[This is a draft of Chapter 6 in a manuscript by Yongsheng Xu and me. My UCR seminar (20 October 2021) on the measurement of multidimensional deprivation will be based on Chapter 7 of the same manuscript. But I am posting preliminary drafts of both Chapter 6 and Chapter 7 on our seminar website since much of the notation and several concepts in Chapter 7 are introduced in Chapter 6.

Prasanta K. Pattanaik]

Chapter 6: Measurement of individual and social well-being

Prasanta K. Pattanaik and Yongsheng Xu

In this chapter, we discuss two important issues in the functioning and capability approach (FCA) to well-being: the measurement of individual well-being and the measurement of social well-being. In the course of our discussion, we also consider certain related issues: complementarity and substitutability of attributes, interpersonal comparisons of well-being, and our choice of individual well-being functions in practical applications of the FCA.

Section 6.1 presents basic notation and definitions. In Section 6.2, we explore the structure of our proposed class of individual well-being functions which, we believe, has some attractive properties. Section 6.3 presents definitions of complementarity and substitutability of attributes based on an individual well-being function. Then, in Section 6.4 we study the structure of a class of social well-being functions that aggregate individual well-being functions. Section 6.5 discusses a further issue relating to interpersonal comparisons of well-being and its implications for the choice of individual well-being functions in practical application of the FCA. The proofs of the results in Chapter 6 are gathered in Section 6.6.

6.1. Preliminary

We assume that there is a fixed set, $F = \{ f_1, \dots, f_m\}$, of positively valued attributes in terms of which the well-being of every individual is assessed. Let $M = \{1, \dots, m\}$. For each attribute f_j, let V_j denote the set of values which f_j can take. For the moment, V_j ($j \in M$) can be either a set of finite values or a closed (non-degenerate) interval of the real line. In the following sections, we will deal with specific cases of the set of values. We assume that, for every $j \in M$, we have an antisymmetric ordering $[\geq_j]$ ("at least as high as, in terms of f_j") over V_j, with $[>_j]$ denoting the asymmetric factor of $[\geq_j]$: a higher value for f_j indicates a higher achievement in terms of the attribute f_j.

As a convention, we shall assume that, for each $j \in M$, the lowest value in V_j is v_j^0 and the highest value in V_j is v_j^*. Let $V = V_1 \times \ldots \times V_m$, and let $v^0 = (v_1^0, \ldots, v_m^0)$ and $v^* = (v_1^*, \ldots, v_m^*)$. Thus, v^0 is the individual achievement vector in which the achievement in every dimension is at the lowest level, and v^* is the individual achievement vector in which the achievement in every dimension is at the highest level. For any two vectors $v, v' \in V$, we shall write $v \geq v'$ to mean $[(v_j [\geq_j] \, v_j' \text{ for all } j \in M]$, and $v \gg v'$ to mean $[(v_j [>_j] \, v_j' \text{ for all } j \in M]$. In the subsequent discussions of this chapter, we shall examine specific forms of V in the context of measuring well-being.

6.2. Individual well-being functions

Let the group of individuals under consideration have n individuals; for convenience we shall call the group the society and denote it by $N = \{1, \ldots, n\}$. A social well-being measure is viewed as an aggregation of various well-being levels of different individuals in the society, the level of an individual's well-being being reflected in her achievement vector, an element in V. Formally, for all $i \in N$, let $\rho_i: V \to [0,1]$ be individual i's well-being measure with the interpretation that, for all $v = (v_1, \ldots, v_j, \ldots, v_m), v' = (v_1', \ldots, v_j', \ldots, v_m') \in V$, $\rho_i(v) \geq \rho_i(v')$ indicates that the level of i's well-being from v is at least as high as the level of i's well-being from v'; $\rho_i(v) > \rho_i(v')$ indicates that the level of i's well-being from v is higher than the level of i's well-being from v'; and $\rho_i(v) = \rho_i(v')$ indicates that i's well-being level corresponding to v is the same as i's well-being level corresponding to v'.

We now introduce certain properties of individual well-being measures.

Definition 6.1. For all $i \in N$, ρ_i satisfies:
normalization iff, for all $i \in N$, $\rho_i(v^0) = 0$, and $\rho_i(v^*) = 1$;
monotonicity iff for all $v = (v_1, \dots, v_j, \dots, v_m), v' = (v'_1, \dots, v'_j, \dots, v'_m) \in V$, if [$v \geq v'$ and $v \neq v'$], then $\rho_i(v) > \rho_i(v')$.

Normalization is a standard property in the literature on measurement of multi-dimensional well-being. It requires that, when an individual's achievement vector is v^0 so that the individual's achievement in every attribute is at the lowest level, the measure of well-being for the individual is at the lowest and is normalized as 0, and when an individual's achievement vector is v^* so that the individual achieves the highest level along every attribute, the measure of well-being for the individual is at the highest and is normalized as 1. Monotonicity is another standard property used in the literature on measurement of multidimensional well-being and deprivation. It basically requires that any increase in an individual's achievement in any attribute increases the well-being level of the individual. Given that each attribute is a desirable attribute, monotonicity is fairly reasonable.

Let an individual $i \in N$ be given in the remainder of this section. Then, for individual i's well-being measure ρ_i, we will omit the subscript i in ρ_i and write simply ρ. We shall now analyze the structure of individual well-being measures for two special cases regarding the values that the attributes can take. In the first case, we assume that every attribute can take only finite and discrete values while, in the second case, we assume that every attribute is cardinally measurable along a closed (non-degenerate) interval of a real line. Clearly, these are two polar cases. The hybrid case, where some attributes (e.g., health) take finite and discrete values while others (e.g., leisure) are cardinally measurable along a real interval, seems to be more realistic, but, as far as we know, this case has not been explored in the literature.

The individual well-being function when each attribute can take only finite and discrete values

We first consider the case in which the measurement of each attribute is discrete, so that for each attribute f_j ($j \in M$), V_j is a finite set of values which f_j can take.

For a given integer $k > 1$, consider any $a^1, \dots, a^k \in V$, $b^1, \dots, b^k \in V$. We say that the two collections, $(a^1, \dots, a^k)$ and $(b^1, \dots, b^k)$, of achievement vectors are *achievement-equivalent* iff, for every $j \in M$, there exists a permutation π_j on the set $\{1, 2, \dots, k\}$ such that $a_j^t = b_j^{\pi_j(t)}$ for all $t \in \{1, 2, \dots, k\}$. Further, we say that $(a^1, \dots, a^k)$ *dominates* $(b^1, \dots, b^k)$ iff, $[\rho(a^t) \geq \rho(b^t)$ for all $t \in \{1, \dots, k\}]$ and $[\rho(a^t) > \rho(b^t)$ for some $t \in \{1, \dots, k\}]$.

To understand the concept of achievement-equivalence of two collections of achievement vectors, let's consider five attributes, each of which can take three values, 0, 1 and 2. Consider the following achievement vectors:

$$a^1 = (0,1,1,0,2), a^2 = (1,0,0,0,1), a^3 = (0,0,2,1,0), a^4 = (0,1,0,0,0)$$

$$b^1 = (1,0,0,1,0), b^2 = (0,1,2,0,0), b^3 = (0,1,0,0,2), b^4 = (0,0,1,0,1)$$

Then, we have:

for attribute 1, we have $a_1^2 = b_1^1 = 1$, $a_1^3 = b_1^2 = 0$, $a_1^4 = b_1^3 = 0$, $a_1^1 = b_1^4 = 0$,

for attribute 2, we have $a_2^1 = b_2^2 = 1$, $a_2^4 = b_2^3 = 1$, $a_2^2 = b_2^1 = 0$, $a_2^3 = b_2^4 = 0$,

for attribute 3, we have $a_3^3 = b_3^2 = 2$, $a_3^1 = b_3^4 = 1$, $a_3^2 = b_3^1 = 0$, $a_3^4 = b_3^3 = 0$,

for attribute 4, we have $a_4^4 = b_4^1 = 1$, $a_4^1 = b_2^2 = 0$, $a_4^2 = b_4^3 = 0$, $a_4^4 = b_3^4 = 0$,

for attribute 5, we have $a_5^1 = b_5^3 = 2$, $a_5^2 = b_5^4 = 1$, $a_5^3 = b_5^1 = 0$, $a_5^4 = b_5^2 = 0$.

Consequently, (a^1, a^2, a^3, a^4) and (b^1, b^2, b^3, b^4) are achievement-equivalent. From this example and, in general, it can be checked that, for every attribute f_j and every value v_j which attribute f_j can take, if $(a^1, \dots, a^k)$ and $(b^1, \dots, b^k)$ are achievement-equivalent, then, the number of vectors in $(a^1, \dots, a^k)$, in which v_j figures as the value of f_j, is the same as the number of vectors in $(b^1, \dots, b^k)$, in which v_j figures as the value of f_j.

With the concept of achievement-equivalence of two collections of achievement vectors and the notion of dominance introduced above, we introduce the following property of ρ, which is a variant of a property of ρ introduced in Pattanaik and Xu (2019):

Definition 6.2. ρ satisfies *non-dominance* iff, for all $k > 1$ and all $a^1, \dots, a^k \in V$, $b^1, \dots, b^k \in V$, [$(a^1, \dots, a^k)$ and $(b^1, \dots, b^k)$ are achievement-equivalent] implies that neither $(a^1, \dots, a^k)$ dominates $(b^1, \dots, b^k)$ nor $(b^1, \dots, b^k)$ dominates $(a^1, \dots, a^k)$.

To see the intuition of non-dominance, consider $(a^1 \in V, \dots, a^k \in V)$, $(b^1 \in V, \dots, b^k \in V)$ $(k > 1)$ such that $(a^1, \dots, a^k)$ and $(b^1, \dots, b^k)$ are achievement-equivalent. Since $(a^1, \dots, a^k)$ and $(b^1, \dots, b^k)$ are achievement-equivalent, as we remarked earlier, for every attribute f_j and every value v_j which attribute f_j can take, the number of vectors in $(a^1, \dots, a^k)$, in which v_j figures as the value of f_j, is the same as the number of vectors in $(b^1, \dots, b^k)$, in which v_j figures as the value of f_j. Intuitively, the "aggregate data" regarding dimensional achievements are the same for $(a^1, \dots, a^k)$ and $(b^1, \dots, b^k)$. Given this, Non-Dominance rules out the possibility of having [$\rho(a^t) \geq \rho(b^t)$ for all $t \in \{1, \dots, k\}$] and [$\rho(a^t) > \rho(b^t)$ for some $t \in \{1, \dots, k\}$].

For example, suppose there are five attributes and each attribute can take exactly three values, 0, 1 and 2. Consider the achievement vectors introduced earlier:

$$a^1 = (0,1,1,0,2), a^2 = (1,0,0,0,1), a^3 = (0,0,2,1,0), a^4 = (0,1,0,0,0)$$

$$b^1 = (1,0,0,1,0), b^2 = (0,1,2,0,0), b^3 = (0,1,0,0,2), b^4 = (0,0,1,0,1)$$

As we demonstrated earlier, (a^1, a^2, a^3, a^4) and (b^1, b^2, b^3, b^4) are achievement-equivalent. Then, non-dominance rules out the following two possibilities of individual well-being measures:

(Possibility 1): $\rho(a^k) \geq \rho(b^k)$ for all $k \in \{1,2,3,4\}$ and $\rho(a^k) > \rho(b^k)$ for some $k \in \{1,2,3,4\}$,
(Possibility 2): $\rho(b^k) \geq \rho(a^k)$ for all $k \in \{1,2,3,4\}$ and $\rho(b^k) > \rho(a^k)$ for some $k \in \{1,2,3,4\}$.

With the help of non-dominance, we can show that the individual well-being function ρ has a particular structure. This result is presented in Proposition 6.1 below and its proof is relegated to Section 6.5.

Proposition 6.1. Suppose, for each attribute f_j ($j \in M$), V_j contains a finite number of values. An individual well-being measure ρ satisfies normalization, monotonicity and non-dominance if and only if

(6.1) for some m positive constants, $w_1, \dots, w_m$, with $w_1 + \cdots + w_m = 1$, there exists, for each $j \in M$, an increasing function $\varphi_j: V_j \rightarrow [0, w_j]$, with $\varphi_j(v_j^0) = 0$ and $\varphi_j(v_j^*) = w_j$, such that, for some increasing function $\sigma: [0,1] \rightarrow [0,1]$, with $\sigma(0) = 0$ and $\sigma(1) = 1$, we have

$$\rho(v) = \sigma(\textstyle\sum_{j=1}^{m} \varphi_j(v_j)) \text{ for all } v \in V.$$

The expression $\sum_{j=1}^{m} \varphi_j(v_j)$ figuring in (6.1) can be interpreted as an index of "overall achievement" (IOA) of an individual whose achievement vector is v; $\sigma(\sum_{j=1}^{m} \varphi_j(v_j))$ then has the obvious interpretation as an individual's well-being given as a function of this individual's IOA. Therefore, normalization, monotonicity and non-dominance restrict the form of the individual well-being measure ρ to be a positive transformation of the individual's IOA. It may be noted that variants of this specific form for measures of an individual's well-being in a multidimensional framework have been studied by several authors including Bossert et al. (2013), Dhongde et al. (2016), and Pattanaik and Xu (2019).

We regard the function σ derived in (6.1) as cardinal so that an individual's well-being, $\rho(v) = \sigma(\sum_{j=1}^{m} \varphi_j(v_j))$ corresponding to the achievement vector $v \in V$ is cardinal as well. Note that Proposition 6.1 says that, among the many individual well-being measures that we can construct, there is one that is cardinal, and we can normalize this cardinal measure so that the individual well-being associated with the achievement vector v^0 is 0 and the individual well-being associated with the achievement vector v^* is 1 (normalization). With this interpretation of a cardinal individual well-being function, we can talk about the well-being differences for an individual moving from one achievement vector to another achievement vector, or the well-being differences between two well-being levels of an individual.

The individual well-being function when all attributes are cardinally measurable.

We have examined above the structure of individual well-being functions when all attributes can only take finite and discrete values. We now consider the case in which all attributes are cardinally measurable and the set of values that each attribute can take is a non-degenerate and closed interval of the real line. With a bit of looseness in notation, for each $j \in M$, let $V_j = [0, v_j^*]$ be the set of values which f_j can take (it is a closed real interval): the lowest value in V_j is 0 and the highest value in V_j is $v_j^* > 0$. Note that in this case, we have $v^0 = (0, \ldots, 0)$.

We now analyze the structure of the individual well-being function ρ. For this purpose, we consider the following two weaker versions of non-dominance.

Definition 6.3. ρ satisfies**:**

non-dominance (3) iff, for all $a^1, a^2, a^3 \in V$, $b^1, b^2, b^3 \in V$, if (a^1, a^2, a^3) and (b^1, b^2, b^3) are achievement-equivalent, then neither (a^1, a^2, a^3) dominates (b^1, b^2, b^3) nor (b^1, b^2, b^3) dominates (a^1, a^2, a^3);

and

non-dominance (2) iff, for all $a^1, a^2 \in V$, $b^1, b^2 \in V$, if (a^1, a^2) and (b^1, b^2) are achievement-equivalent, then neither (a^1, a^2) dominates (b^1, b^2) nor (b^1, b^2) dominates (a^1, a^2).

Non-dominance (3) and non-dominance (2) are weaker versions of non-dominance introduced earlier. Non-dominance (3) restricts the two collections of achievement-equivalent achievement vectors, in the premise of non-dominance, to three achievement vectors in each collection, while non-dominance (2) restricts the two collections of achievement-equivalent achievement vectors, in the premise of non-dominance, to just two achievement vectors in each collection.

It may be of interest to compare some conventionally used axioms in the context of cardinal variables to derive additively separable functions with the above two versions of non-dominance.

Definition 6.4. ρ **satisfies:**

separability iff, for all $j \in M$ and all $x = (x_j; x_{-j}), y = (y_j; y_{-j}), z = (z_j; z_{-j}), a = (a_j; a_{-j}) \in V$ and, if ($x_{-j} = y_{-j}, z_j = x_j, a_j = y_j$, and $z_{-j} = a_{-j}$), then

$$\rho(x_j; x_{-j}) \geq \rho(y_j; y_{-j}) \Leftrightarrow \rho(z_j; z_{-j}) \geq \rho(a_j; a_{-j});$$

and

double cancellation iff, for all $(x_1, x_2), (y_1, y_2), (y_1, z_2), (z_1, x_2), (z_1, y_2) \in V_1 \times V_2$,

$$[\rho(x_1, x_2) \geq \rho(y_1, y_2) \,\&\, \rho(y_1, z_2) \geq \rho(z_1, x_2)] \Rightarrow \rho(x_1, x_2) \geq \rho(z_1, y_2).$$

Separability and double cancellation have been used for deriving additively separable measures when, respectively, there are more than 2 attributes and there are exactly two attributes. To see that non-dominance (2) implies separability, we consider $x = (x_j; x_{-j}), y = (y_j; y_{-j}), z = (z_j; z_{-j}), a = (a_j; a_{-j}) \in V$ with ($x_{-j} = y_{-j}, z_j = x_j, a = y_j$, and $z_{-j} = a_{-j}$). Suppose, without loss of generality, $\rho(x_j; x_{-j}) \geq \rho(y_j; y_{-j})$. Suppose to the contrary that $\rho(z_j; z_{-j}) < \rho(a_j; a_{-j})$. Consider two collections of achievement vectors, ($x = (x_j; x_{-j})$ and $w = (a_j; a_{-j})$) and ($y = (y_j; y_{-j})$ and $z = (z_j; z_{-j})$). From ($x_{-j} = y_{-j}, z_j = x_j, a_j = y_j$, and $z_{-j} = a_{-j}$), we see that ($x = (x_j; x_{-j})$ and $a = (a_j; a_{-j})$) and ($y = (y_j; y_{-j})$ and $z = (z_j; z_{-j})$) are achievement equivalent. If $\rho(x_j; x_{-j}) \geq \rho(y_j; y_{-j})$ and $\rho(z_j; z_{-j}) < \rho(a_j; a_{-j})$, then that ($x = (x_j; x_{-j})$ and $a = (a_j; a_{-j})$) dominates ($y = (y_j; y_{-j})$ and $z = (z_j; z_{-j})$), a contradiction of non-dominance (2). Therefore, in the presence of non-dominance (2), separability must hold. Similarly, we can see that, in the presence of non-dominance (3), double cancellation holds.

With the help of non-dominance (3) and non-dominance (2), we obtain the following result the proof of which can be found in the appendix to this chapter.

Proposition 6.2. Suppose for each $j \in M$, $V_j = [0, v_j^*]$ with $v_j^* > 0$.

If $m = 2$, an individual well-being measure ρ satisfies normalization, monotonicity, and non-dominance (3) if and only if

(6.2) for some m positive constants, $w_1, \dots, w_m$, with $w_1 + \cdots + w_m = 1$, there exists, for each $j \in M$, an increasing function $\varphi_j \colon V_j \to [0, w_j]$, with $\varphi_j(0) = 0$ and $\varphi_j(v_j^*) = w_j$, such that, for some increasing function $\sigma \colon [0,1] \to [0,1]$, with $\sigma(0) = 0$ and $\sigma(1) = 1$, we have

$$\rho(v) = \sigma\left(\sum_{j=1}^{m} \varphi_j(v_j)\right) \text{ for all } v \in V.$$

Further, if $m > 2$, an individual well-being measure ρ satisfies normalization, monotonicity, and non-dominance (2) if and only if (6.2) holds.

Again, for the case in which every attribute is cardinally measurable along an interval, an individual well-being measure is given as a function of this individual's IOA if the individual well-being measure satisfies normalization, monotonicity and certain weaker versions of non-dominance. Note that the individual well-being function is regarded as cardinal as we discussed earlier.

For applied work, it is often necessary to impose further structure on the function $\rho(\cdot)$ that figures in (6.2). Several specific forms for the function $\rho(\cdot)$ have been suggested in the literature.[1] When choosing a specific form for individual well-being functions, a question that often comes up is whether one should build into the form of the function any assumption about substitutability, complementarity or independence of the attributes. We now turn to this issue.

---

[1] See, for instance, Blackorby and Donaldson (1982), Maasoumi (1986), Tsui (1995), Tsui and Weymark (1997), Bourguignon (1999), Decancq and Lugo (2012), and Dhongde et al. (2021).

Complementarity and substitutability of attributes

Given an individual well-being function $\rho(\cdot)$ as specified in (6.1) and (6.2)) , we can now discuss and define the notions of complementarity and substitutability of attributes. Our definitions rely on the Auspitz-Lieben-Edgeworth-Pareto (ALEP) notions of complements and substitutes as defined in Auspitz-Lieben (1889) and quoted by Edgeworth (1925) and Pareto (1906) (see Kannai (1980)).

Definition 6.5. Suppose for every $j \in M$, the attribute f_j is cardinally measurable along the interval $[0, v_j^*]$ and $\rho(v)$ is twice differentiable. Then two attributes, f_j and $f_{j'}$ are said to be: (i) *ALEP substitutes* iff $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}} < 0$; (ii) *ALEP complements* iff $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}} > 0$; and *ALEP independent* iff $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}} = 0$.

Suppose, for every $j \in M$, the attribute f_j is cardinally measurable along the real interval $[0, v_j^*]$ and $\rho(\cdot)$ is a twice differentiable individual well-being function of the type specified in (6.2): for all achievement vectors $v \in V$, $\rho(v) = \sigma(\sum_{j=1}^m \varphi_j(v_j))$, where $\sigma\colon [0,1] \to [0,1]$, is increasing, and $\sigma(0) = 0$ and $\sigma(1) = 1$, and for each $j \in M$, $\varphi_j\colon V_j \to [0, w_j]$, is increasing and $\varphi_j(v_{0j}) = 0$ and $\varphi_j(v_j^*) = w_j > 0$, and $w_1 + \cdots + w_m = 1$. Then, by Definition 6.5, noting $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}} = \sigma''(\sum_{j=1}^m \varphi_j(v_j))\varphi_j'(v_j)\varphi_{j'}'(v_{j'})$ for all $j, j' \in M$ and $\varphi_k'(v_k) > 0$ for all $k \in M$, it follows that two attributes, f_j and $f_{j'}$, are: (i) ALEP complements iff $\sigma''(\sum_{j=1}^m \varphi_j(v_j)) > 0$, (ii) ALEP substitutes iff $\sigma''(\sum_{j=1}^m \varphi_j(v_j)) < 0$, and (iii) ALEP independent iff $\sigma''(\sum_{j=1}^m \varphi_j(v_j)) = 0$.

Thus, when $\varphi_j'(\cdot) > 0$ for each attribute, two attributes are: (i) ALEP complements if σ is strictly convex, (ii) ALEP substitutes if σ is strictly concave, and iii) ALEP independent if σ is linear.

Remark 6.1. Let $h(\cdot)$ be a strictly increasing function and consider $h\big(\rho(v)\big) = h(\sigma(\sum_{j=1}^{m} \varphi_j(v_j)))$ (that is, $h(\sigma(\sum_{j=1}^{m} \varphi_j(v_j)))$ is a strictly monotone transformation of the well-being function $\sigma(\sum_{j=1}^{m} \varphi_j(v_j))$). Note that $\frac{\partial^2 h(\rho(v))}{\partial v_j \partial v_{j'}} = (h''(\sigma')^2 + h'\sigma'')\varphi_j'(v_j)\varphi_{j'}'(v_{j'})$. Thus, assuming that $\varphi_j'\big(v_j\big) > 0, \varphi_{j'}'(v_{j'}) > 0$, ALEP complementarity defined above is invariant under strictly increasing convex transformations of the well-being function $\rho(\cdot)$, ALEP substitutability defined above is invariant under strictly increasing concave transformations of the well-being function $\rho(\cdot)$, and ALEP independence defined above is invariant under strictly increasing linear transformations of the well-being function $\rho(\cdot)$.

Remark 6.2. Note that Remark 6.1 is only appropriate for an individual well-being function $\rho(v)$ taking the form of $\sigma(\sum_{j=1}^{m} \varphi_j(v_j))$ as derived in Section 6.2. For a general well-being function $\rho(v)$, as noted by Allen (1934), Hicks and Allen (1934), and Hicks (1939), the condition on the cross partial $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}}$ is not invariant under increasing transformations of the well-being function $\rho(\cdot)$. In such cases, one would have to be careful in defining notions of complementary, substitutable and independent attributes.[2]

[2] See Kannai (1980) for a modified notion of the ALEP complements/substitutes where he considers the class of least concave well-being functions for the given ranking of achievement vectors to define complements and substitutes. A concave well-being function $\rho(\cdot)$ is said to be least concave if every concave well-being function $\Phi(\cdot)$ representing the giving ranking of achievement vectors is given by $\Phi(\cdot) = \Psi(\rho(\cdot))$, where Ψ is a strictly increasing and concave function of a single variable. It may be noted that the least concave well-being functions are unique up to a positive linear transformation.

Example 6.1. Consider the following individual well-being functions (one can easily normalize each function so that its range is [0,1]): for some $w_j > 0$ $(j = 1, \dots, m)$ with $w_1 + \cdots + w_m = 1$, and for all $v \in V$,

(i)

$$\rho^I(v) = \left(1 + \sum_{j=1}^{m} w_j v_j\right)^{\gamma} + \alpha \ln\left(1 + \sum_{j=1}^{m} w_j v_j\right) - 1,$$

where $\alpha > 0, \gamma > 0$;

(ii)

$$\rho^{II}(v) = \left(\sum_{j=1}^{m} w_j v_j^{\beta}\right)^{\frac{\alpha}{\beta}}$$

where $\alpha \neq 0, \beta \neq 0$ (see Bourguignon (1999); see also Decancq and Lugo (2012) and Maasoumi (1986) for a variant of this individual well-being function where they take $\alpha = 1$);

(iii)

$$\rho^{III}(v) = \prod_{j=1}^{m} v_j^{w_j}$$

(see Decancq and Lugo (2012), and Tsui (1995));

(iv)

$$\rho^{IV}(v) = \left(\sum_{j=1}^{m} w_j v_j\right)^{\gamma}$$

with $\gamma > 0$.

Note that

$$\frac{\partial^2\rho^I(v)}{\partial v_j\partial v_{j'}} = \gamma(\gamma-1)\left(1+\sum_{j=1}^{m} w_j v_j\right)^{\gamma-2} - \frac{\alpha w_j w_{j\prime}}{(1+\sum_{j=1}^{m} w_j v_j)^2}$$

Then, if $\gamma \le 1$, we have $\frac{\partial^2\rho^I(v)}{\partial v_j\partial v_{j'}} < 0$ implying that two attributes, f_j and $f_{j'}$, are ALEP substitutes; and if $\gamma > 1$, depending on $\gamma(\gamma-1)\big(1+\sum_{j=1}^{m} w_j v_j\big)^{\gamma-2} - \frac{\alpha w_j w_{j\prime}}{(1+\sum_{j=1}^{m} w_j v_j)^2}$, two attributes, f_j and $f_{j'}$, can be ALEP substitutes, or ALEP complements, or ALEP independent. For example, for $\gamma = 2, w_1 = \cdots = w_m = \frac{1}{m}$, we have that two attributes, f_j and $f_{j'}$, are (1) ALEP substitutes if $\sum_{j=1}^{m} w_j v_j < \frac{\sqrt{2\alpha}}{2m} - 1$, (2) ALEP complements if $\sum_{j=1}^{m} w_j v_j > \frac{\sqrt{2\alpha}}{2m} - 1$, and (3) ALEP independent if $\sum_{j=1}^{m} w_j v_j = \frac{\sqrt{2\alpha}}{2m} - 1$.

For $\rho^{II}(v)$, tote that

$$\frac{\partial^2\rho^{II}(v)}{\partial v_j\partial v_{j'}} = \alpha(\alpha-\beta) w_j w_{j'} v_j^{\beta-1} v_{j'}^{\beta-1}\left(\sum_{j=1}^{m} w_j v_j^{\beta}\right)^{\frac{\alpha}{\beta}-2}$$

Then, two attributes, f_j and $f_{j'}$, are: (i) ALEP complements iff $\alpha(\alpha-\beta) > 0$, (ii) ALEP substitutes iff $\alpha(\alpha-\beta) < 0$, and (iii) ALEP independent iff $(\alpha-\beta) = 0$ $(\text{note that } \alpha \neq 0)$.

For $\rho^{III}(v)$, tote that

$$\frac{\partial^2\rho^{III}(v)}{\partial v_j\partial v_{j'}} = \frac{w_j w_{j'}}{v_j v_{j'}} \prod_{j=1}^{m} v_j^{w_j}$$

Then, any two attributes, f_j and $f_{j'}$, are always ALEP complements since $\frac{\partial^2\rho(v)}{\partial v_j\partial v_{j'}} > 0$.

For ρ^{IV}, we note that

$$\frac{\partial^2 \rho^{IV}(v)}{\partial v_j \partial v_{j'}} = \gamma(\gamma-1) w_j w_{j'} \left(\sum_{j=1}^{m} w_j v_j \right)^{\gamma-2}$$

Then, two attributes, f_j and $f_{j'}$, are: (i) ALEP complements iff $\gamma > 1$, (ii) ALEP substitutes iff $\gamma < 1$, and (iii) ALEP independent iff $\gamma = 1$.

When an individual well-being function $\rho(\cdot)$ is not differentiable (including when attributes can only take discrete values), we can use supermodular/submodular/modular well-being functions to define complementary, substitutable, and independent attributes.

For two achievement vectors, $v = (v_1, \dots, v_m), u = (u_1, \dots, u_m) \in V$, let $v \uparrow u$ denote the component-wise maximum of v and u, and $v \downarrow u$ denote the component-wise minimum of v and u.

Definition 6.6. Given an individual well-being function $\rho(\cdot)$, two attributes, f_j and $f_{j'}$, are said to be

complements iff $\rho(v \uparrow u) + \rho(v \downarrow u) > \rho(v) + \rho(u)$,

substitutes iff $\rho(v \uparrow u) + \rho(v \downarrow u) < \rho(v) + \rho(u)$,

and

independent iff $\rho(v \uparrow u) + \rho(v \downarrow u) = \rho(v) + \rho(u)$,

for all $v, u \in V$ with $v_k = u_k$ for all $k \neq j, j'$ and $\{v \uparrow u, v \downarrow u\} \neq \{v, u\}$.

It may be remarked that the above notions of complements, substitutes and independent attributes rely on weaker notions of an individual well-being function being supermodular, submodular, and modular, respectively. When an individual well-being function $\rho(\cdot)$ is twice differentiable, the condition $\rho(v \uparrow u) + \rho(v \downarrow u) > \rho(v) + \rho(u)$ for all $v, u \in V$ with $v_k = u_k$ for all $k \neq j, j'$ and $\{v \uparrow u, v \downarrow u\} \neq \{v, u\}$ implies $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}} \geq 0$, and the condition $\rho(v \uparrow u) + \rho(v \downarrow u) < \rho(v) + \rho(u)$ for all $v, u \in V$ with $v_k = u_k$ for all $k \neq j, j'$ and $\{v \uparrow u, v \downarrow u\} \neq \{v, u\}$ implies $\frac{\partial^2 \rho(v)}{\partial v_j \partial v_{j'}} \leq 0$. See, for example, Topkis (1998).

Intuitively, the assumption that attributes are ALEP substitutes of each other seems to be more plausible than the assumption that the attributes are complements of each other. In fact, we are unable to think of plausible examples of attributes which are ALEP complements.

6.4. Social well-being measure

In this section, we discuss measures of social well-being. For each $i \in N$, let $a_i = (a_{i1}, \dots, a_{ij}, \dots, a_{im}) \in V$ be individual i's achievement vector. Let

$$\mathcal{A} = \{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}: a_i \in V, i \in N\}$$

be the set of all $n \times m$ matrices such that, for all $i \in N$, $a_i \in V$ is individual i's achievement vector. Each $A \in \mathcal{A}$ is interpreted as the society's achievement matrix.

For all $v \in V$, all $i \in N$, and all $A \in \mathcal{A}$, $(v;\ A_{-i})$ will denote the matrix $B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathcal{A}$ such that $b_i = v$ and, for all $i' \in N/\{i\}$, $b_{i'} = a_{i'}$; thus, $(v;\ A_{-i})$ is the achievement matrix derived by replacing a_i by v in A, other things remaining the same.

A *social well-being measure* is a function g from $\mathcal{A}$ to $[0,1]$ with the interpretation that, for all $A, B \in \mathcal{A}$, $g(A) \geq g(B)$ indicates that the society's well-being level under A is at least as high as the society's well-being level under B, $g(A) > g(B)$ indicates that the society's well-being level under A is higher than the society's well-being level under B, and $g(A) = g(B)$ indicates that the society's well-being level under A is the same as the society's well-being level under B.

Definition 6.7. The social well-being measure g satisfies *welfarism* iff there exists a symmetric function $\tau: [0, \infty)^n \rightarrow [0, \infty)$, such that, if for all $i \in N, \rho_i: V \rightarrow [0,1]$ is individual i's well-being function, then, for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A}$, $g(A) = \tau\big(\rho_1(a_1), \ldots, \rho_i(a_i), \ldots, \rho_n(a_n)\big)$.

Thus, given an achievement matrix of the society, a social well-being measure satisfying welfarism focuses on individuals' well-being exclusively and assesses the level of social well-being corresponding to the achievement matrix by aggregating the levels of different individuals' well-being in the society as indicated by their respective achievement vectors contained in the achievement matrix. Given the definition of welfarism, the following result, Proposition 6.3, follows directly from our earlier discussion in this section. The proof of Proposition 6.3. follows easily from welfarism and Propositions 6.1 and 6.2 and we omit it.

Proposition 6.3. Suppose the social well-being measure g satisfies welfarism and each individual i's ($i \in N$) well-being function ρ_i satisfies normalization, monotonicity, and

(i) non-dominance when, for each attribute f_j ($j \in M$), V_j contains a finite number of values,

or

(ii) non-dominance (3) if $m = 2$ and non-dominance (2) if $m > 2$ when, for each $j \in M$, $V_j = [v_j^0, v_j^*]$ with $v_j^* > 0$.

Then, for some function $\tau: [0, \infty)^n \rightarrow [0,1]$, for some m positive constants, $w_1, \ldots, w_m$, with $w_1 + \cdots + w_m = 1$, there exists, for each $j \in M$, an increasing function $\varphi_j: V_j \rightarrow [0, w_j]$, with $\varphi_j(v_j^0) = 0$ and $\varphi_j(v_j^*) = w_j$, such that, for some increasing functions $\sigma_i: [0,1] \rightarrow [0,1]$ $(i = 1, \cdots, n)$, with $\sigma_i(0) = 0$ and $\sigma_i(1) = 1$, we have

(6.3) $g(A) = \tau\left(\sigma_1\left(\sum_{j=1}^m \varphi_j(a_{1j})\right), \ldots, \sigma_n\left(\sum_{j=1}^m \varphi_j(a_{nj})\right)\right)$

If we are willing to impose the following properties on a social well-being measure, the aggregating function, τ, when the social well-being measure satisfies welfarism, can be further restricted to be additive.

Definition 6.8. A social well-being measure g satisfies:

S-*normalization* iff, for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A}$, if for all $i \in N$: $a_i = v^0$, then $g(A) = 0$, and if for all $i \in N$: $a_i = v^*$, then $g(A) = 1$;

and

S-independence iff for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, A' = \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix}, B' = \begin{pmatrix} b'_1 \\ \vdots \\ b'_n \end{pmatrix} \in \mathcal{A}$, and for all $i' \in N$, if [(for all $i \in N \setminus \{i'\}$: $a_i = b_i$ and $a'_i = b'_i$), and ($a_{i'} = a'_{i'}$ and $b_{i'} = b'_{i'}$)], then $g(A) - g(B) = g(A') - g(B')$.

S-normalization is a standard axiom in the literature on measurement of multi-dimensional well-being and deprivation, and requires that, when every individual's achievement in every attribute is at the lowest level v_0, the measure of social well-being is also at the lowest and is normalized as 0, and when every individual's achievement in every attribute is at the highest level, the measure of social well-being is at the highest and is normalized as 1. S-independence is another straightforward axiom used in the literature, and stipulates that, starting with a given achievement matrix, when the achievement vector of one individual changes while all other individuals' achievement vectors remain unchanged, the resulting change in the society's well-being is independent of the achievement vectors of those other individuals. A variant of S-independence was proposed in a different context by Chakraborty, Pattanaik and Xu (2008), and was used in Dhongde et al. (2016) in the context of measuring multidimensional deprivation with binary data.

Proposition 6.4. For each $i \in N$, let ρ_i be individual i's well-being function. A social well-being measure g satisfies welfarism, S-normalization and S-independence if and only if:

(6.4) there exists a function $\vartheta\colon [0,\infty) \to [0,\infty)$ such that, for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A}$,

$$g(A) = \vartheta\big(\rho_1(a_1)\big) + \cdots + \vartheta\big(\rho_n(a_n)\big)$$

Thus, according to Proposition 6.4, the properties of welfarism, S-normalization and S-independence imposed on a social well-being measure imply that the social well-being measure is additively separable in the individuals' well-being levels. It may be of interest to note that, when each individual's well-being is viewed as that individual's utility, the form of an additively separable social well-being measure given in (6.4) is known as prioritarian in the philosophical literature (see, for example, Parfit (1995), Hirose (2011)) or as generalized utilitarian in the social choice literature (see, for example, Blackorby, Bossert and Donaldson (2002)).

In the following section, we discuss some conceptual issues relating to the intuitive interpretation of a social well-being function as given by (6.4) and the associated problem of interpersonal comparisons of individual well-being.

6.5. The social well-being measure and interpersonal comparisons of individual well-being

Consider a member of the class of social well-being functions specified by (6.4) in the preceding section. The first thing that we would like to note is that, though we derived Proposition 6.4 in the framework of the FCA where the individual well-being is interpreted as the value attached to the individual's functioning bundle, the formal derivation does not depend on this specific interpretation. For instance, that formal derivation would remain virtually intact if, for all $i \in N$, a_i was interpreted as the commodity bundle consumed by individual i and $\rho_i(\cdot)$ was i's (cardinal) utility function, utility being interpreted as happiness or preference satisfaction. In this case, what can be the interpretation of the functions ϑ figuring in Proposition 6.4? One possible interpretation of these functions is that they reflect the process of rendering the utility difference of different individuals commensurate by translating them into happiness differences. They establish whether an increase in individual 1's happiness associated with an increase of 0.5 recorded by 1's utility function is greater than , smaller than or equivalent to the decrease in individual e's happiness associated with a decrease of 0.4 recorded by e's utility function. Note that interpersonal comparisons of utility differences for one person with that for another person is possible because the notion of happiness has descriptive content independent of the commodity bundle that generates the individual's happiness. While the term "happiness" refers to a mental state, it has a descriptive content of its own. The independent descriptive content of happiness allows us to compare the change in the happiness of one person, i, that goes with a change in i's utility number with the change in happiness of another person, e, that goes with a change in e's utility number. So far we have considered interpersonal comparisons of utility differences by utilizing the descriptive content of the notion of happiness. We can also similarly think of interpersonal comparisons of utility levels when necessary.

Now consider the notion, used in the FCA, of an individual well-being as the value that she attaches to her functioning bundle. It may be tempting to think that interpersonal comparisons of well-being here may be handled in the same way in which we handle interpersonal comparisons of utility when utility is interpreted as happiness or preference satisfaction. There is, however, one major difference between the problem of interpersonal comparisons of well-being changes when well-being is conceived as preference satisfaction or happiness and the corresponding problem when an individual's well-being is interpreted as the value that she attaches to the functioning bundle achieved by her. As Amartya Sen (*Commodities and Capabilities*, 1987) notes with incisive clarity, though the notions of happiness or preference satisfaction refer to mental states, they have independent descriptive content while well-being of an individual, conceived as the value that the individual attaches to her own functioning bundle, does not have any independent descriptive content of its own. While, in practice, it may be difficult to measure change in the happiness (or levels of happiness) of individuals, given that the notion of happiness has descriptive content, in principle we can hope to measure changes in the happiness (resp. levels of happiness) of two individuals i and e and compare the change in the happiness (resp. the level of happiness) of i with that of e. But the notion an individual's well-being as the individual's own valuation of her achieved functioning bundle does not have any such descriptive content. It is simply the value that the individual puts on her achieved functioning bundle. In the absence of any descriptive content of an individual's valuation of a functioning bundle, one cannot compare the difference in the values that an individual attaches to two different functioning bundles with the difference in the values that another individual attaches to two functioning bundles in the same way as one could compare the difference in the happiness of one individual in two different situations with the difference in the happiness of another individual in two situations. As Sen (*Commodities and Capabilities*, 1987, 36-37) observes:

> The mental characteristics of happiness, desire, etc. exist in their own right and the utility function … establishes an empirical connection between commodities and utility. In contrast, the valuation function sees well-being as 'supervenient' on descriptive information (functionings in this case), without an independent descriptive content.

In the absence of any independent descriptive content of the notion of the value that an individual attaches to her functioning bundle, there is no *empirical* basis for deciding whether an increase of 0.5 in the value of i's valuation function represents a bigger or smaller or identical increase in value as compared to an increase of 0.4 in the value of e's valuation function.

A different route has been taken by Sen (1987) who has sought to formulate some widely accepted or acceptable *convention* for comparing the values that two individuals attach to their respective functioning bundles. One such convention suggested by Sen (1987) is the dominance criterion introduced and examined in Chapter 3, Section 3.2. Unfortunately, as shown in Chapter 3, Section 3.2, this convention runs into problems; we have already discussed the conceptual difficulties that arise when, in the FCA, we try to accommodate, in addition to the dominance criterion, even moderate differences in the values that constitute the basis for assessing the different individuals' well-being.

In light of such difficulties, two routes seem to be in front of us. One route is to stick to individuals' self-evaluations of their functioning bundles and invoke the ethical evaluator's value judgments about what relative importance the numbers yielded by the n individual self-evaluation functions should have in the assessment of social well-being as discussed in Section 6.4. Note that, along this route, certain limited conventions for comparing the values that two individuals attach to their respective functioning bundles can be embedded in the ethical evaluator's value judgment. For example, if we take individual well-being functions characterized in Section 6.2, then we have $\rho_i(v^0) = 0 < 1 = \rho_e(v^*)$ for any two individuals $i, e \in N$. Now, if the ethical evaluator's value judgments are such that equal importance is to be attached to the valuations of all the individuals, then any individual having the bundle v^* would be deemed to have a higher level of well-being than any individual having the bundle v^0.

Of course, the dominance axioms formulated in Chapter 3 may not hold in general for self-evaluation functions discussed in Section 6.2.

The second route out of such difficulties is to assume that an individual well-being function ρ_i is invariant with respect to i. It is not surprising that, in practical applications of the FCA, when comparing the well-being of a given society at different points of time or the well-being of different societies, social scientists often use a framework where the well-being functions of all individuals under consideration are identical and have a specific form and social well-being is assessed on the basis of the individual well-being functions with the postulated form. How do we interpret the individual well-being function which is identical for all individuals in such a framework? One can think of alternative possible interpretations here. First, it may be interpreted as reflecting the unanimously held values in the society under consideration (it is possible that unanimity is reached after open discussion and debates in the society). Such unanimity, when it exists, makes the problem of measuring social well-being significantly simpler, but it is highly unlikely that we would have it in practice in any sizable society even after extensive discussions in the society. Second, the specific individual well-being function can be taken to be a well-being function resulting from the use of some voting procedure after discussions fail to resolve differences in the values of different individuals. In Chapter 3, we expressed our concern about this particular route for arriving at a single individual well-being function to be used to evaluate the well-being of all individuals in the society. We do not know of any instance where a social scientist has claimed that the individual well-being function used by her represents the unanimously held values of all individuals in the society or the outcome of a voting rule explicitly used to choose a single individual well-being function for all individuals in a given society. A third possible interpretation is that the individual well-being function assumed to be identical for all individuals reflects the social scientist's own values and is presented as a contribution to the public discourse about how an individual's well-being should be assessed. In the subsequent discussions in the remaining chapters, Chapters 7,8 and 9, we shall assume that all individuals in the society share the same individual well-being function.

To conclude this section, we note that, in the literature on multidimensional well-being and deprivation, various individual well-being functions have been introduced and studied. For example, the following class of measures has been studied by Decancq and Lugo (2012) axiomatically with the help of related results in the theory of social choice (for example, Blackorby and Donaldson (1982) and Tsui and Weymark (1997)) and by Maasoumi (1986) from the perspective of information theory: for a number β, for all $v = (v_1, \ldots, v_m) \in V = [0, \infty)^m$,

$$\rho(v) = \begin{cases} (\sum_{j=1}^{m} w_j v_j^{\beta})^{1/\beta}, \beta \neq 0 \\ \prod_{j=1}^{m} v_j^{w_j}, \qquad \beta = 0 \end{cases}$$

where $w_j > 0$ $(j = 1, \ldots, m)$ and $w_1 + \cdots + w_m = 1$.

Tsui (1995) studied a social well-being measure defined below axiomatically: for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A},\ g(A) = \sum_{i=1}^{n}(a + b\prod_{j=1}^{m} a_{ij}^{r_j})$, where $a, b > 0, r_j > 0 (j = 1, \ldots, m)$ are given parameters and Bourguignon (1999) studied the following social well-being measure: for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A}$, $g(A) = \sum_{i=1}^{n}(\sum_{j=1}^{m} w_j a_{ij}^{\beta})^{\alpha/\beta}$, where $\alpha, \beta \neq 0$ are given parameters.

In studying inequality in multidimensional well-being in the United States, Dhongde et al. (2021) use a social well-being measure defined as follows: for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A},\ g(A) =$

$\sum_{i=1}^{n}(\sum_{j=1}^{m} w_j a_{ij})^{\gamma}$, where $\gamma > 0$ is a given parameter, and $w_j > 0$ $(j = 1, \dots, m)$ and $w_1 + \cdots + w_m = 1$.

6.6. Proofs

This section contains the proofs of Propositions 6.1 and 6.2 in Section 6.2, and Proposition 6.4 in Section 6.4

Proof of Proposition 6.1. Suppose, for each attribute f_j $(j \in M)$, V_j contains a finite number of values. Let ρ be an individual well-being measure satisfying normalization, monotonicity and non-dominance. In what follows, we shall show that the following holds:

(6.7) for some m positive constants, $w_1, \dots, w_m$, with $w_1 + \cdots + w_m = 1$, there exists, for each $j \in M$, an increasing function $\varphi_j : V_j \to [0, w_j]$, with $\varphi_j(v^0) = 0$ and $\varphi_j(v_j^*) = w_j$, such that for all $v, v' \in V$,

$$\rho(v) \geq \rho(v') \Leftrightarrow \sum_{j=1}^{m} \varphi_j(v_j) \geq \sum_{j=1}^{m} \varphi_j(v_j').$$

Let $k > 1$ and let $\{a^1, \dots, a^k\}$ and $\{b^1, \dots, b^k\}$ be two achievement-equivalent collections of individual achievement vectors. By non-dominance, it follows that ρ satisfies the following:

(6.8) if $\rho(a^p) \geq \rho(b^p)$ for all $p = 1, \dots, k-1$, then $\rho(b^k) \geq \rho(a^k)$.

By Theorem 4.1 of Fishburn (1970), for each $j \in M$, there exists a function $h_j : V_j \to [0, \infty)$ such that,

(6.9) for all $v, v' \in V$, $\rho(v) \geq \rho(v') \Leftrightarrow \sum_{j=1}^{m} h_j(v_j) \geq \sum_{j=1}^{m} h_j(v_j')$.

By normalization, $\rho(v_1^*, \dots, v_m^*) = 1 > 0 = \rho(v_0)$. By monotonicity, ρ is increasing in each of its arguments. Then, each h_j $(j \in M)$ is increasing as well. Therefore, for each $j \in M$, $h_j(v_j^*) >$

$h_j(v_j^0) \geq 0$. For each $j \in M$, let $w_j = \frac{h_j(v_j^*) - h_j(v_j^0)}{\sum_{k \in M}(h_k(v_k^*) - h_k(v_j^0))}$ and define the function φ_j as follows: for all $v_j \in V_j$,

$$\varphi_j(v_j) = \frac{h_j(v_j) - h_j(v_j^0)}{\sum_{k \in M}(h_k(v_k^*) - h_k(v_j^0))}.$$

Then, for each $j \in M$, $w_j > 0$, φ_j is increasing with $\varphi_j(v_j^0) = 0$ and $\varphi_j(v_j^*) = w_j$, and $\varphi_j: V_j \to [0, w_j]$. Note also that $w_1 + \cdots + w_m = 1$, and for all $v, v' \in V$,

$$\rho(v) \geq \rho(v') \Leftrightarrow \sum_{j=1}^{m} \varphi_j(v_j) \geq \sum_{j=1}^{m} \varphi_j(v_j').$$

Thus, we have established (6.7).

From (6.7), there exists an increasing function $\sigma: [0,1] \to [0,1]$ such that, for all $v \in V$, $\rho(v) = \sigma(\sum_{j=1}^{m} \varphi_j(v_j))$, $\sigma(0) = 0$ and $\sigma(1) = 1$.

To complete the proof of Proposition 6.1, we note that, if the statement (6.7) holds, then it can be verified that the measure ρ specified in the statement satisfies normalization, monotonicity, and non-dominance. ■

Proof of Proposition 6.2. Suppose for each $j \in M$, $V_j = [0, v_j^*]$ with $v_j^* > 0$. Let ρ be an individual well-being measure satisfying non-dominance (3) when $m = 2$ and non-dominance (2) when $m > 2$. As in the proof of Proposition 6.1, we shall show that the following holds:

(6.10) for some m positive constants, $w_1, \ldots, w_m$, with $w_1 + \cdots + w_m = 1$, there exists, for each $j \in M$, an increasing function $\varphi_j: V_j \to [0, w_j]$, with $\varphi_j(0) = 0$ and $\varphi_j(v_j^*) = w_j$, such that for all $v, v' \in V$,

$$\rho(v) \geq \rho(v') \Leftrightarrow \sum_{j=1}^{m} \varphi_j(v_j) \geq \sum_{j=1}^{m} \varphi_j(v_j').$$

Let $k \in \{3,2\}$ and let $\{a^1, \ldots, a^k\}$ and $\{b^1, \ldots, b^k\}$ be two achievement-equivalent collections of individual achievement vectors. When $m = 2$, by non-dominance (3), and when $m > 2$, by non-domiance (2), it follows that ρ satisfies the following:

(6.11) if $\rho(a^p) \geq \rho(b^p)$ for all $p = 1,2$, then $\rho(b^3) \geq \rho(a^3)$ when $m = 2$;

and

(6.12) if $\rho(a^1) \geq \rho(b^1)$, then $\rho(b^2) \geq \rho(a^2)$ when $m > 2$.

By Fishburn's (1970) Theorem 5.4 when $m = 2$ and Theorem 5.5 when $m > 2$, for each $j \in M$, there exists a function $h_j: V \to [0, \infty)$ such that,

(6.13) for all $v, v' \in V$, $\rho(v) \geq \rho(v') \Leftrightarrow \sum_{j=1}^{m} h_j(v_j) \geq \sum_{j=1}^{m} h_j(v_j')$.

Then, the remainder of the proof is similar to that of Proposition 6.1 and we omit it. ■

Proof of Proposition 6.4. For each $i \in N$, let ρ_i be individual i's well-being function. Let g be a well-being measure of the society satisfying welfarism, S-normalization and S-independence. Let $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ be any given matrix in $\mathcal{A}$. Let $\mathbb{O}$ be the achievement matrix where every individual's achievement vector is v^0.

For every $i \in N$, let $A^i = (a_i;\ \mathbb{O}_{-i})$. Consider $B \equiv (v^0;\ A_{-i})$. Consider the achievement matrices, A, B, A^1 and $\mathbb{O}$. Then, by S-independence, we have

$$(6.14)\ g(A) - g(B) = g(A^1) - g(\mathbb{O}).$$

By S-Normalization, $g(\mathbb{O}) = 0$. We then obtain

$$(6.15)\ g(A) - g(B) = g(A^1) = g(a_1;\ \mathbb{O}_{-1}).$$

Consider $C = (v^0; B_{-2})$, and the achievement matrices B, C, A^2 and $\mathbb{O}$. By S-independence and S-normalization we have

$$(6.16)\ g(B) - g(C) = g(A^2) - g(\mathbb{O}) = g(A^2) = g(a_2;\ \mathbb{O}_{-2}).$$

From (6.15) and (6.16), we then have

$$(6.17)\ g(A) - g(C) = g(a_1; \mathbb{O}_{-1}) + g(a_2; \mathbb{O}_{-2}).$$

By repeating the above procedures with C and beyond and from S-independence and S-normalization, we can obtain

$$(6.18)\ g(A) - g(a_n; \mathbb{O}_{-n}) = g(a_1; \mathbb{O}_{-1}) + \cdots + g\big(a_{n-1}; \mathbb{O}_{-(n-1)}\big).$$

Then

(6.19) $g(A) = g(a_1; \mathbb{O}_{-1}) + \cdots + g\big(a_{n-1}; \mathbb{O}_{-(n-1)}\big) + g(a_n; \mathbb{O}_{-n})$

By welfarism, for some symmetric function $\tau: [0, \infty)^n \rightarrow [0, \infty)$, we have

(6.20) for all $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{A}$, $g(X) = \tau(\rho_1(x_1), \dots, \rho_n(x_n))$

In particular, we have

(6.21) for all $i \in N$: $g(a_i; \mathbb{O}_{-i}) = \tau(\rho_1(v^0), \dots, \rho_{i-1}(v_0),\ \rho_i(a_i), \rho_{i+1}(v^0), \dots, \rho_n(v^0))$

For all $i \in N$, define

$$\vartheta_i\big(\rho_i(a_i)\big) = \tau\big(\rho_1(v^0), \dots, \rho_{i-1}(v^0),\ \rho_i(a_i),\ \rho_{i+1}(v^0), \dots, \rho_n(v^0)\big)$$

then, ϑ_i is a function from $[0, \infty)$ to $[0, \infty)$. Then, from (6.19), (6.20) and (6.21), we obtain:

(6.22) $g(A) = \vartheta_1\big(\rho_1(a_1)\big) + \cdots + \vartheta_n\big(\rho_n(a_n)\big)$

Note that τ is symmetric. Therefore, $\vartheta_1 = \cdots = \vartheta_n$. We denote this common function by ϑ.

On the other hand, if, for each $i \in N$, ρ_i is individual i's well-being function and there exists a function $\vartheta: [0, \infty) \rightarrow [0, \infty)$ such that, for all $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathcal{A}$,

$$g(A) = \vartheta\big(\rho_1(a_1)\big) + \cdots + \vartheta\big(\rho_n(a_n)\big)$$

then it can be verified that g satisfies welfarism, S-normalization and S-independence. ■

[This is a draft of Chapter 7 in a manuscript by Yongsheng Xu and me. My UCR seminar (20 October 2021) on the measurement of multidimensional deprivation will be based on this chapter. But I am posting preliminary drafts of both Chapter 6 and Chapter 7 on our seminar website since much of the notation and several concepts in Chapter 7 are introduced in Chapter 6.

Prasanta K. Pattanaik]

Chapter 7. Measurement of deprivation

Prasanta K. Pattanaik and Yongsheng Xu

This chapter is concerned with the problem of measuring multidimensional deprivation, which has been studied in the FCA at least as extensively as the problem of measuring multidimensional well-being. The approach, which we follow here and the basic features of which are to be found in earlier contributions of several writers including Tsui (2002), Bourguignon and Chakravarty (2003), Permanyer (2014, 2019), and Pattanaik and Xu (2018), differs significantly from the conventional and more widely used approach in the literature on the measurement of deprivation in a society. We first present our approach to the problem (Section 7.1), and then briefly outline the conventional approach (Section 7.2) to the problem of measuring multidimensional deprivation proposed and studied in the literature. In Section 7.3, we compare and contrast our approach with the conventional approach. All the formal proofs of the results in this chapter are gathered in Section 7.4.

7.1. An approach to measuring multidimensional deprivation

In this section, we outline an approach to the measurement of multidimensional deprivation, where the notion of an individual's deprivation is directly based on the notion of her well-being and a " deprived individual" is conceived as a person whose overall well-being falls short of a specified benchmark level of well-being. We shall use the notation introduced in Chapter 6.

We assume that the individual well-being function is the same for all individuals. Given this assumption, earlier (see Chapter 6) we have shown that, under some plausible conditions, the individual well-being function, $\rho(\cdot)$, shared by all individuals, would take the following form:

(7.1) for every achievement matrix A in $\mathcal{A}$, $\rho(a_i) = \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij}))$ for all $i \in N$.

Recall that, for every individual i with an achievement vector a_i, $\sum_{j=1}^{m} \varphi_j(a_{ij})$ can be interpreted as i's "index of overall achievement" (IOA) when her achievement vector is a_i, while $\rho(a_i) \equiv \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij}))$ is i's well-being corresponding to i's achievement vector a_i. As in Chapter 6, the individual well-being function $\rho(\cdot)$ in (7.1) is treated as cardinal. Thus, for all individual achievement vectors, a, b, a', and b', we can compare the well-being differences, $\rho(a) - \rho(b)$ and $\rho(a') - \rho(b')$.

Let $\underline{w} \in (0,1]$ be the fixed benchmark level of well-being such that, for every achievement matrix $A = (a_i)_{i \in N} \in \mathcal{A}$ and for every $i \in N$, i is considered to be *deprived* if and only if i's well-being level, $\sigma(\sum_{j=1}^{m} \varphi_j(a_{ij}))$, falls below $\underline{w}$. More formally, for a given achievement matrix $A = (a_i)_{i \in N} \in \mathcal{A}$, we define the *deprivation status*, to be denoted by $d_i^0(A)$, of an individual $i \in N$ below:

$$(7.2) \qquad d_i^0(a_i) = \begin{cases} 1, & \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij})) < \underline{w} \\ 0, & \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij})) \geq \underline{w} \end{cases}$$

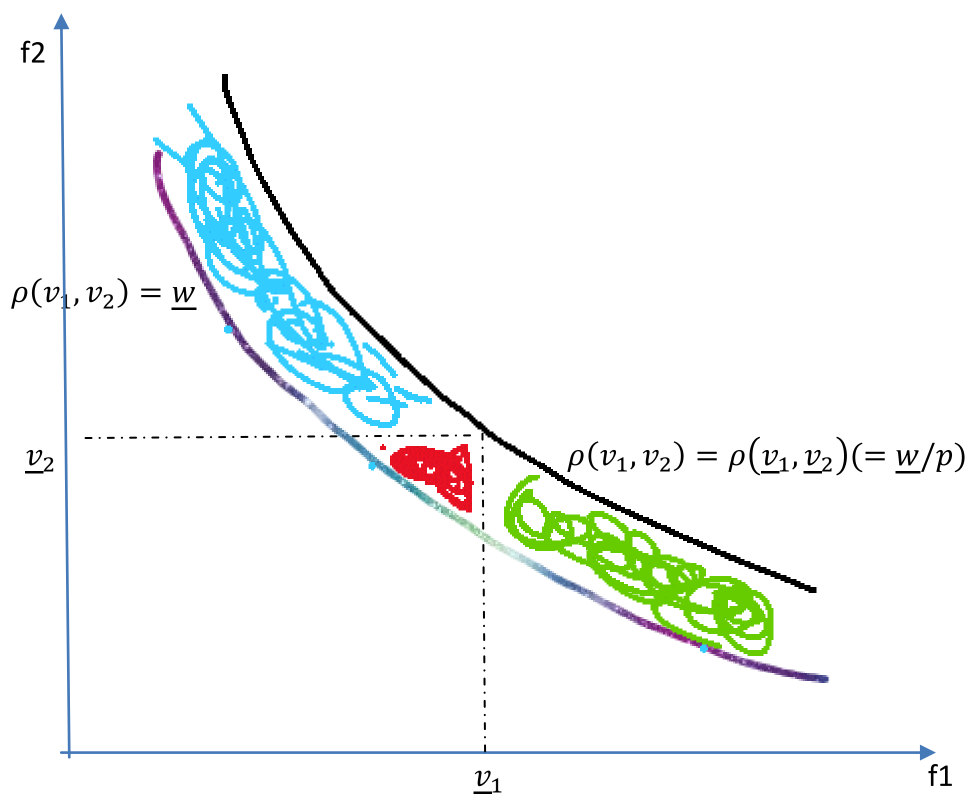
and the *normalized deprivation gap* of i, to be denoted by $d_i(a_i)$, is defined as follows:

$$(7.3) \qquad d_i(a_i) = \begin{cases} \dfrac{\underline{w} - \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij}))}{\underline{w}}, & \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij})) < \underline{w} \\ 0, & \sigma(\sum_{j=1}^{m} \varphi_j(a_{ij})) \geq \underline{w} \end{cases}$$

Since $0 \leq d_i(a_i) \leq 1$ for all $A = (a_i)_{i \in N} \in \mathcal{A}$, it is clear that, given an achievement matrix A, for all $i \in N$, $d_i^0(a_i) = 1$ (resp. $d_i^0(a_i) = 0$) is a sufficient condition for i to be a deprived (resp. non-deprived) individual.

Variants of this method of identifying the deprived and the non-deprived individuals have been discussed in the literature on measurements of multidimensional well-being and deprivation by several authors including Tsui (2002), Bourguignon and Chakravarty (2003), Permanyer (2014, 2019) and Pattanaik and Xu (2018, 2019).

Several points may be noted here about our strategy of identifying deprived and non-deprived individuals. First, the choice of $\underline{w}$ can be the result of judgment made by researchers reflecting the sentiment of the society. In particular, one can start by thinking and using the widely used notion that, for each $j \in M$, there exists $\underline{v}_j \in (0, v_j^*]$ such that an individual's achievement in terms of the attribute f_j is "satisfactory" if and only if it is at least as high as $\underline{v}_j$, $\underline{v}_j$ being the cutoff level of attribute f_j so that, for any individual $i \in N$, i is considered to be deprived in attribute f_j if and only if i's achievement in the attribute f_j falls below $\underline{v}_j$ (see Section 7.2 for the formal definition). Our intuition about such dimensional cutoff values may be firmer than our intuition about the benchmark level of overall individual well-being $\underline{w}$. Once we start thinking along this route, we can then set $\underline{w} = p\rho(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m)$ for some number p with $1 \geq p > 0$. For example, if $p = 0.85$, then an individual is deprived if and only if his well-being is less than eighty five percent of his well-being from the achievement bundle $(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m)$. Since $p < 1$, we may have situations in which an individual's achievement in terms of each attribute is "unsatisfactory" and yet is not classified as deprived: this can happen if an individual's achievement in every dimension f_j falls short of $\underline{v}_j$ but the shortfall in every dimension is very small and the individual's well-being function is continuous, then the individual's actual well-being may still be greater than $\underline{w} = p\rho(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_m)$. In the diagram below, for the case involving just two attributes, we draw two "indifference curves", one for $\rho(v_1, v_2) = \rho(\underline{v}_1, \underline{v}_2)$) and the other for $\rho(v_1, v_1) = \underline{w}$ $(= p\rho(\underline{v}_1, \underline{v}_2))$. An achievement vector in the "red area", where any achievement in each attribute falls short of the corresponding cutoff level for that attribute, is such that an individual's well-being does not fall below the benchmark level of well-being and is not classified as deprived. An achievement vector in either the "blue stripe" or the "green stripe", where the achievement in one attribute falls short of the cutoff level for that attribute while the achievement in the other attribute is above the cutoff level for that attribute, is such that an individual's well-being is above the benchmark level of well-being and is not deprived, even if the shortfall in one dimension is significant: the over-achievement in the other dimension can compensate this shortfall and can still bring the individual's well-being level above the benchmark level of well-being.



Second, in (7.2) and (7.3), we work with the individual well-being function derived in Chapter 6. If one wants to use an individual well-being function with a different form, one can still work with our formulation of deprivation, where an individual's deprivation is simply the individual's normalized shortfall of her relevant well-being from the well-being benchmark. Third, not only is the benchmark level of individual well-being, as we have introduced it, the same for all individuals, but it is also invariant with respect to the achievement matrix. This latter restriction may have to be relaxed if we want to consider relative deprivation. For example, one may like to take the benchmark level, $\underline{w}$, of individual well-being to be two-thirds of the median of individual well-being levels in the society. In that case, one will need to permit dependence of $\underline{w}$ on the achievement matrix under consideration. This can be accommodated in our framework without much difficulty, but we shall ignore this complication here.

For all $i \in N$, let $\mathcal{U}_i = \{d_i(a_i): A \in \mathcal{A}\}$. Given (7.1), it may be noted that $\mathcal{U}_1 = \cdots = \mathcal{U}_n$. Consequently, we shall use $\mathcal{U}$ to denote $\mathcal{U}_i$. It may be noted that, (i) when all attributes are cardinally measurable, and assuming continuity of the individual well-being function ρ, $\mathcal{U} = [0,1]$, and (ii) when each attribute can take discrete values, the possible values that $\mathcal{U}$ can take are discrete as well.

For every achievement matrix $A \in \mathcal{A}$, let $d(A) = (d_1(a_1), \dots, d_i(a_i), \dots, d_n(a_n)) \in \mathcal{U}^n$, and $d(A)$ will be referred to as a profile of individual normalized deprivations for an achievement matrix $A \in \mathcal{A}$. For any achievement matrices $A, B \in \mathcal{A}$, any individuals $i, i' \in N$, $d(A) = (d_1(a_1), \dots, d_i(a_i), \dots, d_n(a_n))$ and $d(B) = (d_1(b_1), \dots, d_i(b_i), \dots, d_n(b_n))$ are said to be $\{i, i'\}$-variant if $d_k(a_k) = d_k(b_k)$ for all $k \in N - \{i, i'\}$.

A deprivation measure of the society (social deprivation measure for short) is a function h from $\mathcal{U}^n$ to $[0,1]$ with the following interpretation: for all $A, B \in \mathcal{A}$, $h(d(A)) \geq h(d(B))$ indicates that the society's deprivation under A is at least as high as the society's deprivation under B, $h(d(A)) > h(d(B))$ indicates that the society's deprivation under A is higher than the society's deprivation under B, and $h(d(A)) = h(d(B))$ indicates that the society's deprivation under A is the same as the society's deprivation under B.

The following definition introduces certain axioms to be imposed on a deprivation measure, h, of the society. The axioms are the counter-parts of the axioms proposed for a social well-being measure considered in Chapter 6.

Definition 7.1. A social deprivation measure h satifies:

D-Normalization if and only if**,** for all $A \in \mathcal{A}$, if $d(A) = \underbrace{(0, \dots, 0)}_{n \text{ times}}$, then $h(d(A)) = 0$, and if $d(A) = \underbrace{(1, \dots, 1)}_{n \text{ times}}$, then $h(d(A)) = 1$;

D-Anonymity if and only if, for every bijection π from N to N and for for all $A = (a_i)_{i \in N}, B = (b_i)_{i \in N} \in \mathcal{A}$, if $[d_i(a_i) = d_{\pi(i)}(b_{\pi(i)})$ for all $i \in N]$, then $h(d(A)) = h(d(B))$;

D-Monotonicity if and only if, for all $A = (a_i)_{i \in N}, B = (b)_{i \in N} \in \mathcal{A}$, if $[d_i(a_i) \geq d_i(b_i)$ for all $i \in N$ and $d(A) \neq d(B)]$, then $h(d(A)) > h(d(B))$;

and

D- independence if and only if**,** for all $A = (a_i)_{i \in N}, B = (b_i)_{i \in N}, A' = (a'_i)_{i \in N}, B' = (b'_i)_{i \in N} \in \mathcal{A}$ and all $i' \in N$, if [for all $i \in N \backslash \{i'\}, d_i(a_i) = d_i(b_i)$ and $d_i(a'_i) = d_i(b'_i)$], and $[d_{i'}(a_{i'}) = d_{i'}\left(a'_{i'}\right)$ and $d_{i'}(b_{i'}) = d_{i'}\left(b'_{i'}\right)]$, then $h(A) - h(B) = h(A') - h(B')$.

With the help of the above axioms, we obtain the following result; its proof can be found in Section 7.4.

Proposition 7.1. Let h be a social deprivation measure. Then, h satisfies D-Normalization, D-Anonymity, D-Monotonicity and D-Independence if and only if:

(7.4) there exists an increasing function $\phi: \mathcal{U} \to [0,1]$, with $\phi(0) = 0, \phi(1) = 1$, such that, for all $A = (a_i)_{i \in N} \in \mathcal{A}$,

$$h(d(A)) = \frac{1}{n} \sum_{i=1}^{n} \phi\big(d_i(a_i)\big).$$

The function ϕ figuring in (7.4) may be called a *deprivation weighting function*, and it transforms, for each individual $i \in N$ and for a given achievement matrix $A = (a_i)_{i \in N} \in \mathcal{A}$, i's normalized deprivation gap, $d_i(a_i)$, to a contribution $\phi\big(d_i(a_i)\big)$ that is used for measuring social deprivation. If the weighting function ϕ is further restricted to be convex, then, the worse-off that people are (in terms of higher normalized deprivation gaps), the higher contributions to social deprivation that they are viewed to make.

Note that (7.4) gives us a class of social deprivation measures. One can think of plausible restrictions on the functional forms of ϕ figuring in Proposition 7.1. For this purpose, we consider the following two properties.

Definition 7.2. A social deprivation measure, h, satisfies:

Uniform Scale-Invariance (USI) if and only if, for all achievement matrices $A, B, X, Y \in \mathcal{A}$, if $[h(d(A)) - h(d(B)) = h(d(X)) - h(d(Y))]$, then, for all $k \in (0,1]$ and for all $A', B', X', Y' \in \mathcal{A}$, $[kd(A) = d(A'), kd(B) = d(B'), kd(X) = d(X')$, and $kd(Y) = d(Y')]$ implies $[h(d(A')) - h(d(B')) = h(d(X')) - h(d(Y'))]$;

and

Transfer Axiom (TA) If and only if, for all $A = (a_i)_{i\in N}, B = (b_i)_{i\in N} \in \mathcal{A}$, all $i, i' \in N$ and for all $t > 0$, if $d(A)$ and $d(B)$ are $\{i, i'\}$-variant, $d_i(a_i) > d_{i'}(a_{i'})$, $d_i(b_i) = d_i(a_i) + t$ and $d_{i'}(b_{i'}) = d_{i'}(a_{i'}) - t$, then $h(d(B)) > h(d(A))$.

The intuition of USI can be explained as follows. Consider two situations with respective normalized deprivation matrices A and B. Suppose there is an equiproportionate decrease in the overall deprivations of all the people in both situations. Then the difference between the levels of social deprivation in the two situations will change by an amount that will depend exclusively on the initial difference in social deprivation and the proportionality factor by which the individual deprivations change. It may be noted that a similar property was introduced in Chakraborty, Pattanaik and Xu (2006) in the context of measuring income poverty.

Transfer Axiom requires that if, initially, individuals i and i' are both deprived, and the normalized deprivation gap of i is higher than that of i', then, other things remaining the same, an increase of t in the normalized deprivation gap of i, together with a decrease of t in the normalized deprivation gap of i', will increase the social deprivation. A similar property was introduced in Chakraborty, Pattanaik and Xu (2006) for measuring income poverty. The transfer axiom is also reminiscent of (though, formally, not identical to) Sen's (1976) transfer axiom which stipulates that a transfer of income from a poor person to anybody richer must increase the society's poverty.

One can prove the following proposition; see Section 7.4 for the proof.

Proposition 7.2. Let h be a social deprivation measure. Then, h satisfies D-Normalization, D-Anonymity, D-Monotonicity, D-Independence, Uniform Scale Invariance, and Transfer Axiom if and only if:

(7.5) for some $\gamma > 1$, for all $A = (a_i)_{i \in N} \in \mathcal{A}$,

$$h(d(A)) = \frac{1}{n}\sum_{i=1}^{n} \left(d_i(a_i)\right)^{\gamma}.$$

7.2. The conventional approach to measuring multidimensional deprivation

In this section, we outline the conventional approach to measuring multidimensional deprivation when every attribute is cardinally measurable. The approach can be readily adapted for the case in which every attribute has two levels of achievement. We can also adapt the approach for the more general case where each attribute can take finite and discrete values.

Again, let $A = (a_{ij})_{n \times m} \in \mathcal{A}$ be a given society's achievement matrix where the i-th row represents individual i's achievement vector a_i and the jth column represents the achievements in dimension f_j by all the individuals. For each individual $i \in N$, $a_i = (a_{i1}, \ldots, a_{im})$ is i's achievement vector with a_{ij} being i's achievement in attribute f_j.

Recall that, when every attribute is cardinally measurable, for each $j \in M$, $V_j = [0, v_j^*]$ is the set of values which f_j can take. Following the traditional literature on measuring multidimensional deprivations, for each $j \in M$, let $\underline{v}_j \in V_j$ be the cutoff level of attribute f_j so that, for any individual $i \in N$, i is considered to be deprived in attribute f_j if and only if i's achievement, a_{ij}, in the attribute f_j, falls below $\underline{v}_j$. More formally, we define the "deprivation status", to be denoted by $d_{ij}^0(a_{ij})$, of an individual $i \in N$ along an attribute $f_j \in F$ below:

$$(7.6) \qquad d_{ij}^0(a_{ij}) = \begin{cases} 1, & a_{ij} < \underline{v}_j \\ 0, & a_{ij} \geq \underline{v}_j \end{cases}$$

and the normalized deprivation gap of i in attribute f_j, to be denoted by d_{ij}, is defined as follows:

$$(7.7) \qquad d_{ij}(a_{ij}) = \begin{cases} \dfrac{\underline{v}_j - a_{ij}}{\underline{v}_j}, & a_{ij} < \underline{v}_j \\ 0, & a_{ij} \geq \underline{v}_j \end{cases}$$

Note that $0 \leq d_{ij}(a_{ij}) \leq 1$. Let $\mathcal{D}$ denote the set of all $n \times m$ matrices $D = (d_{ij})_{n \times m}$ such that each entry $d_{ij} \in [0,1]$ of D is the normalized deprivation gap of i in attribute f_j, and $\mathcal{D}^0$ denote the set of all $n \times m$ matrices such that each entry of the matrix indicates the deprivation status of an individual along an attribute and therefore takes a value of either 0 or 1.

To measure social deprivation, the conventional approach takes two steps (see Alkire et al. (2015)): the first step is to identify those individuals who are deprived (the identification problem), and the second step is to aggregate individual deprivations of the deprived into social deprivation (the aggregation problem).

We first discuss the problem of identification of deprived individuals. For this purpose, let $\omega = (\omega_1, \dots, \omega_m)$ be the vector of the positive weights attached to the different attributes, with $\sum_{j \in M} \omega_j = 1$. For each individual $i \in N$ and given the vector, $d_i^0 = (d_{i1}^0, \dots, d_{im}^0)$, describing the dimensional deprivation statuses of individual i, the deprivation score is defined as $\sum_{j \in M} \omega_j d_{ij}^0$. Let k $(1 \geq k > 0)$ be the cutoff value for the deprivation score such that a person is *deprived* if and only if her deprivation score is at least as great as the cutoff value k, i.e., $\sum_{j \in M} \omega_j d_{ij}^0 \geq k$ (see Alkire et al. (2015)). For convenience, let's call this approach to the identification problem as the *generalized counting method*. The set, $N^*(A)$, of deprived individuals for a given achievement matrix A, is defined below: for any given achievement matrix $A = (a_{ij})_{n \times m} \in \mathcal{A}$

$$N^*(A) = \{i \in N: \sum_{j \in M} \omega_j d_{ij}^0(a_{ij}) \geq k\}$$

It may be noted that the generalized counting method is an extension of a historically important measure for multidimensional deprivation, the "counting" method, which has been developed and used in empirical research in two regions, Latin America and Europe, since the 1970s, though the motivations and applications are different: in Latin America, it was developed in the context of basic needs approach, and in Europe, it was developed in the context of social exclusion (see Chapter 4 of Alkire et al. (2015) for more details).

The generalized counting method extends the two commonly used methods of identifying the deprived, the union method and the intersection method (see, for example, Atkinson (2003)), in the literature on measuring multidimensional deprivation. The union method of identifying the deprived classifies an individual as deprived when she experiences deprivation in at least one attribute. The intersection method, on the other hand, classifies an individual as deprived when she has dimensional deprivation in every attribute. Therefore, if $k = 1$, then the generalized counting method yields the intersection method, and if $k \leq \min\{\omega_1, \dots, \omega_m\}$, the generalized counting method is the same as the union method.

Once the identification problem is taken care of, the next step is to aggregate individual deprivations into social deprivation. Formally, let $d: \mathcal{D} \to [0,1]$ be a measure of social deprivation. Thus, for any normalized deprivation matrix $D \in \mathcal{D}$, $d(D)$ measures the overall deprivation of the society under D: for any normalized deprivation matrices $D, D' \in \mathcal{D}$, $d(D) \geq d(D')$ is to be interpreted as indicating that the society's deprivation level under D is at least as high as the society's deprivation level under D'; $[d(D) > d(D')]$ and $[d(D) = (D')]$ have corresponding interpretations.

There are various measures proposed along this approach. One prominent class of measures of multidimensional deprivation that has been proposed and used often in empirical studies is due to Alkire et al. (2015) and is defined as below:

Let $\alpha \geq 0$ and let $\omega = (\omega_1, \ldots, \omega_m)$ be the vector of the positive weights attached to the different attributes, with $\sum_{j \in M} \omega_j = 1$. Then, for every normalized deprivation matrix $D \in \mathcal{D}$,

(7.8) $$\alpha > 0: d(D) = \frac{1}{n} \sum_{i \in N^*} \sum_{j \in M} \omega_j (d_{ij})^\alpha$$

(7.9) $$\alpha = 0: d(D) = \frac{1}{n} \sum_{i \in N^*} \sum_{j \in M} \omega_j d_{ij}^0$$

7.3. The two approaches: comparisons and contrasts

The approach that we introduced and studied in Section 7.1 for measuring multidimensional deprivation has several features as listed below:

(i) an individual's overall deprivation is viewed simply as an unacceptably low level of his/her overall well-being,

(ii) social deprivation is conveniently the sum of the overall deprivations of those individuals who have unacceptably low levels of overall well-being,

(iii) in defining the deprivation status of an individual, an individual's "overachievement" along some dimensions can compensate for "underachievement" along some other dimensions,

(iv) social deprivation is independent of well-being levels of those who are identified as non-deprived even if they or some of them suffer some "dimensional deprivations"—the achievement levels along some dimensions are "unacceptably low"—but are compensated by overachievement along some other dimensions (so that they are classified as non-deprived),

(v) the measures of social deprivation given in (7.4) and (7.5) are separable across deprived individuals but non-separable across dimensions,

(vi) it can deal with the case where all attributes are cardinally measurable, as well as the case where all attributes are ordinally measurable, provided that the individual well-being function is the same for all individuals in the society.

It may also be noted that the conventional approach outlined in Section 7.2 for measuring multidimensional deprivation has several features as summarized below.

(a) an individual's overall deprivation is viewed as an independent concept from his/her overall well-being and is to be assessed by looking at the individual's deprivation in terms of the different dimensions,
(b) in defining the deprivation status of an individual, an individual's "overachievement" along some dimensions cannot compensate for "underachievement" along some other dimensions,
(c) social deprivation is independent of well-being levels of those who are identified as non-deprived even if they or some of them suffer some dimensional deprivations,
(d) it can encounter difficulties with ordinally measurable attributes, when there are three or more achievement levels, even if there is an agreement on a set of weights to be attached to various attributes,
(e) the specific measures of social deprivation given in (7.8) are separable across deprived individuals and across dimensions.

Clearly, our approach to measuring multidimensional deprivation discussed in Section 7.1 is through the notion of an individual's well-being defined with reference to the same space of attributes. The analytical cores of this approach are: (1) the choice of the well-being function $\sigma(\sum_{j=1}^{m} \varphi_j(a_{ij}))$ of individuals, where $a_i = (a_{i1}, \dots, a_{ij}, \dots, a_{im})$ is an individual i's achievement vector along the specified m attributes; and (2) the choice of a well-being threshold $\underline{w}$. It may be remarked that the choice of the individual well-being function is not crucial for our analytical framework, but, in practice, to use that analytical framework, it is necessary to choose a specific individual well-being function. Once those two choices are made, the measurement of social deprivation based on thus defined individual deprivations (in terms of well-being) becomes exactly similar to the aggregation problem in the income poverty or unidimensional deprivation. One advantage of this approach is that it brings together the analytical framework for measuring well-being and that for measuring deprivation.

On the other hand, the conventional approach to measuring social deprivation takes a route that views an individual's overall deprivation as an independent concept to be assessed by looking at the individual's deprivation in terms of the different attributes. In particular, the conventional approach ignores an individual's over-achievements over the respective deprivation benchmarks in some dimensions, a point raised by several authors including Permanyer (2014, 2019) and Pattanaik and Xu (2018). We recall the two major features of the conventional approach and then illustrate the difficulties that it may encounter: (I) social deprivation does not in any way depend on the magnitude of an individual's over-achievement along any dimension, and (II) the notion of an individual's overall deprivation is not directly or indirectly linked to the notion of that individual's well-being, and as a matter of fact, the notion of an individual's well-being does not figure in the formal structure of measuring social deprivation at all.

Consider feature (I) first. In the conventional approach to measuring multidimensional deprivation, the generalized counting method is used for classifying deprived and non-deprived individuals. The generalized counting method of identifying the deprived is an important contribution as it draws our attention to the limitations of the "union approach" and the "intersection approach" to the problem of identifying the deprived. The method permits the possibility that an individual may not be deprived even when he is deprived in some dimensions, and the possibility that an individual may be deprived even when she is not deprived in all dimensions. However, it causes some intuitive difficulties when an individual's normalized deprivations in terms of the different attributes are cardinally measurable. When we have cardinal information about dimensional deprivations, intuitively, it seems that whether some individual is deprived or not also depends on the depths of the individual's deprivations along various dimensions. The generalized counting method completely ignores such information about the depths and magnitudes of dimensional deprivations. Consider the following example from Pattanaik and Xu (2018). Suppose there are five attributes and the vector of weights attached to different attributes is (1/5, 1/5, 1/5, 1/5, 1/5). Suppose further the cutoff value k is 3/5. Then, an individual is deprived if and only if he is deprived in at least three out of five attributes. Consider two individuals, 1 and 2, whose normalized deprivation vectors are, $d_1 = (\varepsilon, \varepsilon, \varepsilon, 0,0)$ with $\varepsilon > 0$ and $d_2 = (0,0,0,1,1)$, respectively. Then, according to the generalized counting method, individual 1 is considered deprived while individual 2 is not considered deprived. This is true for any arbitrarily small but positive ϵ. This seems counterintuitive: when we have information about the magnitudes or depths of dimensional deprivation, in assessing whether an individual is deprived, we should not only take into account the list of the dimensions in which the individual is deprived, but we also should consider the extent to which the individual is deprived in terms of each attribute in that list. The generalized counting method disregards the information about the magnitudes and depths of an individual's dimensional deprivations.

The problem discussed above concerning the identification of deprived individuals runs into the aggregation stage as well. Consider our earlier example where we have two individuals, 1 and 2, with their respective normalized deprivation vectors $d_1 = (\varepsilon, \varepsilon, \varepsilon, 0,0)$ with $\varepsilon > 0$ and $d_2 = (0,0,0,1,1)$. From the identification stage, individual 2 is not identified as deprived while individual 1 is. Note that individual 2's overall deprivation, given the equal weights to be attached to the five attributes, is $\frac{2}{5}$, and individual 1's overall deprivation is $\frac{3\varepsilon}{5}$. If $\varepsilon > 0$ is sufficiently small, the fact that individual 1's overall deprivation, however small, contributes to social deprivation while individual 2's overall deprivation, which is much larger than individual 1's overall deprivation, does not contribute to social deprivation at all seems a bit difficult to swallow intuitively.

Given the above difficulties faced by the conventional approach, a question naturally arises: what are the possible justifications for this approach? One can think of at least two possible defenses for using the conventional approach to measuring multidimensional deprivation (see Pattanaik and Xu (2018) for a more detailed discussion). First, one may argue that the damage to an individual's quality of life, caused by even a very small shortfall from the deprivation benchmark in one attribute cannot be mitigated to any extent by the contributions to the individual's quality of life made by over-achievements, however large, in other attributes. This argument seems to run into difficulty when one is, at the same time, prepared to allow tradeoffs between deprivations in different attributes. A second line of argument may be this. One may be prepared to make tradeoffs between over-achievements and under-achievement in different attributes by admitting the possibility that the damage to an individual's quality of life caused by some dimensional deprivations can be partially compensated by the same individual's over-achievements in other attributes. But then, one may also argue that, presently, one is only interested in measuring the damage to an individual's quality of life made by the dimensional deprivation that the individual is experiencing, and, if one wants to know to what extent such damage is compensated by the individual's over-achievements in other attributes, one would have to undertake a different exercise where the individual's well-being or overall quality of life would be evaluated. This seems a reasonable defense of the conventional approach to measuring multidimensional deprivation. If, however, one resorts to this argument, then one has to accept the possibility that a policy which puts resources to promote people's over-achievements in certain dimensions where they do not have any deprivation instead of devoting the same resources to fight already existing deprivations in other dimensions, may be justified since that may be the most cost-effective way of increasing people's well-being and quality of life.

Our final comparison of the two approaches is on the separability of measures of social deprivation across dimension. As noted earlier, the specific measure of social deprivation given in (7.8) is separable across dimensions, while the measure of social deprivation given in (7.4) proposed in our approach is not separable across dimensions. Separability across dimensions and across deprived individuals makes the measure easy and simple to be implemented. However, as noted in Pattanaik and Xu (2018), separability across dimensions has some undesirable implications and creates certain intuitive difficulties. To illustrate this aspect of separability across dimensions, we consider the following example. There are four attributes, all attributes have the same weight (1/4), the cut-off value k is 1/4. Let $0 < \underline{a}_j < v_j^*$ every attribute $j = 1,2,3,4$. Consider an individual with the following achievement vectors, $a = (0, \underline{a}_2, \underline{a}_3, \underline{a}_4)$, $b = (0, 0, \underline{a}_3, \underline{a}_4)$, $c = (0, \underline{a}_2, 0, \underline{a}_4)$, $x = (0, \underline{a}_2, \underline{a}_3, 0)$ and $y = (0, 0, 0, 0)$.

Then, the deprivation statuses are: $d^0(a) = (1,0,0,0), d^0(b) = (1,1,0,0), d^0(c) = (1,0,1,0), d^0(x) = (1,0,0,1)$, $d^0(y) = (1,1,1,1)$. Given the cut-off level $k = 1/4$, the individual is identified as deprived under each achievement vector. We then obtain the following normalized deprivation vector: $(a) = (1,0,0,0), d(b) = (1,1,0,0), d(c) = (1,0,1,0), d(x) = (1,0,0,1)$, $d(y) = (1,1,1,1)$. To see intuitive difficulties of separability across dimensions embedded in the measure given in (7.8) of the conventional approach to measuring social deprivation, we recall the idea espoused in the Stiglitz, Sen, and Fitoussi Commission's (2009) report and cited earlier: "the consequences for quality of life having multiple disadvantages far exceed the sum of their individual effects". Most people will probably agree with this idea. Now, let's examine the implications of the measure of a deprived individual's overall deprivation, $\sum_{j\in M} \omega_j (d_{ij})^\alpha$, where $\alpha > 0$, figuring in (7.8) used for measuring social deprivation. Take the above five normalized deprivation vectors, $(a) = (1,0,0,0), d(b) = (1,1,0,0), d(c) = (1,0,1,0)$, $d(x) = (1,0,0,1), d(y) = (1,1,1,1)$. Given the separability across dimensional deprivations, it is clear that the addition to the individual's overall deprivation when his normalized deprivation vector changes from $d(a) = (1,0,0,0)$ to $d(y) = (1,1,1,1)$ is exactly equal to the sum of: (i) the addition to the individual's overall deprivation when his normalized deprivation vector changes from $d(a) = (1,0,0,0)$ to $d(b) = (1,1,0,0)$, (ii) the addition to the individual's overall deprivation when his normalized deprivation vector changes from $d(a) = (1,0,0,0)$ to $d(c) = (1,0,1,0)$, and (iii) the addition to the individual's overall deprivation when his deprivation vector changes from $d(a) = (1,0,0,0)$ to $d(x) = (1,0,0,1)$. Clearly, this is in sharp contrast to the Stiglitz, Sen, and Fitoussi Commission's position cited earlier.

7.4. Proofs

This section contains proofs of all the results in Chapter 7.

Proof of Proposition 7.1: Let h be a deprivation measure of the society satisfying D-Normalization, D-Monotonicity, D-Anonymity and D-Independence. Let $A = (a_i)_{i\in N}$ be a given achievement matrix in $\mathcal{A}$, and let $\overline{A} = (\overline{a}_i)_{i\in N}$ be such that $\overline{a}_i = (v_1^*, \dots, v_m^*)$ for all $i \in N$ (that is, for each individual i, i's achievement along any dimension $j \in M$ is at the highest level). It may be noted that, for each $i \in N$, $d_i(\overline{a}_i) = 0$ for any choice $\underline{w} \in (0,1]$ of a cutoff level of nominal overall individual achievement.

For every $i \in N$, let $A^i = (a^i_k)_{k \in N}$ be such that $a^i_i = a_i$ and $d_k(a^i_k) = 0$ for all $k \in N \backslash \{i\}$. Consider $B \equiv (b_i;\ A_{-i})$ with $d_i(b_i) = 0$. Consider the achievement matrices, A, B, A^1 and $\overline{A}$. Then, by D-Independence, we have

$$(7.10)\quad h(A) - h(B) = h(A^1) - h(\overline{A}).$$

By D-Normalization, $h(\overline{A}) = 0$. We then obtain

$$(7.11)\quad h(A) - h(B) = h(A^1) - h\left(a_1;\ \overline{A}_{-1}\right).$$

Consider $C = (c_2; B_{-2})$ with $d_2(c_2) = 0$, and the achievement matrices B, C, A^2 and $\overline{A}$. By D-Independence and D-Normalization we have

$$(7.12)\quad h(B) - h(C) = h(A^2) - h\left(\overline{A}\right) = h(A^2) = h\left(a_2;\ \overline{A}_{-2}\right).$$

From (7.11) and (7.12), we then have

$$(7.13)\quad h(A) - h(C) = h\left(a_1; \overline{A}_{-1}\right) + h\left(a_2; \overline{A}_{-2}\right).$$

By repeating the above procedures with C and beyond and from D-Independence and D-Normalization, we can obtain

$$(7.14)\quad h(A) - h\left(a_n; \overline{A}_{-n}\right) = h\left(a_1; \overline{A}_{-1}\right) + \cdots + h\left(a_{n-1}; \overline{A}_{-(n-1)}\right).$$

Then

$$(7.15)\quad h(A) = h\left(a_1; \overline{A}_{-1}\right) + \cdots + h\left(a_{n-1}; \overline{A}_{-(n-1)}\right) + h\left(a_n; \overline{A}_{-n}\right)$$

For each $i \in N$, let $\phi_i: \mathcal{U} \to [0,1]$ be such that, for all $v \in V$, $\phi_i(d_i(v)) = h(v; \overline{A}_{-i})/n$. Clearly, for each $i \in N$, ϕ_i is a real-valued function. By D-Monotonicity, for each $i \in N$, ϕ_i is increasing. Then,

$$(7.16)\quad h(A) = \frac{1}{n}[\phi_1(d_1(a_1)) + \cdots + \phi_n(d_n(a_n))]$$

By D-Anonymity, $\phi_1 = \cdots = \phi_n$. Let $\phi = \tau_i$ for all $i \in N$. Then,

$$(7.17)\quad h(A) = \frac{1}{n}[\phi(d_1(a_1)) + \cdots + \phi(d_n(a_n))]$$

By D-Normalization, $\phi(0) = 0$ and $\tau(1) = 1$.

On the other hand, if there is an increasing function $\phi: \mathcal{U} \to [0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$, we have $h(A) = \frac{1}{n}\sum_{i=1}^{n} \phi(d_i(a_i))$ for all $A \in \mathcal{A}$, then it can be verified that h satisfies D-Normalization, D-Monotonicity, D-Anonymity and D-Independence. ■

Proof of Proposition 7.2. Let h be a deprivation measure of the society satisfying D-Normalization, D-Monotonicity, D-Anonymity, D-Independence, Uniform Scale Invariance, and Transfer Axiom.

From Proposition 7.1, we have that

(7.18) there exists an increasing function $\phi: \mathcal{U} \to [0,1]$, with $\phi(0) = 0, \phi(1) = 1$, such that, for all $A = (a_i)_{i \in N} \in \mathcal{A}$,

$$h(d(A)) = \frac{1}{n}\sum_{i=1}^{n} \phi\big(d_i(a_i)\big).$$

Consider $A = (a_i)_{i \in N}, B = (b_i)_{i \in N}, X = (x_i)_{i \in N}, Y = (y_i)_{i \in N} \in \mathcal{A}$ with

(7.19) $[\forall i, i' \in N: a_i = a_{i'},\ b_i = b_{i'}, x_i = x_{i'}, y_i = y_{i'}]$

and let $h(d(A)) - h\big(d(B)\big) = h\big(d(X)\big) - h(d(Y))$. Then, from (7.9), we have $h(d(A)) = \frac{1}{n}\sum_{i=1}^{n} \phi\big(d_i(a_i)\big) = \phi\big(d_1(a_1)\big)$, $h(d(B)) = \frac{1}{n}\sum_{i=1}^{n} \phi\big(d_i(b_i)\big) = \phi\big(d_1(b_1)\big)$, $h(d(X)) = \frac{1}{n}\sum_{i=1}^{n} \phi\big(d_i(x_i)\big) = \phi\big(d_1(x_1)\big)$, $h(d(Y)) = \frac{1}{n}\sum_{i=1}^{n} \phi\big(d_i(y_i)\big) = \phi\big(d_1(y_1)\big)$, and therefore $h(d(A)) - h\big(d(B)\big) = h\big(d(X)\big) - h(d(Y))$ implies

(7.20) $\phi\big(d_1(a_1)\big) - \phi\big(d_1(b_1)\big) = \phi\big(d_1(x_1)\big) - \phi\big(d_1(y_1)\big)$.

Consider $A' = (a_i')_{i \in N}, B' = (b_i')_{i \in N}, X' = \left({x'}_i\right)_{i \in N}, Y' = (y_i')_{i \in N} \in \mathcal{A}$ such that

(7.21) $[\forall i \in N:\ kd_i(a_i) = d_i(a_i'), kd_i(b_i) = d_i(b_i'), kd_i(x_i) = d_i(x_i')$, and $kd_i(y_i) = d_i(y_i')]$.

By Uniform Scale-Invariance, we obtain

(13) $h\big(d(A')\big) - h\big(d(B')\big) = h\big(d(X')\big) - h\big(d(Y')\big)$.

From (7.9) and noting (7.19), (7.20), (7.21) and (7.22), it then follows that

(7.23) $\phi\big(d_1(a_1')\big) - \phi\big(d_1(b_1')\big) = \phi\big(d_1(x_1')\big) - \phi\big(d_1(y_1')\big)$,

which is equivalent to

(7.24) $\phi\big(kd_1(a_1)\big) - \phi\big(kd_1(b_1)\big) = \phi\big(kd_1(x_1)\big) - \phi\big(kd_1(y_1)\big)$.

Therefore, we have shown that

(7.25) For all $s, t, s', t' \in [0,1]$ and $k > 0$ such that $ks, kt, ks', kt' \in [0,1]$, if $\phi(s) - \phi(t) = \phi(s') - \phi(t')$ then $\phi(ks) - \phi(kt) = \phi(ks') - \phi(kt')$.

Then, following the reasoning in Chakraborty, Pattanaik and Xu (2006), we must have

(7.26) $\phi(s) = \beta s^{\gamma}$ for some $\beta > 0$ and $\gamma > 0$.

Note that $\phi(1) = 1$. Then, $\beta = 1$. Therefore,

(7.27) for some $\gamma > 0$, for all $A = (a_i)_{i \in N} \in \mathcal{A}$, $h(d(A)) = \frac{1}{n}\sum_{i=1}^{n} \big(d_i(a_i)\big)^{\gamma}$.

It can then be checked that, by Transfer Axiom, $\gamma > 1$.

On the other hand, if (7.9) holds, then it can be verified that h satisfies D-Normalization, D-Monotonicity, D-Anonymity, D-Independence, Uniform Scale Invariance, and Transfer Axiom.

■