allocation with mean 1 and cumulative distribution function F_Y. Assume Y has an absolutely continuous distribution, and quantile function F_Y^{-1}. Then $V := F_Y(Y) \sim U[0,1]$ is the rank of individuals in the distribution. The value $L_Y(q)$ at rank q of the Lorenz curve associated with allocation Y is the cumulative share of individuals with rank $V \leq q$. Hence

$$L_Y(q) \;=\; \mathbb{E}[F_Y^{-1}(V)\,1\{V \leq q\}] \;=\; \int_0^q F_Y^{-1}(v)\,dv. \tag{1.1}$$

To analyze inequality in allocation X, we can first look at inequality in each marginal allocation X_1 and X_2, using the Lorenz curves $L_1 := L_{X_1}$ and $L_2 := L_{X_2}$. However, this strategy disregards the effect of dependence. The latter is relevant to inequality, as can be trivially illustrated by the fact that for given wealth and income marginal allocations, the co-monotonic allocation (the wealthier individuals have higher incomes) is more unequal than the admittedly unrealistic counter-monotonic allocation (the wealthier individuals have lower income).

To take dependence into account, we propose to emulate the Gastwirth [1971] formulation by measuring cumulative shares of both resources for all individuals, whose *vector rank* U is below $r = (r_1, r_2)$. The vector rank we adopt is a uniform random vector on $[0,1]^2$. It is obtained from the vector allocation X via a unique transformation analogous to the probability integral transform $V = F_Y(Y)$ in the scalar case described above. Its associated *vector quantile*, inverse of the vector rank, can be integrated over $[0, r_1] \times [0, r_2]$ to obtain a vector analogue of the Gastwirth [1971] formulation of the Lorenz curve.

We adopt the vector quantile and rank notions proposed in Chernozhukov et al. [2017], based on optimal transport theory. The vector quantile of random vector X is defined, to emulate the traditional quantile function, as the unique cyclically monotone function that pushes the uniform distribution on $[0,1]^2$ into the distribution of random vector X. Existence and uniqueness follow from Theorem 1 in McCann [1995], which extends the Brenier [1991] polar factorization theorem (see also Rachev and Rüschendorf [1990]) beyond finite second moments. We give a brief review of the theory of vector ranks and quantiles, and the underlying notions in optimal transport theory, in Appendix A.1. Here we only give the elements required to define our vector Lorenz map.

Definition 1 (Vector quantiles)**.** Let λ be the uniform distribution on $[0,1]^d$, and let P_X be the distribution of a random vector X on $\mathbb{R}^2$. There exists a convex function $\psi_X : [0,1]^2 \to \mathbb{R} \cup \{+\infty\}$ such that $(\nabla\psi_X)_\# \lambda = P_X$. The function $\nabla\psi_X$ exists and is unique, λ-almost everywhere. We define the vector quantile of X as the function $\nabla\psi_X$, and we call the function ψ_X a *potential* of X.

Existence and uniqueness in Definition 1 follow from Proposition 3 in Appendix A.1. In case $d = 1$, gradients of convex functions are nondecreasing functions, hence the vector quantile of Definition 1 reduces to the classical quantile function.

By construction, if a random vector U has uniform distribution on $[0,1]^2$, then the random vector $\nabla\psi_X(U)$ has distribution P_X. Take any point $r = (r_1, r_2) \in [0,1]^2$. The proportion of the resources held by individuals ranked below r is the cumulative integral of the vector quantile, as in the scalar case.

Definition 2 (Lorenz map)**.** Let $U \sim U[0,1]^2$. The *Lorenz map* of an allocation X is the vector-valued function $\mathcal{L}_X : [0,1]^2 \to [0,1]^2$ defined by

$$\mathcal{L}_X(r) \;=\; \mathbb{E}[\nabla\psi_X(U)\, 1\{U \le r\}] \;=\; \begin{bmatrix} \displaystyle\int_{[0,r_1]\times[0,r_2]} \frac{\partial\psi_X(u_1,u_2)}{\partial u_1}\, du_1\, du_2 \\ \displaystyle\int_{[0,r_1]\times[0,r_2]} \frac{\partial\psi_X(u_1,u_2)}{\partial u_2}\, du_1\, du_2 \end{bmatrix}. \tag{1.2}$$

As stated in Proposition 3 in Appendix A.1, when X has absolutely continuous distribution P_X, its quantile function is P_X-almost everywhere invertible with inverse $\nabla\psi_X^*$, where ψ_X^* is the convex conjugate of ψ_X (see the introduction for notation and definitions from convex analysis). In that case, the transformation $U = \nabla\psi_X^*(X)$ is the vector analogue of the integral probability transform $V = F_Y(Y)$ discussed above. The random vector $U = \nabla\psi_X^*(X) \sim U[0,1]^2$ is the vector rank of the individual with endowment X, in the terminology of Chernozhukov et al. [2017]. In this case, the Lorenz map of Definition 2 can be rewritten:

$$\mathcal{L}_X(r) \;=\; \mathbb{E}[X\ 1\{\nabla\psi_X^*(X) \leq r\}].$$

This clarifies the interpretation of $\mathcal{L}_X(r)$ as the cumulative endowment X of all individuals with vector rank $\nabla\psi_X^*(X)$ below r.

We now explore three special endowments and compute the corresponding Lorenz maps. First, we consider the case, where all individuals are endowed with the same quantity of resources. We call the corresponding Lorenz map *egalitarian*.

Example 1 (Egalitarian)**.** If $X = (1,1)$ almost surely, then $\nabla\psi(r) = (1,1)$ for all $r \in [0,1]^2$, and $\mathcal{L}_X(r) = (r_1r_2, r_1r_2)$. The image of $\mathcal{L}_X$ is the diagonal in $[0,1]^2$.

Second, we check that our definition is compatible with scalar definitions when both resources are independently distributed.

Example 2 (Independent Resources)**.** Let the components X_1 and X_2 of X be independent with univariate Lorenz curves L_1 and L_2 respectively. Then, the Lorenz map is $\mathcal{L}_X(r_1,r_2) = (r_2L_1(r_1), r_1L_2(r_2))$. When $r_1 = 1$, the Lorenz map takes values $\mathcal{L}_X(1,r_2) = (r_2, L_2(r_2))$. That is, the image of $\{(1,r_2) : 0 \leq r_2 \leq 1\}$ under $\mathcal{L}_X$ is the Lorenz curve L_2 of the second resource X_2 (and symmetrically when $r_2 = 1$).

Third, we derive the Lorenz map in the case of endowments $X = (X_1, X_2)$ with identical components, i.e., $X_1 = X_2$ almost surely.

Example 3 (Comonotonic Resources)**.** Let the components X_1 and X_2 of the endowment X be almost surely equal. Then, X_1 and X_2 have identical distributions. The optimal transport map from the uniform distribution on $[0,1]^2$ to the distribution of $X = (X_1, X_2)$ is then $(u_1,u_2) \mapsto (\psi'(u_1+u_2), \psi'(u_1+u_2))$, where $z \mapsto \psi'(z)$ is the optimal map from σ to the distribution of X_1, where $\sigma := (u_1,u_2) \mapsto (u_1+u_2)_\#\lambda$, which has density on $[0,2]$ given by $1-|1-z|$. Each component of the Lorenz curve is then given by

$$\begin{aligned}\mathcal{L}_1(r_1,r_2)=\mathcal{L}_2(r_1,r_2) &= \int_0^{r_2}\int_0^{r_1}\psi'(u_1+u_2)du_1du_2\\ &= \int_0^{r_2}[\psi(u_2+r_1)-\psi(u_2)]du_2.\end{aligned}$$

In case X_1 and X_2 are uniformly distributed on $[0,2]$, the optimal transport map ψ' for $z<1$ is given by $\psi'(z)=z^2$, so that $\psi(z)=z^3/3$. We then have,

$$\begin{aligned}\mathcal{L}_1(r_1,r_2) \;=\; \mathcal{L}_2(r_1,r_2) &= \int_0^{r_2}\left[\frac{(u_2+r_1)^3}{3}-\frac{(u_2)^3}{3}\right]du_2\\ &= \frac{(r_2+r_1)^4}{12}-\frac{(r_2)^4}{12}-\frac{(r_1)^4}{12}\\ &= \frac{r_1^3r_2}{3}+\frac{r_1r_2^3}{3}+\frac{r_1^2r_2^2}{2},\end{aligned}$$

when $r_1+r_2\leq 1$, and $\mathcal{L}_1(r_1,r_2)=\mathcal{L}_2(r_1,r_2)=$

$$=\frac{2}{3}(r_1+r_2)^3-\frac{1}{12}(r_1+r_2)^4-(r_1+r_2)^2-\frac{r_1^4}{12}-\frac{r_2^4}{12}+\frac{2}{3}(r_1+r_2)-\frac{1}{6},$$

when $r_1+r_2>1$.

1.2. **Inverse Lorenz Function and α-Lorenz Curves.** In the scalar case discussed above, with absolutely continuous endowment distribution, the Lorenz curve L_Y defined in (1.1) is invertible and its inverse is equal to

$$L_Y^{-1}(y) \;=\; \int_0^{L_Y^{-1}(y)}dv \;=\; \int_0^1 1\{L_Y(v)\leq y\}\,dv \;=\; \mathbb{P}(L_Y(V)\leq y). \tag{1.3}$$

The scalar inverse Lorenz curve at y is therefore shown in (1.3) to be equal to the maximum proportion of the population with cumulative share of the resource equal to y. In the vector case, the right-hand side of (1.3) becomes

$$l_X(z) \;:=\; \mathbb{P}(\mathcal{L}_X(R)\leq z), \tag{1.4}$$

where $z=(z_1,z_2)\in[0,1]^2$, inequality $\leq$ is understood componentwise, and the probability is taken with respect to the uniform random vector R on $[0,1]^2$. The expression above is no longer the inverse of the Lorenz map $\mathcal{L}_X$, but it can still be interpreted as the share of the population with cumulative shares of both resources equal to $z = (z_1, z_2)$. This motivates the following definition.

Definition 3 (Inverse Lorenz Function)**.** (1) The Inverse Lorenz Function (ILF) of a bivariate random vector X is the function $l_X : [0,1]^2 \to [0,1]$ defined by (1.4). (2) We call α-Lorenz curves the curves $\mathcal{C}_X^\alpha = \{z \in [0,1]^2 : \, l_X(z) = \alpha\}$ for each $\alpha \in (0,1)$.

The α-Lorenz curves are the objects involved in our proposed multivariate inequality visualization technique. We will see how to compare the inequality of different endowments based on the shape and relative positions of their respective α-Lorenz curves. First, we revisit the three special cases of the previous section.

Example 1 continued [Egalitarian] The inverse Lorenz curve $l_X(z)$ of the endowment $X = (1,1)$ almost surely is 0 when $z_1 z_2 = 0$ and

$$\begin{aligned} l_X(z) &= \mathbb{P}(R_1 R_2 \leq \min\{z_1, z_2\}) \\ &= \min\{z_1, z_2\}(1 - \log(\min\{z_1, z_2\})), \text{ for } (z_1, z_2) \in (0,1]^2. \end{aligned}$$

Example 2 continued [Independent Resources] The inverse Lorenz curve $l_X(z)$ of endowment X with independent components is

$$\begin{aligned} l_X(z) &= \mathbb{P}(\, R_2 L_1(R_1) \leq z_1, R_1 L_2(R_2) \leq z_2) \\ &= \int_0^1 \min\left\{ \frac{z_1}{L_1(u)}, l_2\left(\frac{z_2}{u}\right)\right\} du, \end{aligned}$$

where l_2 is the univariate inverse Lorenz Function of X_2.

Example 3 continued [Comonotonic Resources] The inverse Lorenz curve $l_X(z)$ of endowment $X = (X_1, X_2)$ with $X_2 = X_1$ almost surely, is

$$\begin{aligned} l_X(z) &= \mathbb{P}(\mathcal{L}_1(R) \leq z_1, \mathcal{L}_2(R) \leq z_2) \\ &= \mathbb{P}(\mathcal{L}_1(R) \leq \min(z_1, z_2)) \\ &= h(\min(z_1, z_2)), \end{aligned}$$

where $R \sim U[0,1]^2$, $\mathcal{L}_j(z)$ is the j-th component, $j = 1, 2$, of $\mathcal{L}_X(r)$, and h is the distribution function of $\mathcal{L}_1(R)$. By definition, the α-Lorenz curves $\mathcal{C}_X^\alpha$ are the curves with equation $h(\min(z_1, z_2)) = \alpha$.

Figure 1 shows the egalitarian α-Lorenz curve compared to the comonotonic and independent cases, with lognormal marginals, and $\alpha = 0.75$.

1.3. **Properties of Lorenz maps, and α-Lorenz curves.** The traditional scalar Lorenz curve enjoys many well-known properties that justify its interpretation as an inequality measurement, comparison and visualization tool. We emulate many of these properties and provide new ones to justify the interpretation of our Lorenz map, Inverse Lorenz Function and α-Lorenz curves as multi-attribute inequality measurement, comparison and visualization tools.

First, the Lorenz map characterizes the distribution of the endowment it's associated with.

Property 1. The Lorenz map $\mathcal{L}_X$ characterizes the distribution of X in the sense that X and X' are identically distributed if and only if $\mathcal{L}_X = \mathcal{L}_{X'}$.

The Lorenz map is a map from $[0,1]^2$ to $[0,1]^2$. Hence, unlike the traditional scalar Lorenz curve, it cannot be a cdf. However, the Inverse Lorenz Function trivially is a cdf by construction.

Property 2. The Inverse Lorenz Function is the cumulative distribution function of a random vector on $[0,1]^2$.

In the univariate case, the Lorenz curve of the uniform endowment $Y = 1$ almost surely, is $L_Y(q) = q$, which is sometimes called the egalitarian line. The Lorenz curve of any other endowment $Y \geq 0$ is below the egalitarian line, i.e., $L_Y(q) \leq q$, for all $q \in [0,1]$. We show that the egalitarian Lorenz map and the egalitarian Inverse Lorenz Function provide similar bounds in the multi-attribute case. We restrict attention to endowments with components that display a form of positive association defined below.

Assumption 1. *The vector quantile $\nabla\psi_X$ of X is such that*

$$\mathbb{E}\left[\frac{\partial\psi_X}{\partial u_1}(U_1,U_2)\,\middle|\, U_2 = u_2\right], \quad 0 \leq u_2 \leq 1, \ \textit{and}$$
$$\mathbb{E}\left[\frac{\partial\psi_X}{\partial u_2}(U_1,U_2)\,\middle|\, U_1 = u_1\right], \quad 0 \leq u_1 \leq 1,$$

are monotonically increasing as functions of u_2 and u_1, respectively, where the vector (U_1, U_2) is uniformly distributed on $[0, 1]^2$.

This assumption imposes a type of positive dependence between X_1 and X_2 through their ranks. More precisely, Assumption 1 imposes a form of *positive regression dependence*, as in Lehmann [1966], between one resource and the other's rank. A sufficient condition for Assumption 1 is supermodularity of the potential function ψ_X of endowment X, as shown in Lemma 1 below. We also show in Lemma 1, that supermodularity of the potential function ψ_X also implies positive quadrant dependence of the two components X_1 and X_2 of X i.e., $\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) \geq \mathbb{P}(X_1 \leq x_1)\mathbb{P}(X_2 \leq x_2)$, for all $x_1, x_2 \in \mathcal{X}$, see Lehmann [1966].

Lemma 1 (Supermodular potential)**.** Suppose X has a supermodular potential function, i.e., $\mathbb{P}(\partial^2 \psi_X(U)/\partial u_1 \partial u_2 \geq 0) = 1$, with $U \sim U[0, 1]^2$. Then Assumption 1 holds, and X_1 and X_2 are positive quadrant dependent.

For endowments satisfying Assumption 1, we show that egalitarian Lorenz map and Inverse Lorenz Function serve as bounds.

Property 3. The Lorenz map of any endowment X satisfying Assumption 1 is component-wise dominated by the egalitarian Lorenz map of Example 1. Moreover, the Inverse Lorenz Function of endowment X is bounded below by the egalitarian ILF.

Without Assumption 1, some allocations may have a Lorenz map that is component-wise larger than the egalitarian Lorenz map for some ranks. This apparent departure from properties of the scalar Lorenz curve is due to the restricted notion of egalitarianism employed here. Other notions of egalitarianism will produce Lorenz maps that are not dominated by $(r_1 r_2, r_1 r_2)$. We discuss the specific example of income egalitarianism, in the terminology of Kolm [1977].

Suppose the population has heterogeneous marginal rates of substitution between the two resources. If the allocation is Pareto efficient, the implicit relative price p of good two is equal to the marginal rate of substitution of each individual at their allocation. Then, an egalitarian allocation is one in which all individuals have the same total budget $X_1 + pX_2$, not necessarily the same endowment (X_1, X_2).

To illustrate the point, consider the potential $\psi_X(u) = (u_1 - u_2)^2/2 + u_1 + u_2$. It corresponds to an endowment X, whose distribution is supported on the line $X_1 + X_2 = 2$, as discussed. Calculating the Lorenz map, we obtain

$$
\begin{aligned}
\mathcal{L}_1(r) &= \int_0^{r_2}\int_0^{r_1} \frac{\partial \psi_X}{\partial u_1}(u_1, u_2) du_1 du_2 \\
&= \int_0^{r_2}\int_0^{r_1} (u_1 - u_2 + 1) du_1 du_2 \\
&= \int_0^{r_2} [r_1^2/2 - u_2 r_1 + r_1] du_1 du_2 \\
&= r_2 r_1^2/2 - r_2^2 r_1/2 + r_2 r_1 = \frac{1}{2} r_1 r_2 (r_1 - r_2) + r_1 r_2.
\end{aligned}
$$

Symmetrically, $\mathcal{L}_2(r) = \frac{1}{2} r_1 r_2 (r_2 - r_1) + r_1 r_2$. Notice, in particular, that $\mathcal{L}_1(r) > r_1 r_2$ in the region where $r_1 > r_2$.

If the implicit relative price of resource 2 is 1, endowment X is an egalitarian endowment, since all individuals have equal budgets. However, this endowment does not satisfy Assumption 1 and its Lorenz map is not dominated by $(r_1 r_2, r_1 r_2)$ as we have shown.

Visually, inequality can be assessed by the departure of α-Lorenz curves from their egalitarian analogues. This visual comparison is facilitated by the fact that they are shaped like indifference curves.

Property 4. (i) The α-Lorenz curves $\mathcal{C}_X^\alpha$ are the level curves of a bivariate cdf, hence they are downward sloping, non decreasing in α and they do not cross. In addition, (ii) The α-Lorenz curves $\mathcal{C}_X^\alpha$ are convex if

$$
\frac{\partial l_X}{\partial z_2} \frac{\partial l_X^2}{\partial z_1 \partial z_2} - \frac{\partial l_X}{\partial z_1} \frac{\partial l_X^2}{\partial z_2^2} \geq 0.
$$

Figure 2 shows α-Lorenz curves for allocations with lognormally distributed marginals and Plackett copulas (as in Bonhomme and Robin [2009]) with dependence parameter $\kappa = 2$, for $\alpha = 0.5$ 0.75, 0.9 and 0.99. The markings on the horizontal and vertical scales indicate the level α of the egalitarian α-Lorenz curves for comparison.

2. Inequality orderings

2.1. **Lorenz ordering.** Consider two allocations X and $\tilde{X}$, with respective Lorenz maps $\mathcal{L}_X$ and $\mathcal{L}_{\tilde{X}}$. If $\mathcal{L}_X(r) \geq \mathcal{L}_{\tilde{X}}(r)$ for some vector rank r, the same proportion of the population with vector ranks below r commands a larger share of both resources in allocation X than in allocation $\tilde{X}$. If this is true for any vector rank r in $[0,1]^2$, then, we say that allocation $\tilde{X}$ is more unequal than allocation X.

Definition 4. An allocation $\tilde{X}$ is said to be more unequal in the Lorenz order than an allocation X, denoted $\tilde{X} \succcurlyeq_{\mathcal{L}} X$ if $\mathcal{L}_X(r) \geq \mathcal{L}_{\tilde{X}}(r)$ for all $r \in [0,1]^2$.

As a partial ordering based on cumulative sums of vector quantiles, the relation $\tilde{X} \succcurlyeq_{\mathcal{L}} X$ is a multivariate extension of the concept of majorization of Hardy et al. [1934]. It is different from existing multivariate notions of majorization reviewed in Marshall et al. [2011] and Arnold and Sarabia [2018], in that it relies on a multivariate reordering of the random vector allocation. The relation $\tilde{X} \succcurlyeq_{\mathcal{L}} X$ is equivalent to stochastic dominance of the random vector $\mathcal{L}_X(R)$, with $R \sim U[0,1]^2$, over $\mathcal{L}_{\tilde{X}}(R)$ (see Section 3.8 of Müller and Stoyan [2002]). As in the scalar case, it is a partial ordering. Under Assumption 1, the minimal element of $(\mathcal{X}, \preccurlyeq_{\mathcal{L}})$ is the egalitarian allocation.

Similarly, consider two allocations X and $\tilde{X}$, with respective Inverse Lorenz Functions l_X and $l_{\tilde{X}}$. If $l_X(z) \leq l_{\tilde{X}}(z)$ for some vector of shares z, a larger proportion of the population commands the same share of resources in allocation $\tilde{X}$ than in allocation X. If this is true for any vector z of resource shares in $[0,1]^2$, then, we say that allocation $\tilde{X}$ is more unequal than allocation X.

Definition 5. An allocation $\tilde{X}$ is said to be more unequal in the weak Lorenz order than an allocation X, denoted $\tilde{X} \succcurlyeq_l X$ if $l_{\tilde{X}}(z) \geq l_X(z)$ for all $z \in [0,1]^2$.

The relation $\tilde{X} \succcurlyeq_l X$ is equivalent to lower orthant dominance of the random vector $\mathcal{L}_X(R)$, with $R \sim U[0,1]^2$, over $\mathcal{L}_{\tilde{X}}(R)$ (see Section 3.8 of Müller and Stoyan [2002]). In the scalar case, the orderings of Definitions 4 and 5 both coincide with the traditional Lorenz ordering. In higher dimensions, however, the equivalence doesn't hold. Nonetheless, as the name indicates, the weak Lorenz inequality order of Definition 5 is weaker than the Lorenz order of Definition 4, as we show in the following proposition.

Proposition 1. An allocation $\tilde{X}$ is more unequal in the weak Lorenz order than an allocation X, i.e., $\tilde{X} \succcurlyeq_l X$ (Definition 5) if $\tilde{X}$ is more unequal in the Lorenz order, i.e., $\tilde{X} \preccurlyeq_{\mathcal{L}} X$ (Definition 4).

This inequality dominance of Definition 5 can be visualized on $[0,1]^2$ through the relative positions of α-Lorenz curves. Indeed, the α-Lorenz curves are, by definition, curves with equation $l_X(z) = \alpha$. In other words,

$$\mathcal{C}_X^\alpha = \{z \in [0,1]^2 : \; l_X(z) = \alpha\}.$$

Suppose $\tilde{X}$ is less unequal in the Lorenz order than X. For any $z \in \mathcal{C}_X^\alpha$, by definition of the Lorenz order, $l_{\tilde{X}}(z) \leq l_X(z) = \alpha$. So $z \in \mathcal{C}_{\tilde{X}}^{\tilde{\alpha}}$ with $\tilde{\alpha} \leq \alpha$. This can be visualized as a shift to the north-east of the α-Lorenz curves of the less unequal allocation X relative to the α-Lorenz curves of the less unequal allocation $\tilde{X}$.

Figure 3 compares α-Lorenz curves of 6 different allocations with lognormally distributed marginals and Plackett copulas, for $\alpha = 0.9$. The coefficient σ of the lognormal distributions takes value $\sigma = 1$ or 1.5, whereas the dependence parameter κ of the Plackett copula takes value $\kappa = 2$ (lower dependence) or 10 (higher dependence). In case of marginals with different σ, the asymmetry is reflected in the α-Lorenz curves. Moreover, other things equal, inequality increases with σ, which measures inequality in the marginals, and with κ, which measures dependence of the copula.

2.2. **Multivariate Gini Index.** Lorenz ordering is a partial ordering of multivariate distributions. For a complete inequality ordering, we also propose an extension of the classical Gini index to compare inequality in multi-attribute allocations.

The univariate Gini index can be interpreted as the average deviation from perfect equality. We emulate this interpretation and define a multivariate Gini based on an average deviation from the the egalitarian Lorenz map $r \mapsto (r_1 r_2, r_1 r_2)$. The deviation measure we propose is

$$\int_0^1 \int_0^1 \left[r_1 r_2 - \mathcal{L}_1\left(r_1, r_2\right)\right] dr_1 dr_2 + \int_0^1 \int_0^1 \left[r_1 r_2 - \mathcal{L}_2\left(r_1, r_2\right)\right] dr_1 dr_2, \tag{2.1}$$

where, as before, $\mathcal{L}_1$ and $\mathcal{L}_2$ are the first and second components of $\mathcal{L}_X$, respectively. Note that the expression can be extended to allow for weights to reflect relative importance of the two resources, without changing the analysis below. After normalization, (2.1) becomes

$$G(X) \;=\; 1 - 2\left(\int_{r\in[0,1]^2} \left[\mathcal{L}_1(r) + \mathcal{L}_2(r)\right]\, dr\right), \tag{2.2}$$

which yields the following definition.

Definition 6 (Gini Index)**.** (2.2) defines the Gini index of allocation X.

The traditional Gini index of a univariate allocation can also be characterized as a weighted sum of outcomes, where the weights are increasing linearly in the rank of the individual in the population. Hence, the negative of the Gini, seen as a social evaluation function displays inequality aversion by giving more weight to the outcomes of lower ranked individuals than to those of higher ranked ones. We show in the appendix that the same interpretation is valid for our multivariate Gini, which takes the form

$$G(X) \;=\; 2\int_0^1\!\!\int_0^1 \left[(u_1 + u_2 - u_1 u_2)\left(\frac{\partial \psi_X(u)}{\partial u_1} + \frac{\partial \psi_X(u)}{\partial u_2}\right)\right] du_1 du_2 \;-\; 3. \tag{2.3}$$

In Expression (2.3), $\partial\psi_X(u)/\partial u_1 + \partial\psi_X(u)/\partial u_2$ is the sum of the two resource allocations of the individual in the population with vector rank (u_1, u_2). Hence, the Gini index is indeed a weighted sum of outcomes, with weights $(u_1 + u_2 - u_1 u_2)$ increasing with the vector ranks (u_1, u_2). It is a genuinely multivariate extension in that the weighting scheme, hence the social evaluation of inequality, depends on multivariate ranks of individuals.

Examples 2 and 3 continued We compare Gini indices in the independent case with the perfect comonotonicity case, where X_1 and X_2 have the same (uniform on $[0, 2]$) marginal distributions. We verify (analytically for $r_1 + r_2 < 1$ and numerically using Wolfram for $r_1 + r_2 \geq 1$) that $\partial\psi_X(u)/\partial u_1 + \partial\psi_X(u)/\partial u_2$ is smaller in the comonotonic case, than in the independent case. Hence the Gini index (and the measure of inequality) is larger in the comonotonic case.

Example 4 (Anti-monotone Resources)**.** If we have $X_1 + X_2 = 2$ a.s., then $\nabla\psi_1(u) + \nabla\psi_2(u) = 2$ for almost all u, and we obtain $\mathcal{L}_1(r_1, r_2) + \mathcal{L}_2(r_1, r_2) = 2r_1r_2 \geq r_2L_1(r_1) + r_1L_2(r_2)$, so that, in particular, the Gini index in the anti-monotone case is the same as in the case of the egalitarian allocation, and both are smaller than the Gini of the allocation with independent resources.

The Gini index of Definition 6 is in $[0, 1]$ under Assumption 1. It equals 0 for the egalitarian allocation. It tends to 1, when the Lorenz map tends to 0 (extreme inequality). The Gini index of an allocation with independent components reduces to the average of classical scalar Ginis of both components. Like the classical scalar Gini index, it satisfies a set of desirable properties for an inequality index, which we call *axioms*. First, it doesn't depend on labeling of individuals in the population.

Axiom 1 (Anonimity)**.** *The Gini index of allocation X is a function of the distribution P_X of X.*

Second, the Gini index of Definition 6 preserves the Lorenz inequality ordering, in the sense that higher inequality according to $\succcurlyeq_{\mathcal{L}}$ implies a larger value of the Gini index.

Axiom 2 (Monotonicity)**.** *The Gini index is non decreasing in the Lorenz increasing inequality ordering. In other words, $X \preccurlyeq_{\mathcal{L}} \tilde{X}$ implies $G(X) \leq G(\tilde{X})$.*

Finally, it satisfies a multivariate extension of comonotonic independence proposed in Galichon and Henry [2012], as a generalization of the scalar requirement in Weymark [1981]. Two scalar endowments Y and $\tilde{Y}$ are comonotonic if they are both increasing transformations of the same uniform random variable V on $[0, 1]$. Suppose Y and $\tilde{Y}$ have respective cdfs F and $\tilde{F}$. Then Y and $\tilde{Y}$ are comonotonic if they have the same ranks $V := F(Y) = \tilde{F}(\tilde{Y})$.

Now, in case of random vectors X and $\tilde{X}$, comonotonicity is defined in the same way by the fact that X and $\tilde{X}$ have the same vector ranks. The following definition is due to Galichon and Henry [2012] and Ekeland et al. [2012], where it is called μ-comonotonicity[1].

Definition 7 (Vector comonotonicity)**.** Random vectors $X_1, \ldots, X_J$ on $\mathbb{R}^2$ are said to be *comonotonic* if there exists a uniform random vector U on $[0,1]^2$ such that $X_j = \nabla\psi_j(U)$ almost surely, where $\nabla\psi_j$ is the vector quantile of Definition 1 associated with the distribution of X_j, for each $j \leq J$.

The following axiom states that mixing two equally comonotonic endowments with the same third endowment, comonotonic with the first two, does not change the inequality ordering.

Axiom 3 (Comonotonic Independence)**.** *If X, $\tilde{X}$ and Z are comonotonic allocations, and $G(X) \leq G(\tilde{X})$, then, for all $\mu \in (0,1)$, $G(\mu X + (1-\mu)Z) \leq G(\mu\tilde{X} + (1-\mu)Z)$.*

The interpretation of Axiom 3 is very compelling for an inequality index. If individuals are ranked identically in endowments X and $\tilde{X}$, and $\tilde{X}$ is more unequal than X, then, adding to both X and $\tilde{X}$ a third endowment Z cannot reverse the inequality ordering, if Z ranks individuals as X and $\tilde{X}$ do.

We bring these statements together in the following lemma:

Proposition 2. The Gini index of Definition 6 satisfies Axioms 1 to 3.

Proposition 2 is not a characterization of our proposed Gini index, even when we extend the definition to allow for different weights given to the different resources. In other words, there may be other inequality indices that also satisfy Axioms 1-3.

Axiom 2 is the preservation of the multivariate majorization order we introduce, based on multivariate rearrangements. Our multivariate Gini index preserves this majorization order, but not the traditional multivariate majorization of Kolm [1977]. An example of multivariate Gini that satisfies Axioms 1, 3 and preserves the multivariate majorization order of Kolm [1977] is $\tilde{G}(X) := \mathbb{E}[U^\top \nabla\psi_X(U)] - 1$, proposed in Galichon and Henry [2012], where the expectation is taken with respect to $U \sim U[0,1]^2$. However, $\tilde{G}$ is unsuitable as an inequality index in our context as can be seen with the

[1]A related notion, namely c-comonotonicity, was proposed by Puccetti and Scarsini [2010].

following two observations. First, the following expression shows that $\tilde{G}$ only depends on $\mathcal{L}_1(p,1)+\mathcal{L}_2(1,p)$, hence, on very specific features of the dependence between the two components X_1 and X_2 of allocation X.

$$\tilde{G}(X) \;=\; \int_0^1 \Big([p-\mathcal{L}_1\,(p,1)] + [p-\mathcal{L}_2\,(1,p)] \Big)\, dp. \tag{2.4}$$

Second, and more troubling still, for any given fixed marginals for X_1 and X_2, $\tilde{G}$ is maximized when the two components X_1 and X_2 of endowment X are independent, which runs against the intuition that increased dependence can increase inequality[2].

$$\tilde{G}(X) \;\leq\; \frac{1}{2}\,[\,G\,(X_1) + G\,(X_2)\,]\,, \tag{2.5}$$

where $G(X_1)$ and $G(X_2)$ denote the classical scalar Gini index of components X_1 and X_2 respectively.

3. Empirical Illustration

In this section, we propose estimators for the Lorenz map, Inverse Lorenz Function, α-Lorenz curves and Gini index. We then apply them to the analysis of wealth-income inequality in the U.S. with data from the Survey of Consumer Finances (SCF) between the years 1989-2019.

3.1. **Estimation of the Lorenz map, Inverse Lorenz Function, and Gini.** Suppose we have a sample $X_1,\ldots,X_n$ from a probability distribution P_X with sampling weights w_i such that $\Sigma_i w_i = 1$ and normalized to have sample mean one. Let P_n be the empirical distribution relative to the weighted sample $\{(X_1,w_1),\ldots,(X_n,w_n)\}$. Let $\nabla\widehat{\psi}_n$ be an estimator for the population vector quantile $\nabla\psi_X$ of Definition 1. The empirical Lorenz map is defined as the plug-in estimator

$$\widehat{\mathcal{L}}_n(r) \;=\; \int_{[0,r_1]\times[0,r_2]} \nabla\widehat{\psi}_n(u)\, du. \tag{3.1}$$

By Proposition 3, There exists a convex function $\widehat{\psi}_X : [0,1]^2 \to \mathbb{R}\cup\{+\infty\}$ such that $\nabla\widehat{\psi}_{n\#}\lambda = P_n$. The function $\nabla\widehat{\psi}_n$ exists and is unique, λ-almost everywhere.

[2]Note that we are considering inequality over outcomes, not welfare inequality. Hence, the point made in Atkinson and Bourguignon [1982], that increased correlation may decrease utilitarian welfare inequality when resources are complements, doesn't apply here.

Since $\nabla\widehat{\psi}_n$ pushes λ forward to P_n, it takes a.e. finitely many values in $\{X_i\}_{i=1}^n$. Hence, $\widehat{\psi}_n$ is a piecewise affine convex function, and there exists a unique (up to scalar addition) $\widehat{h} \in \mathbb{R}^n$ such that

$$\widehat{\psi}_n(u) \quad = \quad \begin{cases} \max\limits_{i=1,\dots,n} \{u^\top X_i + \widehat{h}_i\} & \text{if } u \in [0,1]^2 \\ +\infty \quad \text{if otherwise,} \end{cases}$$

and $\widehat{h}$ is the solution of a convex optimization problem; see for instance Proposition 6 of Ekeland et al. [2012]. The resulting estimator $\nabla\widehat{\psi}_n$ induces a cell decomposition of the domain $[0,1]^2$ into n convex polytopes, one for each observation in the sample, and each with area equal to the value of the corresponding sample weight w_i. This partitioning is called a power diagram[3]; see Aurenhammer [1987]. The cells of the diagram are defined by

$$\widehat{W}_i \quad = \quad \{r \in [0,1]^2 : \nabla\widehat{\psi}_X(r) = X_i\}.$$

We can therefore write the estimator of the Lorenz map as

$$\widehat{\mathcal{L}}_n(r) \quad = \quad \sum_{i=1}^{n} X_i\, \lambda(\widehat{W}_i \cap [0,r_1] \times [0,r_2]).$$

The estimator of the Lorenz map $\widehat{\mathcal{L}}_n$ can then be used to generate a pseudo sample $\{\widehat{\mathcal{L}}_n(R_1), \dots, \widehat{\mathcal{L}}_n(R_m)\}$, where $\{R_1, \dots, R_m\}$ is a uniform random sample on $[0,1]^2$. The inverse Lorenz map l_X can then be estimated with the empirical distribution of this pseudo-sample:

$$\widehat{l}(z) = \frac{1}{m}\sum_{j=1}^{m} 1\{\widehat{\mathcal{L}}_n(R_j) \leq z\}, \quad z \in [0,1]^2, \tag{3.2}$$

and the α-Lorenz curves are estimated by the level sets of $\widehat{l}(z)$.

[3]Viewing this transport problem as solving for the power diagram of X_i with weights h has computational and geometric advantages, which are exploited by Aurenhammer et al. [1998]. They connected solving for the optimal weight vector $\widehat{h}$ to an unconstrained convex optimization problem. This allows for efficient (damped) Newton or quasi-Netwon methods as in Mérigot [2011] and Kitagawa et al. [2019] for which the gradient and Hessian of the objective function are easily computed, see Lévy [2015]. Implementations for these methods are available in the Geogram and CGAL libraries.

Finally, the Gini index is estimated directly from the Lorenz map with

$$\widehat{G}_n \;\; := \;\; 1 - 2\left(\int_{(r_1,r_2)\in[0,1]^2} \sum_{i=1}^{n} (X_{i1} + X_{i2})\, \lambda\left(\widehat{W}_i \cap [0, r_1] \times [0, r_2]\right)\, dr_1 dr_2\right),$$

where X_{i1} and X_{i2} are the two components of individual i's endowment in the sample.

3.2. **Data and descriptive statistics.** The data is sourced from the public version of the Survey of Consumer Finances (SCF) between the years 1989-2019. It is (normally) a triennial cross-sectional survey that collects micro-level data of U.S families including income, balance sheets, pensions, assets, debts, and expenditures. We combine all assets, including financial, in our wealth variable. The latter and the income variable observed in all waves of the survey between 1989 and 2019 form our data set. The survey over-samples higher income families who are likely to be wealthy. This is intended to counteract low survey response from high income and wealthy households. Details of the sampling technique and a discussion of specific features and issues with the data set are given in Appendix B. Figure 4 shows univariate Lorenz curves for income and for wealth between 1989 and 2019. Univariate inequality increases by decade in both income and wealth. Another notable feature is that 2007 (resp. 1989) displays more top (resp. bottom) inequality for both income and wealth. Figure 5 displays univariate Gini indices for income and for wealth, the multivariate Gini index, as well as Kendall's tau for the dependence between income and wealth over time. Between 1992 and 1995, the drop in wealth Gini is more than offset by the increased income Gini and increased dependence, so that the multivariate Gini increased. Conversely, between 2007 and 2010, the rise in wealth Gini is more than offset by the decreased income Gini and decreased dependence, so that the multivariate Gini decreased.

3.3. **Discussion of α-Lorenz curves for the SCF data.** Figure 6 and 7 display the 0.99- and 0.90-Lorenz curves for the SCF data between the years 1989 and 2019. Most of the 0.99-Lorenz curves lie below the egalitarian 0.90-Lorenz curve, and all the 0.90-Lorenz curves lie below the egalitarian 0.70-Lorenz curve, indicating great disparity. The curves are generally pulled towards the wealth axis. This indicates that inequality in the marginal distribution of wealth is more severe than that of the marginal distribution of income, as could be seen on the univariate Lorenz curves of Figure 4. Figure 6 and 7 display an apparent increase in inequality by decade. More precisely, Figure 8 shows that the multivariate inequality ranking 2016 $>$ 2007 $>$ 1992 holds at all displayed levels, namely $\alpha = 0.1, 0.5, 0.75, 0.9$ and 0.99. These observations are confirmed by the evolution of the multivariate Gini index values plotted in Figure 5. Figure 8 also shows that the inequality discrepancies between 1992, 2007 and 2016 are more pronounced at the top than the bottom of the distribution.

Appendix A. Additional Details and Results

A.1. **Vector ranks and quantiles.** Proposition 3 below, a seminal result in the theory of measure transportation, states essential uniqueness of the gradient of a convex function (hence cyclically monotone map) that pushes the uniform distribution on $[0,1]^2$ into the distribution of an allocation X.

Proposition 3 (McCann 1995)**.** Let P and λ be two distributions on $\mathbb{R}^d$. (1) If λ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, with support contained in a convex set $\mathcal{U}$, the following statements hold: there exists a convex function $\psi : \mathcal{U} \to \mathbb{R} \cup \{+\infty\}$ such that $\nabla\psi_{\#}\lambda = P$. The function $\nabla\psi$ exists and is unique, λ-almost everywhere. (2) If, in addition, P is absolutely continuous on $\mathbb{R}^d$ with support contained in a convex set $\mathcal{X}$, the following holds: there exists a convex function $\psi^* : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ such that $\nabla\psi^*_{\#}P = \lambda$. The function $\nabla\psi^*$ exists, is unique and equal to $(\nabla\psi)^{-1}$, P-almost everywhere.

Proposition 3 is an extension of Brenier [1991] (see also Rachev and Rüschendorf [1990]). It removes the finite variance requirement, which is undesirable in our context. Proposition 3 is the basis for the definition of vector quantiles and ranks in Chernozhukov et al. [2017]. In our context, it is applied with uniform reference measure.[4]

In case $d = 1$, gradients of convex functions are nondecreasing functions, hence vector quantiles and ranks reduce to classical quantile and cumulative distribution functions. As the notation indicates, the function ψ^* of Proposition 3 is the convex

[4]This vector quantile notion was introduced in Galichon and Henry [2012] and Ekeland et al. [2012] and called μ-quantile.

conjugate of ψ. In case of absolutely continuous distributions P on $\mathbb{R}^d$ with finite variance, the vector rank function solves a quadratic optimal transport problem, i.e., vector rank R minimizes, among all functions T such that $T(X)$ is uniform on $[0,1]^d$, the quantity $\mathbb{E}\|X - T(X)\|^2$, where $X \sim P$.

Appendix B. Specific features and issues with the data source

We review some known issues with the data set that impact our analysis. See Hanna et al. [2018] more a more in-depth account.

B.1. **Sampling strategy.**

The over sampling of high income and wealthy households is achieved by applying two distinct sampling techniques. The first sample is the core representative sample selected by a standard multi-stage area-probability design. The second is the high income supplement from statistical records derived from tax data by the Statistics of Income (SOI) division of the Internal Revenue Service of the U.S. The stages sample disproportionately– usually one-third of the final sample is from the high income supplement. Sampling in this way retains characteristic information of the population while also addressing the known selection biases of the wealthy not responding to surveys. In order to represent the population with this sample, weights must be constructed for each unit of observation. For more details on the construction of weights and their implications on the distribution of wealth, see Kennickell and Woodburn [1999].

B.2. **Unit of observation and timing of interviews.**

The observations in this data set are not households, but rather a subset called the *primary economic unit* (PEU) that may be individuals or couples and their financial dependents. For example in the 2016 data set 13% of PEUs were in a household that contained one or more members not in their PEU. Additionally, the respondent is not necessarily the head of the household, so special care must be taken if analyzing attitudes in relation to some demographic characteristics such as age. The interviews start in May of the survey year, after most income taxes are filed and usually finish by the end of the calendar year, see Kennickell [2017b] for challenges at the end of the interview period. Questions also may change over time so it is important to review the codebook each year when making comparisons across time.

B.3. **Multiple Imputation.** During interviews, respondents may omit answers or provide a range of values for which their response belongs. This missing data impacts analysis and so the SCF contains 5 imputed values for each PEU, creating a sample 5 times larger than the actual number of respondents and forms 5 data sets called implicates. Imputation is done by the Federal Reserve Imputation Technique Zeta model (FRITZ), details can be found in Kennickell [2017a] based upon the ideas of Little and Rubin [2019]. Multiple imputation for missing data provide multiple probable values. Each of these form a data set from which sample statistics can be found. The technique of Repeated Imputation Inference (RII) is applied in our analysis. For each implicate $\ell = 1, 2, 3, 4, 5$, the empirical ILF $\widehat{l}_{\ell *}$ is calculated using the appropriate quantile map estimator taking into account sample weights. Then the repeated-imputation estimate of l is

$$\widehat{l}(z) = \frac{1}{5}\sum_{\ell=1}^{5} \widehat{l}_{\ell *}(z).$$

Calculation of the Gini index follows a similar procedure. Accounting for the multiple imputation in the calculation of standard errors is an important issue, but is not revelant to our visualization technique. For more information on multiple imputation and inference with imputed values, see Rubin [1996].

B.4. **Computational issues.** At times within the data, values can be repeated (e.g. many zeros), which have been found to pose issues calculating the quantile maps required for our Lorenz curves. Typically if this happens a few empty cells will be produced, which implies that the solution is invalid. A work-around for this is to introduce some noise to separate the data points. We added a random number between 0 and 10 for most implicates and added a random number between 0 and 200 to the data in 2007 and 2016. These numbers are too small to affect disparity when income and wealth are well over the tens of thousands in most of the stratum of the population so the visuals will not be influenced by this correction.

B.5. **Definition of Wealth.** In the literature, there is no consensus on what factors should be included in wealth measurement. Wolff [2021] defines wealth as marketable weath, which is the sum of marketable or fungible assets less the current value of all debts. Bricker et al. [2017] define wealth as net worth including those assets which are not readily transformed into consumption: properties, vehicles, etc. In our analysis we consider all assets, including financial, as our wealth variable.

Appendix C. Proofs of results in the main text

In this section, we omit the X subscript of ψ_X for notational compactness.

Proof of Property 1. The off diagonal elements of the Jacobian of $\mathcal{L}(r)$ are $\psi(r_1, r_2) - \psi(r_1, 0)$ and $\psi(r_1, r_2) - \psi(0, r_2)$. From the latter, by differentiation, we can recover $\nabla\psi(r_1, r_2)$. The result then follows from the fact that $\nabla\psi$ characterizes P_X, see for instance Chernozhukov et al. [2017]. □

Proof of Lemma 1. Let $U \sim U[0,1]^2$. Let $\tilde{X}_j := \dfrac{\partial\psi}{\partial u_j}(U_1, U_2)$, $j = 1, 2$. Then $(\tilde{X}_1, \tilde{X}_2)$ is distributed identically to (X_1, X_2). Since $\widetilde{X}_2 = \dfrac{\partial\psi}{\partial u_2}(U_1, U_2)$ is monotonically increasing in U_2, we have $U_2 = \left(\dfrac{\partial\psi}{\partial u_2}\right)^{-1}(U_1, \widetilde{X}_2)$. Hence

$$\widetilde{X}_1 = \frac{\partial\psi}{\partial u_1}\left(U_1, \left(\frac{\partial\psi}{\partial u_2}\right)^{-1}(U_1, \widetilde{X}_2)\right).$$

Under the stated assumption, $\widetilde{X}_1$ is increasing in U_1 and $\widetilde{X}_2$. Since $\left(\widetilde{X}_1, \widetilde{X}_2\right) \stackrel{d}{=} (X_1, X_2)$, we have

$$\begin{aligned}
F_X(x_1, x_2) &= \mathbb{P}\left(\widetilde{X}_1 \leq x_1, \widetilde{X}_2 \leq x_2\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\frac{\partial\psi}{\partial u_1}\left(U_1, \left(\frac{\partial\psi}{\partial u_2}\right)^{-1}(U_1, \widetilde{X}_2)\right) \leq x_1, \widetilde{X}_2 \leq x_2\right) \middle| U_1\right] \\
&= \mathbb{E}\left[\min\left\{F_1(x_1|U_1), F_2(x_2|U_1)\right\}\right] \\
&\geq \mathbb{E}\left[F_1(x_1|U_1)\, F_2(x_2|U_1)\right].
\end{aligned}$$

Now $F_1\left(x_1|U_1\right)$ is increasing in U_1, since

$$\begin{aligned} F_1\left(x_1|U_1\right) &= \mathbb{P}\left(\frac{\partial\psi}{\partial u_1}(U_1,U_2)\leq x_1\middle|\, U_1\right) \\ &= \mathbb{P}\left(U_2\leq\left(\frac{\partial\psi}{\partial u_1}\right)^{-1}(U_1,x_1)\middle|\, U_1\right) \\ &= \left(\frac{\partial\psi}{\partial u_1}\right)^{-1}(U_1,x_1). \end{aligned}$$

Similarly $F_2\left(x_2|U_1\right)$ is increasing in U_1. We conclude that $F_X\left(x_1,x_2\right)\geq F_1\left(x_1\right)F_2\left(x_2\right)$, see e.g. Joe [1997]. □

Proof of Property 3. We only need to show for one component of the Lorenz map and the second follows with similar reasoning. We have for the first component

$$\begin{aligned} \mathcal{L}_1(r) &= \int_0^{r_2}\int_0^{r_1}\frac{\partial\psi}{\partial u_1}(u_1,u_2)du_1du_2 \\ &= \int_0^{r_2}[\psi(r_1,u_2)-\psi(0,u_2)]du_2 \\ &\leq \int_0^{r_2}r_1[\psi(1,u_2)-\psi(0,u_2)]du_2,\text{ by convexity} \\ &= r_1\int_0^{r_2}\int_0^{1}\frac{\partial\psi}{\partial u_1}(u_1,u_2)du_1du_2 \end{aligned}$$

Now define $H(r_2)=\int_0^{r_2}\int_0^1\frac{\partial\psi}{\partial u_1}(u_1,u_2)du_1du_2$. $H'(r_2)=\int_0^1\frac{\partial\psi}{\partial u_1}(u_1,r_2)du_1$ is monotone increasing by assumption, so H is convex. Therefore, $H(r_2)\leq H(0)+r_2(H(1)-H(0))$. Note that $H(0)=0$ as an integral over a degenerate interval, and $H(1)=1$, as this is the mean of the normalized X_1. So $H(r_2)\leq r_2$ and so $\mathcal{L}_1(r)\leq r_1r_2$, as desired. □

Proof of Property 4. (i) Let $l_X(z_1,z_2)=\alpha$. Since l_X is a cdf, hence non decreasing in both arguments, the generalized inverse $z_2:=l_X^-(z_1;\alpha)$ is non increasing in z_1 non decreasing in α. (ii) See Claim 1 in Brock and Thomson [1966]. □

Proof of Proposition 1. $\tilde{X}\succcurlyeq_{\mathcal{L}}X$ is equivalent to first order stochastic dominance of $\mathcal{L}_X(R)$ over $\mathcal{L}_{\tilde{X}}(R)$, where $R\sim U[0,1]^d$ (see Section 6.B page 266 of Shaked and

Shanthikumar [2007]). Hence, $\tilde{X} \succcurlyeq_{\mathcal{L}} X$ implies $\mathbb{P}(\mathcal{L}_X(R) \in S) \leq \mathbb{P}(\mathcal{L}_{\tilde{X}}(R) \in S)$ for any lower set S, so that $\tilde{X} \succcurlyeq_{\mathcal{L}} X$ implies $\tilde{X} \succcurlyeq_l X$, given that the sets $[0, z]$ are lower sets. □

Proof of Proposition 2. Axiom 1 is automatically satisfied since G is defined as a function of the allocation's distribution. Axiom 2 follows immediately from Equation 2.2. We now show that Axiom 3 is satisfied. Take $\mu \in (0, 1)$. Let X, $\tilde{X}$ and Z be comonotonic according to Definition 7, and assume $G(X) \leq G(\tilde{X})$, i.e., $\mathbb{E}[\mathcal{L}_X(R)] \geq \mathbb{E}[\mathcal{L}_X(R)]$. We have

$$\begin{aligned}
\mathbb{E}[(1\ 1)^\top \mathcal{L}_{\mu X+(1-\mu)Z}(R)] &= \int_{r\in[0,1]^2}\int_{u\in[0,r]} (1\ 1)^\top \nabla\psi_{\mu X+(1-\mu)Z}(u)\, dudr \\
&= \int_{r\in[0,1]^2}\int_{u\in[0,r]} (1\ 1)^\top \left(\mu\nabla\psi_X(u) + (1-\mu)\nabla\psi_Z(u)\right)\, dudr \\
&= \mu\mathbb{E}[(1\ 1)^\top \mathcal{L}_X(R)] + (1-\mu)\mathbb{E}[(1\ 1)^\top \mathcal{L}_Z(R)] \\
&\geq \mu\mathbb{E}[(1\ 1)^\top \mathcal{L}_{\tilde{X}}(R)] + (1-\mu)\mathbb{E}[(1\ 1)^\top \mathcal{L}_Z(R)] \\
&= \mathbb{E}[(1\ 1)^\top \mathcal{L}_{\mu\tilde{X}+(1-\mu)Z}(R)]
\end{aligned}$$

where the second and last equalities hold by Lemma 2. The result follows. □

Lemma 2 (Galichon and Henry [2012])**.** If X and Z with respective vector quantiles $\nabla\psi_X$ and $\nabla\psi_Z$ are comonotonic, then, for any $\mu \in (0, 1)$, the vector quantile of $\mu X + (1-\mu)Z$ is $\mu\nabla\psi_X + (1-\mu)\nabla\psi_Z$.

Proof of (2.4). Let U be uniformly distributed on $[0, 1]^2$. Note that

$$\tilde{G}(X) = \mathbb{E}\left[U_1 \frac{\partial\psi}{\partial u_1}(U_1, U_2)\right] + \mathbb{E}\left[U_1 \frac{\partial\psi}{\partial u_1}(U_1, U_2)\right] - 1.$$

Now,

$$\begin{aligned}
\mathbb{E}\left[U_1 \frac{\partial\psi}{\partial u_1}(U_1, U_2)\right] &= \int_0^1 \left[\int_0^1 u_1 \frac{\partial\psi}{\partial u_1}(u_1, u_2)\, du_1\right] du_2 \\
&= \int_0^1 \left[\int_0^1 u_1\, d\left(\frac{\partial\mathcal{L}_1}{\partial u_2}(u_1, u_2)\right]\right) du_2
\end{aligned}$$

where the last equality follows from interchangeability of the order of integration and $\mathcal{L}_1$ is the first component of the Lorenz map.

Note that

$$\frac{\partial \mathcal{L}_1}{\partial r_2}(r_1, r_2) = \int_0^{r_1} \frac{\partial \psi}{\partial u_1}(u_1, u_2)\, du_1$$

and

$$\frac{\partial}{\partial r_1}\left(\frac{\partial \mathcal{L}_1(r_1, r_2)}{\partial r_2}\right) = \frac{\partial \psi}{\partial u_1}(u_1, r_2).$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[U_1 \frac{\partial \psi}{\partial u_1}(U_1, U_2)\right] &= \int_0^1 \left(u_1 \frac{\partial \mathcal{L}_1}{\partial u_2}(u_1, u_2)\bigg|_0^1 - \int_0^1 \frac{\partial \mathcal{L}_1}{\partial u_2}(u_1, u_2)\, du_1 \right) du_2 \\
&= \int_0^1 \left(\frac{\partial \mathcal{L}_1}{\partial u_2}(1, u_2) - \int_0^1 \frac{\partial \mathcal{L}_1}{\partial u_2}(u_1, u_2)\, du_1 \right) du_2 \\
&= \int_0^1 \frac{\partial \mathcal{L}_1}{\partial u_2}(1, u_2)\, du_2 - \int_0^1 \int_0^1 \frac{\partial \mathcal{L}_1}{\partial u_2}(u_1, u_2)\, du_2\, du_1 \\
&= \mathcal{L}(1,1) - \mathcal{L}(1,0) - \int_0^1 \mathcal{L}_1(u_1, 1) - \mathcal{L}_1(u_1, 0)\, du_1 \\
&= 1 - \int_0^1 \mathcal{L}_1(u_1, 1)\, du_1.
\end{aligned}$$

Similarly, we have $\mathbb{E}\left[U_2 \frac{\partial \psi}{\partial u_2}(U_1, U_2)\right] = 1 - \int_0^1 \mathcal{L}_2(1, u_2)\, du_1$, as desired. □

Proof of (2.5). $0 \le \tilde{G}(X)$ because $p \ge \mathcal{L}_1(p, 1)$ and $p \ge \mathcal{L}_2(1, p)$. The second inequality follows from $\mathcal{L}_1(p, 1) \ge L_1(p)$ and $\mathcal{L}_2(1, p) \ge L_2(p)$. We now prove the latter. Letting $\nabla\psi$ be the vector quantile function of (X_1, X_2), note that since $\frac{\partial \psi}{\partial u_1}$ pushes uniform measure on $[0,1]^2$ forward to law(X_1), we can write

$$L_1(r_1) = \int_{\{u : \frac{\partial \psi}{\partial u_1}(u) \le z_1(r_1)\}} \frac{\partial \psi}{\partial u_1}(u) du$$

where $z_1(r_1)$ is the quantile of the random variable X_1. Note that the area of the domain $\{u : \frac{\partial \psi}{\partial u_1}(u) \le z_1(r_1)\}$ of integration must be r_1. On the other hand,

$$\mathcal{L}_1(r_1, 1) = \int_0^{r_1} \int_0^1 \frac{\partial \psi}{\partial u_1}(u) du_1 du_2$$

is an integral of the same function over a region with the same area. Writing $A := \{u : \frac{\partial\psi}{\partial u_1}(u) \leq z_1(r_1)\}$, we have $A = B \cup C$, where $B = A \cap ([0, r_1] \times [0, 1])$ and $C = A \cap ((r_1, 1] \times [0, 1])$ and the union is disjoint. Similarly, $[0, r_1] \times [0, 1] = B \cup D$ where $D = ([0, r_1] \times [0, 1]) \cap A^c$. Note that the areas of C and D must be the same, $|C| = |D|$, and $\frac{\partial\psi}{\partial u_1}(u) \leq z_1(r_1)$ throughout C while $\frac{\partial\psi}{\partial u_1}(u) > z_1(r_1)$ throughout D. We have

$$\begin{aligned} L_1(r_1)1 = L_1(r_1) &= \int_B \frac{\partial\psi}{\partial u_1}(u)du + \int_C \frac{\partial\psi}{\partial u_1}(u)du \\ &\leq \int_B \frac{\partial\psi}{\partial u_1}(u)du + z_1(r_1)|C| \\ &= \int_B \frac{\partial\psi}{\partial u_1}(u)du + z_1(r_1)|D| \\ &\leq \int_B \frac{\partial\psi}{\partial u_1}(u)du + \int_D \frac{\partial\psi}{\partial u_1}(u)du \\ &= \mathcal{L}_1(r_1, 1). \end{aligned}$$

Note that this inequality holds for any dependence structure between X_1 and X_2. $\square$

C.1. **Alternative expression for the Gini index.**

Lemma 3. For non negative bivariate random vectors X with $\mathbb{E}[X] = (1, 1)$, we have

$$\begin{aligned} \mathbb{E}[\mathcal{L}_1(U_1, U_2)] &= 1 - \mathbb{E}\left[U_2 \frac{\partial\psi(U)}{\partial U_1}\right] - \mathbb{E}\left[U_1 \frac{\partial\psi(U)}{\partial U_1}\right] + \mathbb{E}\left[U_1 U_2 \frac{\partial\psi(U)}{\partial U_1}\right] \\ \mathbb{E}[\mathcal{L}_2(U_1, U_2)] &= 1 - \mathbb{E}\left[U_1 \frac{\partial\psi(U)}{\partial U_2}\right] - \mathbb{E}\left[U_2 \frac{\partial\psi(U)}{\partial U_2}\right] + \mathbb{E}\left[U_1 U_2 \frac{\partial\psi(U)}{\partial U_2}\right] \end{aligned}$$

where $(U_1, U_2) \sim U([0, 1]^2)$.

Proof of Lemma 3. This is seen by applying integration by parts a few times. Start with

$$\begin{aligned}
\mathbb{E}[\mathcal{L}_1(U_1,U_2)] &= \int_0^1\int_0^1\int_0^{r_2}\int_0^{r_1}\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_1du_2dr_1dr_2\\
&= \int_0^1\int_0^{r_2}\int_0^1\int_0^{r_1}\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_1dr_1du_2dr_2\\
&= \int_0^1\int_0^{r_2}\left(r_1\int_0^{r_1}\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_1\bigg|_0^1-\int_0^1 r_1\frac{\partial\psi(r_1,u_2)}{\partial u_1}\,dr_1\right)du_2dr_2\\
&= \underbrace{\int_0^1\int_0^{r_2}\int_0^1\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_1\,du_2dr_2}_{A}-\underbrace{\int_0^1\int_0^{r_2}\int_0^1 r_1\frac{\partial\psi(r_1,u_2)}{\partial u_1}\,dr_1\,du_2dr_2}_{B}.
\end{aligned}$$

Then

$$\begin{aligned}
A &= \int_0^1\int_0^1\int_0^{r_2}\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_2dr_2du_1\\
&= \int_0^1\left(r_2\int_0^{r_2}\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_2\bigg|_0^1-\int_0^1 r_2\frac{\partial\psi(u_1,r_2)}{\partial u_1}\,dr_2\right)du_1\\
&= \int_0^1\int_0^1\frac{\partial\psi(u_1,u_2)}{\partial u_1}\,du_2\,du_1-\int_0^1\int_0^1 r_2\frac{\partial\psi(u_1,r_2)}{\partial u_1}\,dr_2du_1\\
&= \mathbb{E}\left[\frac{\partial\psi(U)}{\partial u_1}\right]-\mathbb{E}\left[U_2\frac{\partial\psi(U)}{\partial u_1}\right]\\
&= 1-\mathbb{E}\left[U_2\frac{\partial\psi(U)}{\partial u_1}\right]
\end{aligned}$$

and

$$\begin{aligned}
B &= \int_0^1\int_0^1\int_0^{r_2} r_1\frac{\partial\psi(r_1,u_2)}{\partial u_1}\,du_2\,dr_2dr_1\\
&= \int_0^1 r_1\left(r_2\int_0^{r_2}\frac{\partial\psi(r_1,u_2)}{\partial u_1}\,du_2\bigg|_0^1-\int_0^1 r_2\frac{\partial\psi(r_1,r_2)}{\partial u_1}\,dr_2\right)dr_1\\
&= \int_0^1\int_0^1 r_1\frac{\partial\psi(r_1,u_2)}{\partial u_1}\,du_2dr_1-\int_0^1\int_0^1 r_1r_2\frac{\partial\psi(r_1,r_2)}{\partial u_1}\,dr_2dr_1\\
&= \mathbb{E}\left[U_1\frac{\partial\psi(U)}{\partial u_1}\right]-\mathbb{E}\left[U_1U_2\frac{\partial\psi(U)}{\partial u_1}\right].
\end{aligned}$$

Combining these,

$$\mathbb{E}[\mathcal{L}_1(U_1,U_2)] = 1 - \mathbb{E}\left[U_2 \frac{\partial\psi(U)}{\partial u_1}\right] - \mathbb{E}\left[U_1 \frac{\partial\psi(U)}{\partial u_1}\right] + \mathbb{E}\left[U_1 U_2 \frac{\partial\psi(U)}{\partial u_1}\right].$$

It is a similar line of reasoning for the second component. □

Proof of (2.3). Using Lemma 3, we can rewrite G as

$$\begin{aligned} G(X) &= 2\Bigg(\mathbb{E}[U^\top \nabla\psi(U)] + \mathbb{E}\left[U_2(1-U_1)\frac{\partial\psi(U)}{\partial u_1}\right] + \mathbb{E}\left[U_1(1-U_2)\frac{\partial\psi(U)}{\partial u_2}\right]\Bigg) - 3 \\ &= 2\,\mathbb{E}\left[(U_1+U_2-U_1U_2)\left(\frac{\partial\psi(U)}{\partial u_1} + \frac{\partial\psi(U)}{\partial u_2}\right)\right] - 3, \end{aligned}$$

which is the desired result. □

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Figure 1. α-Lorenz curves lognormal marginals with $\sigma = 1$ and $\alpha = 0.75$. Egalitarian, independent and comonotonic cases. The markings on the vertical scale correspond to the α-level of egalitarian α-Lorenz curves.

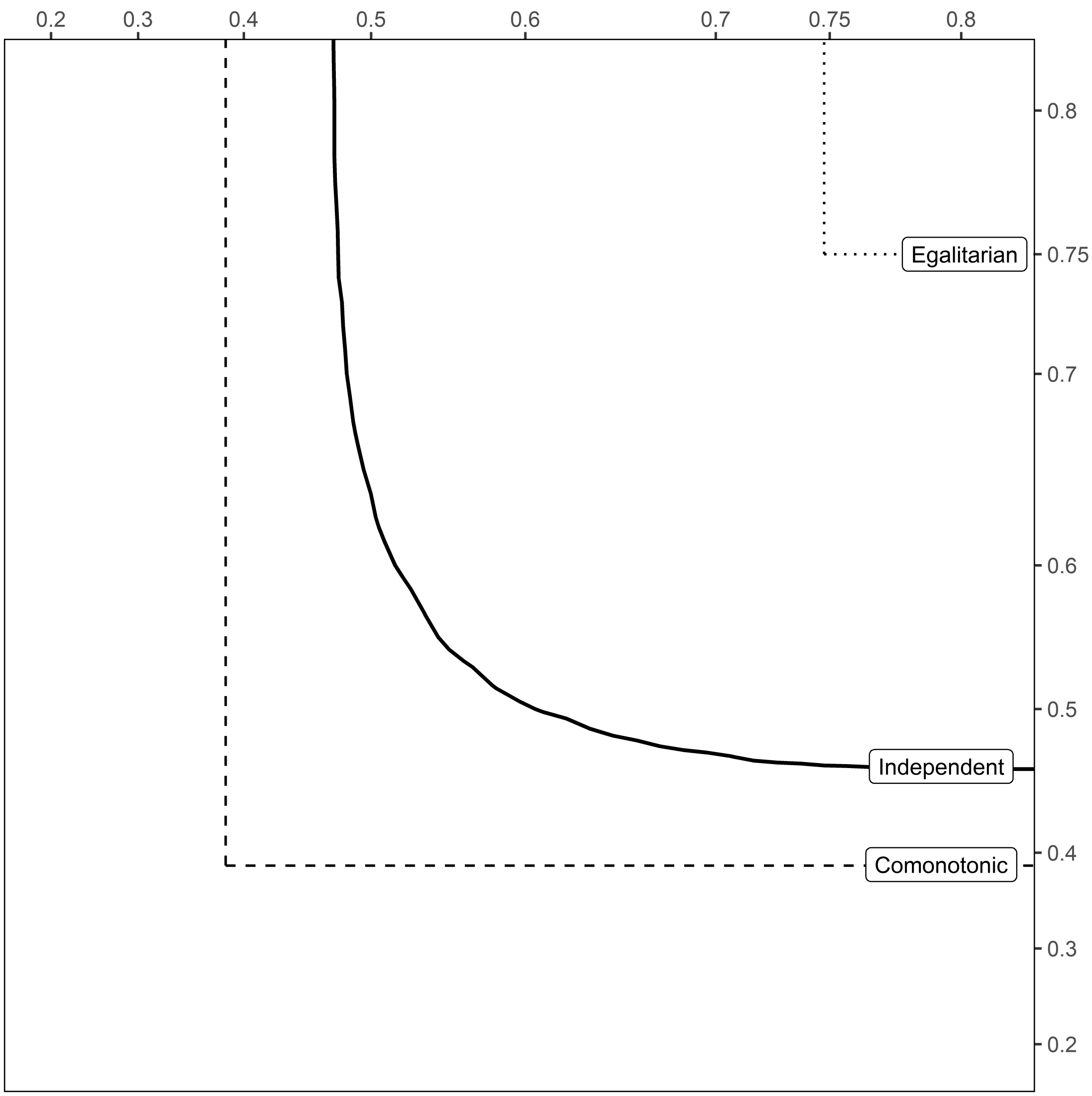

Figure 2. Lorenz curves for lognormal marginals and Plackett copula with dependence parameter 2, for $\alpha = 0.25, 0.5, 0.75, 0.95$.The markings on the horizontal and vertical scales correspond to the α-level of egalitarian α-Lorenz curves.

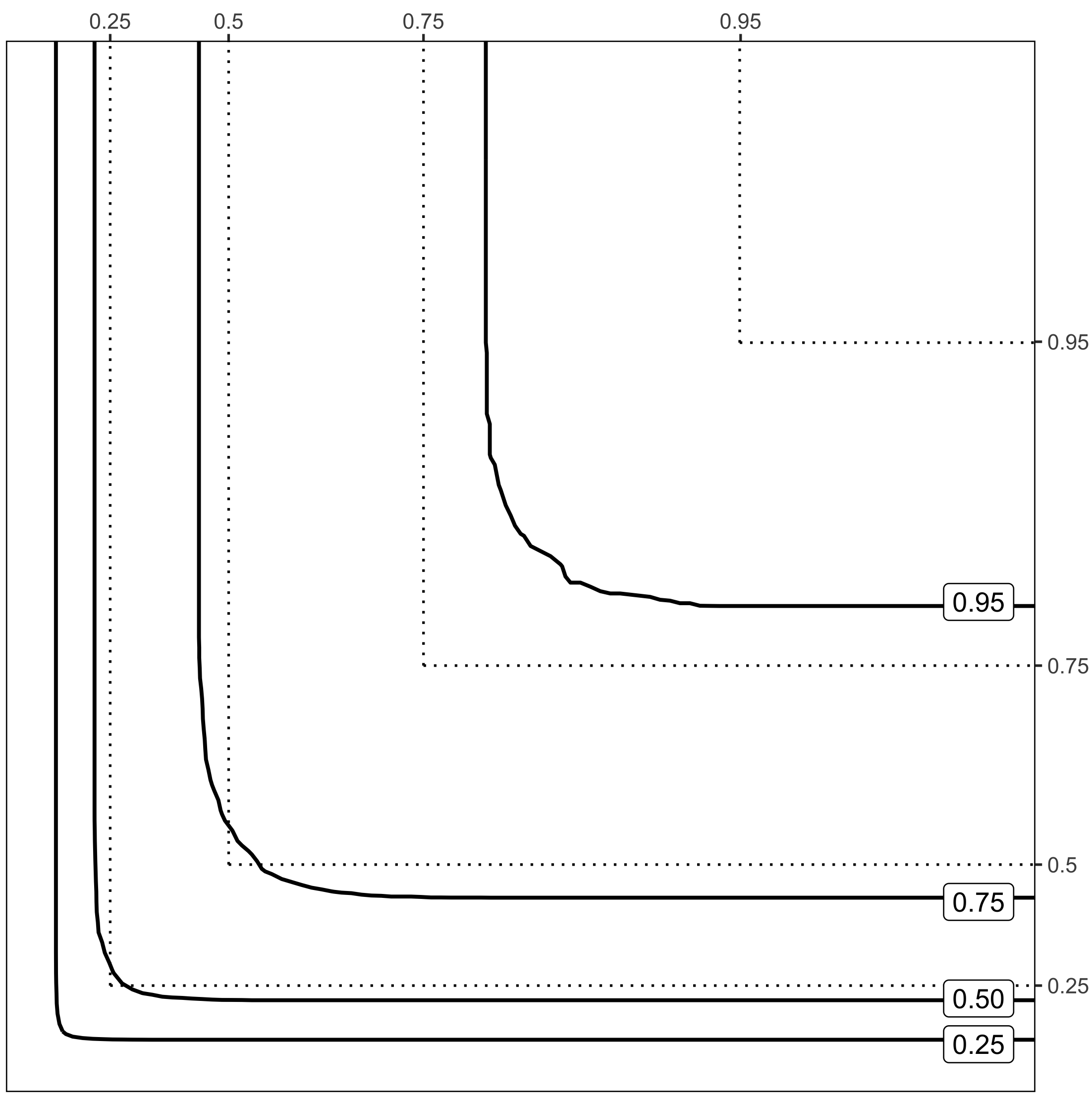

Figure 3. Lorenz curves for lognormal marginals with parameter $\sigma = 1$ or 1.5, and Plackett copula with dependence parameter 2 or 10, for $\alpha = 0.9$.The markings on the vertical scale correspond to the α-level of egalitarian α-Lorenz curves.

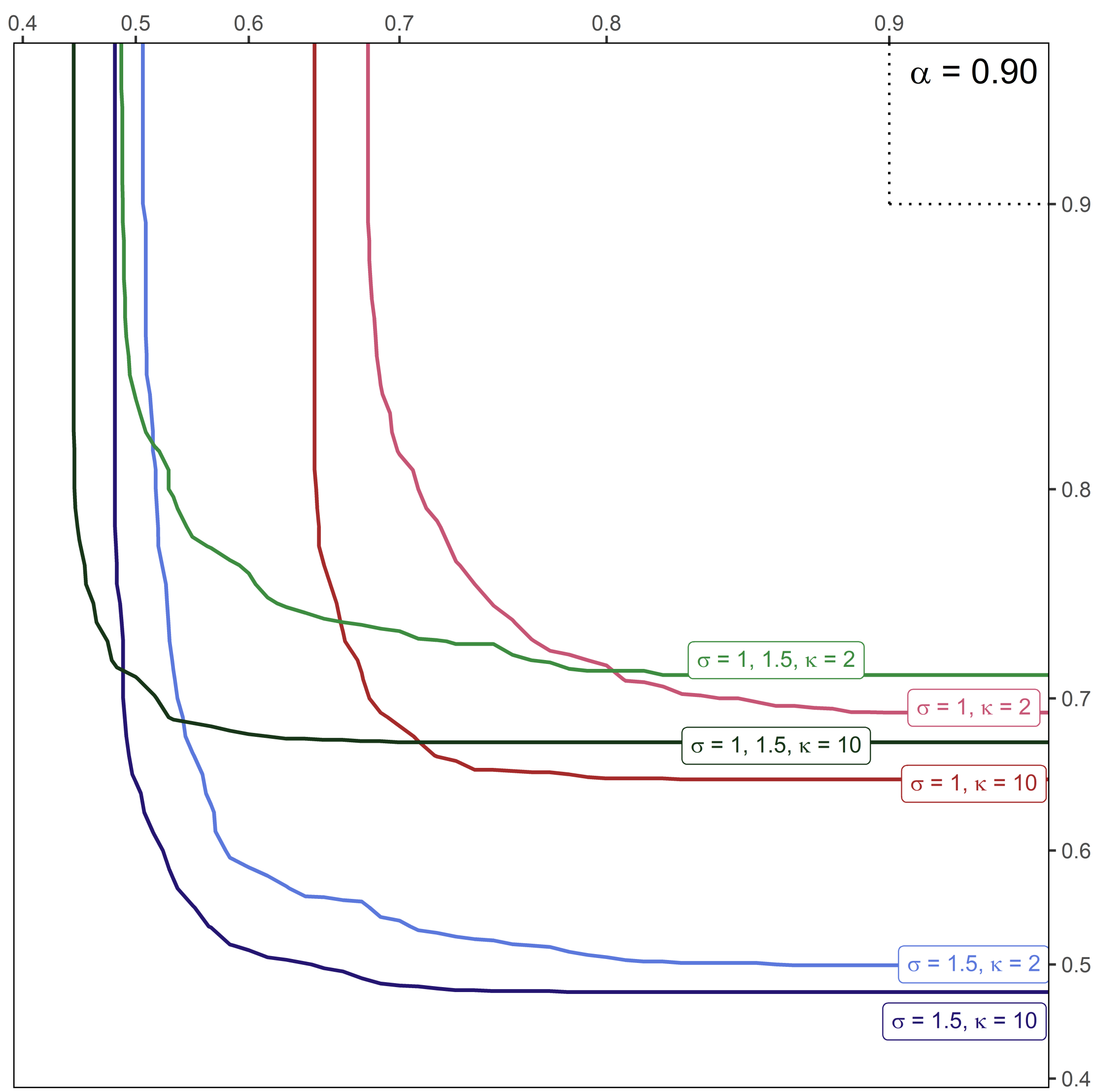

Figure 4. Univariate Lorenz curves for income (top cluster) and for wealth (bottom cluster) in the US between 1989 and 2019.

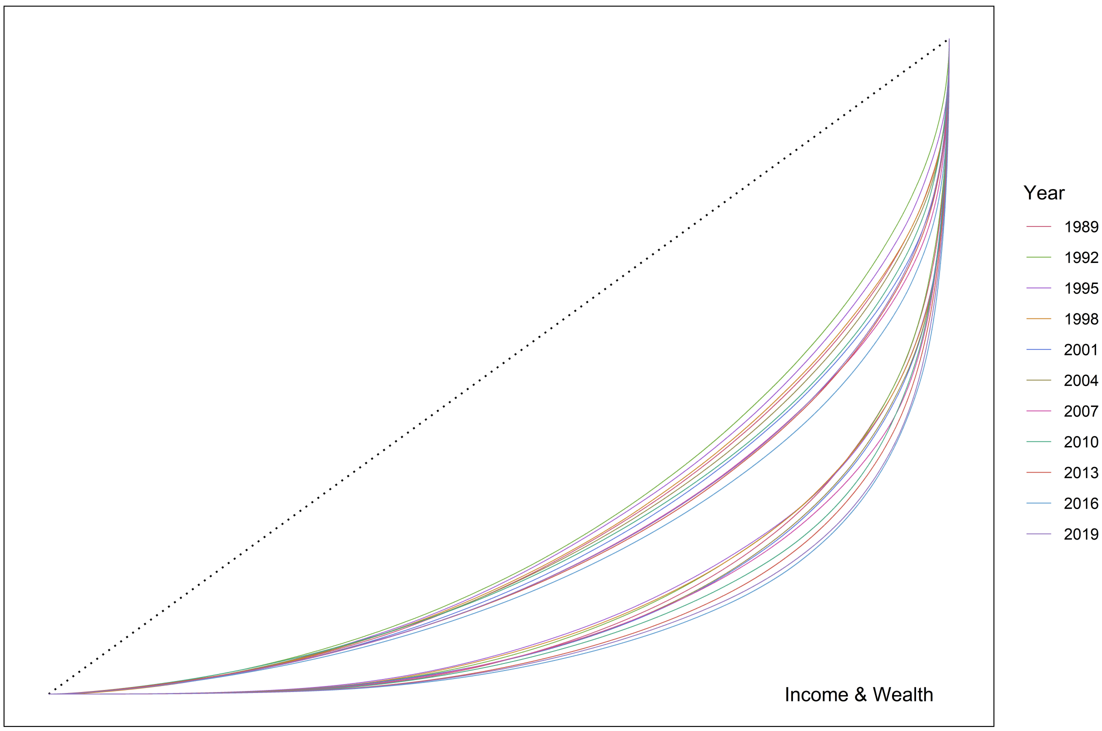

Figure 5. Univariate Gini indices for income (bottom curve in red) and for wealth (top curve in blue), multivariate Gini index, (second to top curve in violet) and Kendall's τ for Income-Wealth (dashed curve) in the US between 1989 and 2019.

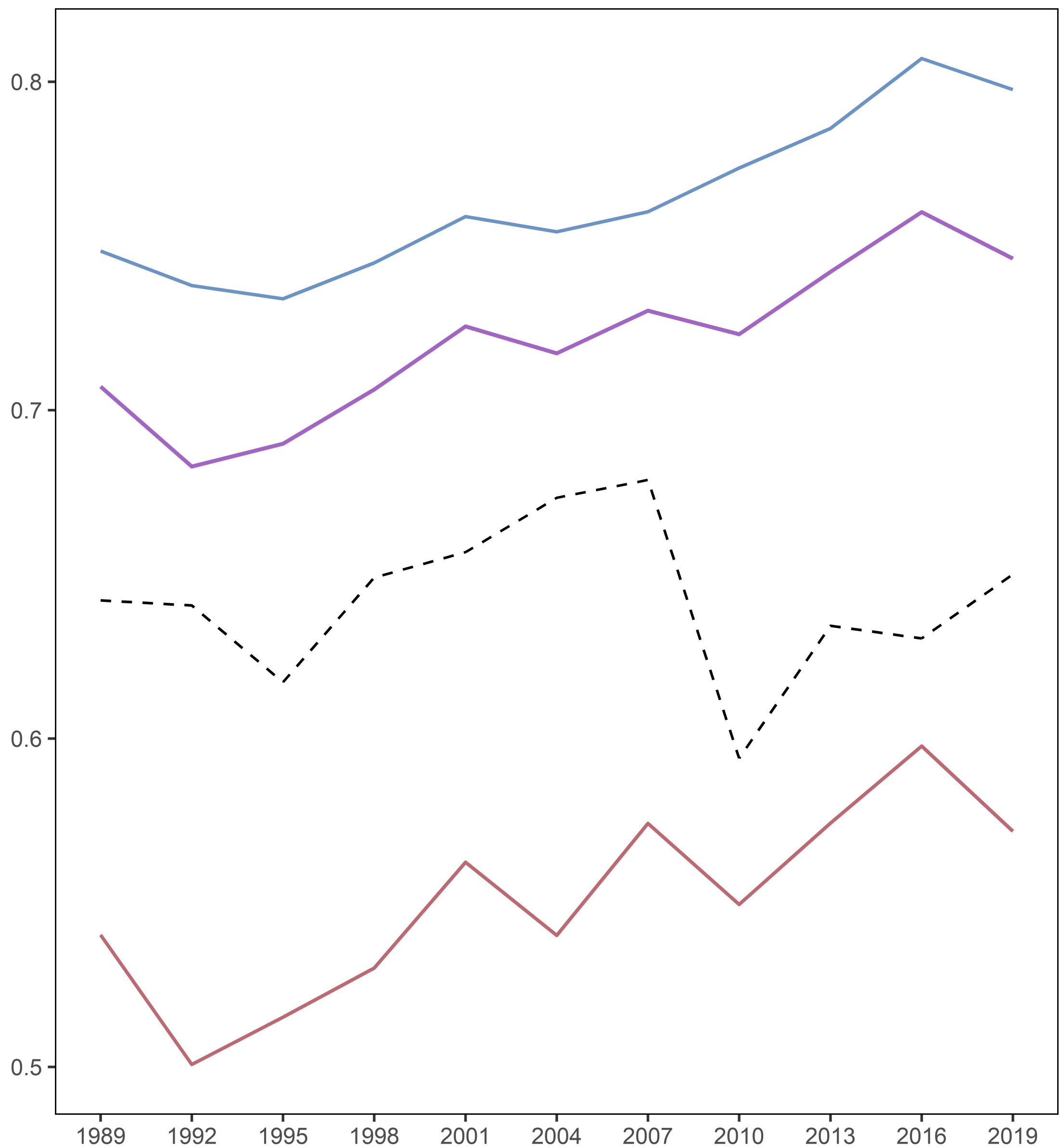

Figure 6. α-Lorenz curves for Income-Wealth in the US between 1989 and 2019, for $\alpha = 0.99$. The markings on the horizontal and vertical scales correspond to the α-level of egalitarian α-Lorenz curves.

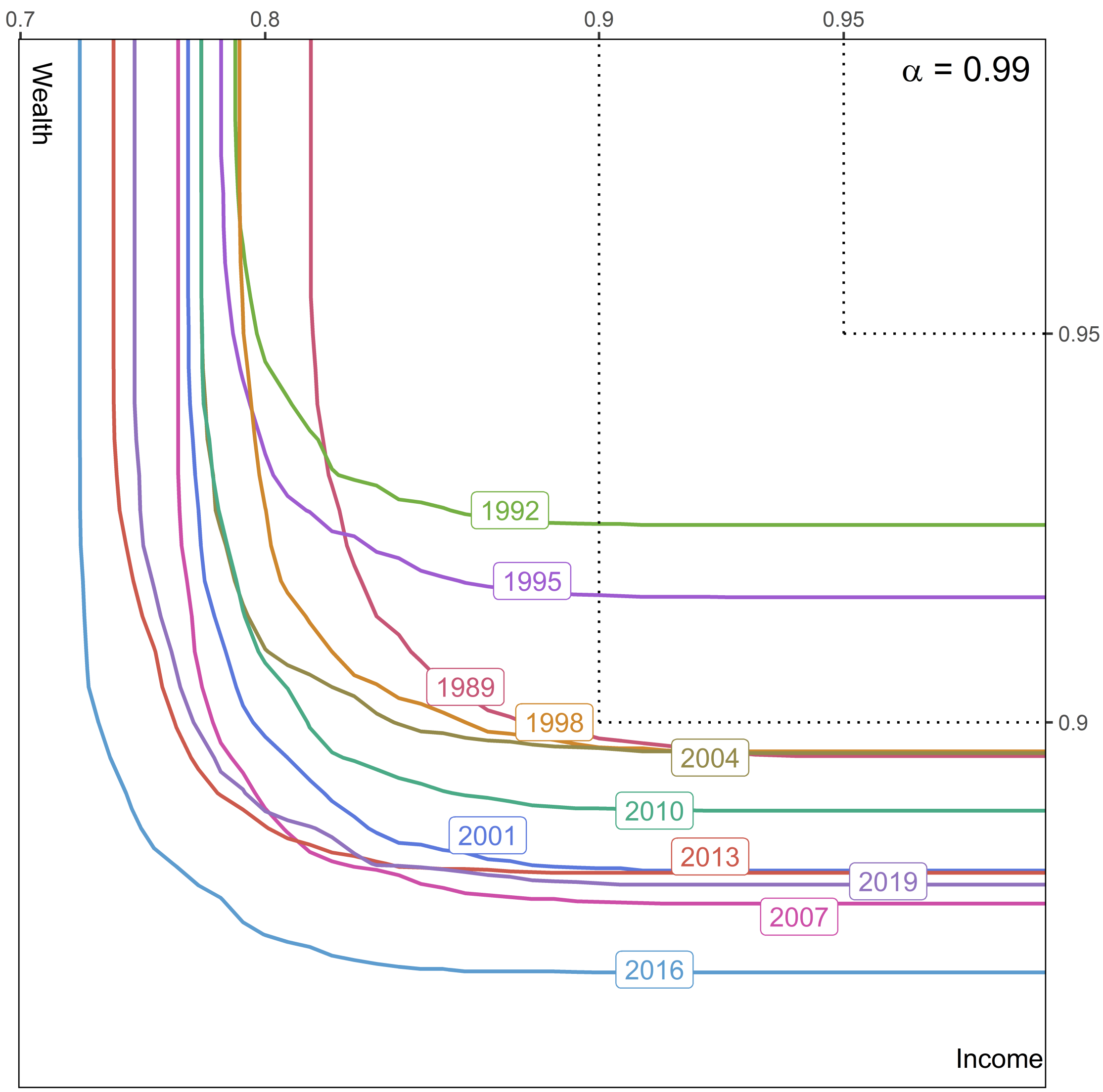

Figure 7. α-Lorenz curves for Income-Wealth in the US between 1989 and 2019, for $\alpha = 0.90$. The markings on the horizontal and vertical scales correspond to the α-level of egalitarian α-Lorenz curves.

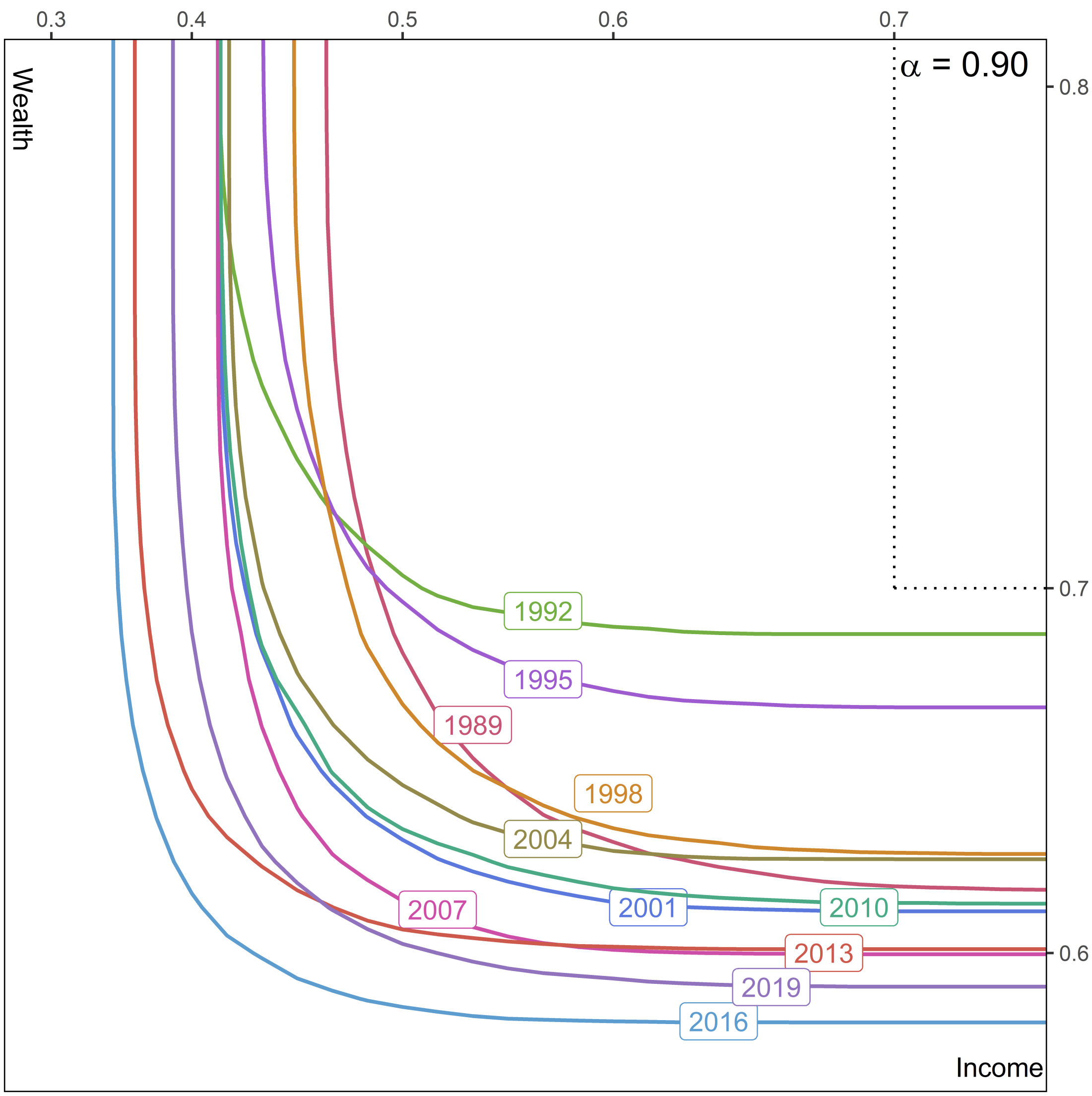

Figure 8. α-Lorenz curves for Income-Wealth in the US in 1992, 2007, and 2016, for $\alpha = 0.1, 0.5, 0.9, 0.99$. The markings on the horizontal and vertical scales correspond to the α-level of egalitarian α-Lorenz curves.