Design Limits and Dynamic Policy Analysis

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Abstract

This paper characterizes the frequency domain properties of feedback control rules in linear systems in order to better understand how different rules affect outcomes frequency by frequency. We are especially concerned in understanding how reductions of variance at some frequencies induce increases in variance at others. Tradeoffs of this type are known in the control literature as design limits. Design limits are important in understanding the full range of effects of stabilization policies. We extend existing results to account for discrete time bivariate systems with rational expectations. Application is made to the evaluation of monetary policy rules.

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1. Introduction

This paper explores a set of constraints on the effects of control policies on fluctuations from the perspective of the frequency domain. Aspects of these constraints were initially discussed in Brock and Durlauf (2004,2005) but otherwise do not appear to have been previously explored in economics contexts. The constraints we study represent fundamental limits on the effects of alternative policies in the sense that they describe how frequency-specific tradeoffs in volatility generically apply to linear feedback rules.

The sorts of constraints we explore may be illustrated in the following example. Suppose one is considering how different controls affect the variance of a state variable x_t. Underlying the statistic $\mathrm{var}\left(x_t \left| C\right.\right)$, the variance of the process given a control, is the spectral density of x given the rule, $f_{x|C}\left(\omega\right)$, because the variance is the integral of the spectral density, i.e.

$$\mathrm{var}\left(x_t \left| C\right.\right) = \int_{-\pi}^{\pi} f_{x|C}\left(\omega\right) d\omega . \tag{1}$$

In fact, the spectral representation of the variance of the state means one can understand the sum of the variances from random and orthogonal sine and cosines of different frequencies. By implication, calculations of the effects of a rule on the overall variance mask the effects on fluctuations at the different frequencies in $[-\pi,\pi]$. Further, eq. (1) hints at the idea that a rule that minimizes the overall variance may exacerbate fluctuations at certain frequencies. A major goal of this paper is to determine under what circumstances this must happen and what forms such fundamental tradeoffs take. In the control literature, these tradeoffs are known as *design limits*.

Design limits are a well established area of study in control theory.[1] An important class of results of this type are sometimes known as Bode integral constraints, after Hendrik Bode who first proposed them in the 1930's. The great bulk of the work in

[1]Our description of linear systems owes much to the formulation in Kwakernaak and Sivan (1972), especially chapter 6.

control theory focuses on single-input, single-output (SISO) systems. One methodological contribution of this paper is that we derive frequency tradeoffs for multiple-input multiple-output (MIMO) systems. While there does exist a set of disparate results in the control literature on frequency tradeoffs for multivariate systems, this work has largely been done for continuous time systems.[2] Some of our discrete time results for backwards-looking systems appear to be new, although they naturally follow from existing results. A second methodological contribution is that we study these tradeoffs when expectations of future state variables affect current values; a property that, while of course natural for economic models, does not arise in engineering contexts. A third contribution of our analysis is that we consider SIMO (single-input, multiple-output) systems as well as MIMO ones. We defer consideration of systems with arbitrary dimensions to future work, noting here that the 2×2 cases we study capture a range of important contexts, most notably the evaluation of macroeconomic stabilization policy.

Why should frequency-specific tradeoffs be of interest to a policymaker? One reaction to the recognition that policymakers face frequency-by-frequency constraints might be that these constraints are irrelevant if the objective of a policymaker is to minimize the overall variance of some combination of states and controls of the system; such loss functions are standard in the literature on evaluating monetary policy rules. We argue that our results are of interest for several reasons. First, there is no principled reason why policymaker loss functions should only depend on the overall variances of variables of interest, and in fact time-nonseparable preferences for policymakers can lead to the assignment of different loss function weights across frequency-specific fluctuations. Examples of this property are found in Otrok (2001) and Otrok, Ravikumar, and Whiteman (2002). Second, differences in the approximation value of a given model to fluctuations at different frequencies may lead to a focus on higher versus lower frequency fluctuations using a model to assess policies; this type of reasoning is developed in Onatski and Williams (2003). Third, there are classes of problems for which the frequency restrictions matter, even if loss functions only depend on unconditional variances. Specifically, evaluating the robustness of policy rules in the face of model uncertainty may

[2] See Seron, Braslavsky, and Goodwin (1997), and Skogestad and Postlethwaite (1996) for surveys. Examples of discrete time analyses include Chen and Nett (1993,1995).

be facilitated using the constraints we describe; an initial example of such an analysis is Brock and Durlauf (2005).

In our judgment, the most important contribution of this paper is its introduction of the idea that macroeconomic stabilization policies involve tradeoffs that are hidden when a policy is evaluated by calculation of its effects on variance. In this sense, when a policy maker chooses a control it must face (for backwards-looking models) the inevitable result that the controlled system will exhibit frequency bands that are robust in the sense that shocks to the system at those frequency bands will be moderated while simultaneously there will always be frequency bands that are fragile in the sense that shocks at those frequency bands will be magnified, not moderated[3]. This kind of result is sometimes also called a "conservation law" or "waterbed" result in the engineering literature. Indeed we will exhibit various conservation laws and waterbed results and illustrate their consequences for a set of two sector macroeconomic models of inflation and the output gap that are commonly used in the macroeconomics literature.

The use of frequency domain methods is not original per se, of course. One classic example is Hansen and Sargent (1980,1981) use of $z-$transform methods to translate time domain expectations into the frequency domain and thereby solve for testable restrictions of rational expectations models. Another important contribution is Bowden's (1977) and Whiteman's (1985,1986) work on spectral utility and the frequency domain analysis of the effects of policies; Whiteman's work is close in spirit to ours, although it does not address the issue of frequency-specific tradeoffs. More recently, frequency methods have proven to be important in the development of the growing macroeconomic literature on robustness, cf. Sargent (1999), Hansen and Sargent (2007, Chapter 8)). That being said, frequency domain approaches continue to be far less popular than time domain methods for analyzing macroeconomic dynamics. We believe the methods developed here complement these other papers in demonstrating that frequency domain approaches have an important role in understanding stabilization policy. While, in principle, one can always translate results from the frequency domain to the time domain and vice versa, the results we exploit are an example in which working in the frequency domain is relatively

[3]The notion that a system may be "robust yet fragile" appears in the control literature, notably in writings of John C. Doyle, e.g. Doyle and Carlson (2000).

straightforward whereas it would appear that the same analysis in the time domain may well be intractable.[4]

Section 2 provides an analysis of four classes of models: backwards-looking MIMO (multiple input, multiple output) systems and hybrid backwards- and forward-looking MIMO systems. We characterize Bode integral-type results for each type of model. Section 3 moves beyond Bode integral constraints to a broader consideration of how design limit occur in MIMO and SIMO systems. Section 4 applies our methods to the evaluation of monetary policy rules. Section 5 contains summary and conclusions. Appendices follow which contains proofs of various claims made in the text.

2. Design limits in multivariate systems

i. backwards-looking models

We first consider a backwards-looking system, i.e. one where expectations do not directly enter into the law of motion for the states. Letting, x_t denote a 2×1 vector of states, u_t a 2×1 vector of controls, and ε_t a 2×1 vector of disturbances that is second-order stationary across time, the canonical law of motion for a backwards-looking system is

$$A_0 x_t = A(L)x_{t-1} + B(L)u_t + \varepsilon_t . \tag{2}$$

In general, the matrix A_0 possesses off diagonal elements because of contemporary interdependences between the states; without loss of generality, we write the matrix as

$$A_0 = \begin{pmatrix} 1 & a_{0,12} \\ a_{0,21} & 1 \end{pmatrix}.$$

[4]For example, the Bode integral constraint, which we exploit in the subsequent analysis, has an extremely convoluted time domain representation for a SISO system, cf. Iglesias (2001) equation 3.2 and surrounding discussion.

The moving average representation of ε_t is

$$\varepsilon_t = W(L) w_t . \tag{3}$$

We assume that each element of $W(L)$ may be written as the ratio of two finite dimensional polynomials,[5] i.e.

$$W(L) = \begin{pmatrix} \frac{w_{n,11}(L)}{w_{d,11}(L)} & \frac{w_{n,12}(L)}{w_{d,12}(L)} \\ \frac{w_{n,21}(L)}{w_{d,21}(L)} & \frac{w_{n,22}(L)}{w_{d,22}(L)} \end{pmatrix}. \tag{4}$$

We do not require the moving average representation to be fundamental. The reason for this is that our interpretation of the backwards-looking model is that it is a structural description of a system.[6]

Our analysis focuses on linear feedback rules of the form

$$u_t = U(L) x_{t-1} . \tag{5}$$

where $U(L)$ is a one-sided polynomial in nonnegative powers of L. Each choice of this polynomial produces a law of motion for the state vector

$$A_0 x_t = A(L) x_{t-1} + B(L) U(L) x_{t-1} + \varepsilon_t . \tag{6}$$

with an associated moving average representation

[5]This assumption means that ε_t possesses a rational spectral density matrix. See Hansen and Sargent (1983) and Ito and Quah (1989) for examples of how rational spectral densities have been used to facilitate time series analyses.

[6]See Fernandez-Villaverde, Rubio-Ramirez, Sargent and Watson (2007) for a comprehensive analysis of the relationship between unrestricted vector autoregressions and structural models, in which invertibility of analogs to $W(L)$ plays a key role.

$$x_t = \left(A_0 - \left(A(L) + B(L)U(L)\right)L\right)^{-1} W(L)w_t . \tag{7}$$

We assume that the elements of $A\left(L\right), B\left(L\right)$ and $U\left(L\right)$ can always be written as the ratio of two finite degree polynomials so that x_t possesses a rational spectral density (see note 5). We will work with the $z-$transform of the moving average coefficients of this system

$$D^C\left(z\right) = \left(A_0 - \left(A\left(z\right) + B\left(z\right)U\left(z\right)\right)z\right)^{-1} W\left(z\right). \tag{8}$$

Associated with $D^C\left(z\right)$ is

$$f_{x|C}\left(z\right) = \frac{1}{2\pi} D^C\left(z\right)\Sigma_w D^C\left(z\right)', \tag{9}$$

where Σ_w is the variance covariance matrix of w. Note that for any matrix function $N\left(z\right)$, $N\left(z\right)'$ is its conjugate transpose. If $z = e^{-i\omega}$, then $f_{x|C}\left(z\right) \equiv f_{x|C}\left(\omega\right)$ is a spectral density. The superscript C is used because of the dependence of the moving average representation on the choice of control. Each choice of the polynomial $U(L)$ will produce a different spectral density matrix for the state variable vector.

Before we continue we need to discuss technical issues of existence of the mathematical objects under scrutiny. First, all the design limit expressions we discuss are integrals of the logarithm of the modulus of a finite degree polynomial over the unit circle in the complex plane. Sufficient conditions for the existence of these integrals are extremely modest. In particular, we do not need the existence of spectral density matrices in order to ensure existence of these integrals. Second, in order for a spectral density matrix to exist it is necessary that the state variables under scrutiny are jointly weakly stationary. Assume $\det\left(A_0\right) \neq 0$. Following Priestley (1982, p. 798, eq. (10.4.51)), given our assumption that ε_t is second order stationary, existence of a spectral density for the no control case requires that $\det\left(I - A_0^{-1}A\left(z\right)z\right)$ has no zeroes inside or on the unit circle in the complex plane. The requirement for the control case is that $\det\left(I - A_0^{-1}\left(A\left(z\right) + B\left(z\right)U\left(z\right)\right)z\right)$ has no zeroes inside or on the unit circle in the complex plane. We shall always choose controls so this latter requirement holds.

One way to understand the effects of a control role is via by considering the way that $f_{x|C}\left(z\right)$ depends on the $z-$transform of the feedback rule, $U\left(z\right)$. The feasible set of control rules determines the feasible set of moving average coefficients in the controlled system. Our goal is to use the feasible set for $f_{x|C}\left(z\right)$ to understand the opportunity set faced by a policymaker.

In the case of restrictions on the moving average polynomial $D^C\left(z\right)$, we will need to focus on the properties of $W\left(z\right)$, specifically

$$\det\left(W\left(z\right)\right) = \overline{w}\,\frac{\prod_{i=1}^{w_{MA}}\left(1 - w_i z\right)}{\prod_{i=1}^{w_{AR}}\left(1 - \rho_i z\right)}. \tag{10}$$

where w_{MA} is the degree of the polynomial

$$w_{n,11}\left(L\right)w_{n,22}\left(L\right)w_{d,12}\left(L\right)w_{d,21}\left(L\right) - w_{n,12}\left(L\right)w_{n,21}\left(L\right)w_{d,11}\left(L\right)w_{d,22}\left(L\right),$$

w_{AR} is the degree of the polynomial

$$w_{d,11}\left(L\right)w_{d,22}\left(L\right)w_{d,12}\left(L\right)w_{d,21}\left(L\right)$$

and $\overline{w}$ is the ratio of the zero degree coefficients on the two polynomials. Since ε_t is second-order stationary, the roots ρ_i all lie inside the unit circle. However, the roots w_i may lie outside the unit circle as we have not assumed the shocks are fundamental.

Our first result characterizes the feasible values of $D^C\left(z\right)$.

Theorem 1. Design limits on the MA polynomial in a backwards-looking MIMO model

For the system described by eq. (2), if the control rule produces stable state variables, the Fourier transform of the associated controlled system matrix of moving average coefficients, $D^C\left(z\right)$, must fulfill

$$\int_{-\pi}^{\pi} \log\left(\left|\det D^C\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_{w,B} \tag{11}$$

where

$$K_{w,B} = 4\pi\left(\log \overline{w} - \log\left|a\right| + \sum_{u_i} \log\left|w_{u_i}\right|\right),\ \ i \in \{u_i\} \text{ if } \left|w_i\right| > 1,\ \ a = \det\left(A_0\right). \tag{12}$$

Pf. See Appendix 1.

The idea that any set of moving average coefficients must fulfill an integral equation of the form described by (11) and (12) is a key idea in the study of design limits as it means that the feasible representations of the state vector are identified by the set of moving average representations defined by the integral constraint.

The policy implications of restrictions on possible moving average representations for a controlled system may be elucidated by comparing the properties of the law of motion for the state vector when a control is present with the law of motion when there is no control, i.e. $u_t = 0\ \ \forall t$. The uncontrolled system is therefore

$$A_0 x_t = A(L) x_{t-1} + \varepsilon_t . \tag{13}$$

In parallel to the controlled system case, define (assuming the system (13) is stable)

$$D^{NC}(z) = \left(A_0 - A(z)z\right)^{-1} W(z). \tag{14}$$

and

$$f_{x|NC}(z) = \frac{1}{2\pi} D^{NC}(z) \Sigma_w D^{NC}(z)'. \tag{15}$$

Notice that while one would typically expect a policymaker to choose a control rule that ensures that the state vector x is stable, it is possible that the uncontrolled system is not. Hence it may not be the case that $D^{NC}(z)$ exists. Our analysis will cover the case where the no control system is unstable, but for intuition we assume stability for the moment.

A stabilization policy may be interpreted as the transformation of $f_{x|NC}(z)$ into $f_{x|C}(z)$. To understand this transformation, we follow the control theory literature and define a sensitivity matrix $S(z)$ via the way in which the control transforms $D^{NC}(z)$ into $D^{C}(z)$, i.e.

$$S(z) = D^{C}(z) D^{NC}(z)^{-1}. \tag{16}$$

if $D^{NC}(z)$ exists, which in turn means that

$$f_{x|C}(z) = S(z) f_{x|NC}(z) S(z)'. \tag{17}$$

This formulation makes clear why, in the control literature, the sensitivity function is said to shape the behavior of the state vector.

As each $D^C\left(z\right)$ corresponds to some $S\left(z\right)$, one can think of the choice of control as the choice of a sensitivity function; any constraints on $D^C\left(z\right)$ in turn may be translated into constraints on $S\left(z\right)$. If one considers (16), it is evident the constraints on the sensitivity function can be derived if it is the case that

$$\begin{aligned}&\int_{-\pi}^{\pi}\log\left(\left|\det S\left(e^{-i\omega}\right)\right|^2\right)d\omega = \\ &\int_{-\pi}^{\pi}\log\left(\left|\det D^C\left(e^{-i\omega}\right)\right|^2\right)d\omega - \int_{-\pi}^{\pi}\log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right)d\omega\end{aligned}. \tag{18}$$

If (18) holds, then one can simply apply Theorem 1 to the terms $\int_{-\pi}^{\pi}\log\left(\left|\det D^C\left(e^{-i\omega}\right)\right|^2\right)d\omega$ and $\int_{-\pi}^{\pi}\log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right)d\omega$ and deduce constraints on $S\left(z\right)$. The difficulty in doing this, as noted above, is that $D^{NC}\left(z\right)$ may not exist because the no control system may be unstable. In fact, (18) holds even if the no control case is unstable, given the following argument, which is similar to that used in Wu and Jonckheere (1992, pg. 1801). Recall that by the fundamental theorem of algebra

$$\det\left(I - A_0^{-1}A\left(z\right)z\right) = \prod_{i=1}^{m_{NC}}\left(1 - \lambda_i^{NC}z\right) \tag{19}$$

where m_{NC} is the degree of the characteristic polynomial of the uncontrolled system and λ_i^{NC} are the eigenvalues of the uncontrolled system. Lemma 5 of Wu and Jonckheere (1992) shows that the integral of the log of $\prod_{i=1}^{m_{NC}}\left(1 - \lambda_i^{NC}z\right)\prod_{i=1}^{m_{NC}}\left(1 - \lambda_i^{NC}z^{-1}\right)$ on the unit circle, i.e. for $z = e^{-i\omega}$, is well defined even if some of the eigenvalues λ_i^{NC} are on or outside the unit circle, i.e. $\left|\lambda_i^{NC}\right| \geq 1$. Therefore, in order to handle the no control system in the presence of instability, one simply *defines*

$$\int_{-\pi}^{\pi} \log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right) d\omega = \\ -\int_{-\pi}^{\pi} \log\left(\left|\det\left(A_0\left(I - A_0^{-1} A\left(e^{-i\omega}\right) e^{-i\omega}\right)\right)\right|^2\right) d\omega + \int_{-\pi}^{\pi} \log\left(\left|\det W\left(e^{-i\omega}\right)\right|^2\right) d\omega \tag{20}$$

We will use this convention throughout.

Applying this argument to Theorem 1, the RHS of (20) is

$$\int_{-\pi}^{\pi} \log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_{w,B} - 4\pi \sum_{v_i} \log\left|\lambda_{v_i}^{NC}\right| \;\; i \in \{v_i\} \text{ if } \left|\lambda_i^{NC}\right| > 1. \tag{21}$$

The combination of Theorem 1, (18) and (21) immediately leads to Theorem 2.

Theorem 2. Design limits on the sensitivity matrix for a backwards-looking MIMO model

For the system described by eq. (2), the associated sensitivity matrix $S\left(e^{-i\omega}\right)$ must fulfill

$$\int_{-\pi}^{\pi} \log\left(\left|\det S\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_B, \tag{22}$$

where

$$K_B = 4\pi \sum_{v_i} \log\left|\lambda_{v_i}^{NC}\right| \;\; i \in \{v_i\} \text{ if } \left|\lambda_i^{NC}\right| > 1. \tag{23}$$

This expression has several properties of interest.

First, $K_B = 0$ whenever the unconstrained system is stable. This means that for a large class of models, the constraint on the sensitivity function is identical to the constraint on the uncontrolled system. More generally, different models may be sorted into equivalence classes with respect to K_B as its value is entirely determined by the unstable roots in the $A(L)$ polynomial. Notice as well that the value of the constraint does not depend on the control rule nor does it depend on $W(L)$, i.e. the (second-order) time series structure of ε_t.

Second, taken together, the facts that a nonzero constraint only occur when the uncontrolled system is unstable and that the magnitudes and number of the unstable roots determine the value of the constraint, indicate that the use of a control to eliminate unstable roots in a system does have a cost in terms of the ability of the policymaker to stabilize fluctuations after these roots have been eliminated. This provides a new perspective on the idea that trends and cycles do not represent independent aspects of stabilization policy.

Third, policymakers inevitably must trade off variance at different frequencies. Since $\left|\lambda_i^{NC}\right| \geq 1$, it is immediate from (23) that $K_B \geq 0$. This implies, given (22), that it is impossible for $\left|\det S\left(e^{-i\omega}\right)\right|^2 < 1 \ \forall \omega \in [-\pi, \pi]$ and therefore it is impossible to reduce the variance contributions at all frequencies when one moves from the uncontrolled system to a controlled one. Further, the integral constraint implies that $\left|\det S\left(e^{-i\omega}\right)\right|^2 > 1$ for some interval of frequencies if $\left|\det S\left(e^{-i\omega}\right)\right|^2 < 1$ for another. In order to reduce the variance contributions of one interval of frequencies, it is necessary to increase the variance contributions of some other interval. This tradeoff is fundamental as it cannot be avoided by the choice of control. By implication, minimizing a linear combination of the variances of the elements of x_t will involve trading off frequency specific variance contributions. In other words, variance minimization implies that, even though overall variance is reduced when one integrates across frequencies, for some frequencies, a control that is optimal in this sense leads to greater variance.

Fourth, the issue of whether the shocks are or are not fundamental is irrelevant to the constraints on the sensitivity function. The reason for this is that the sensitivity function compares the effects of control to no control in such a way that this part of the constraint in Theorem 1 cancels out.

Theorems 1 and 2 are examples of the conservation laws of fragility or waterbed effects that we mentioned in the introduction. As we have discussed, these types of tradeoffs have been studied in the control theory literature. The control theory literature naturally does not consider how expectations affect current state variables. We next consider how to understand design limits for forward-looking (e.g. hybrid) models. As we will show, these are very different from those that exist for the backwards-looking case.

ii. hybrid systems

How does the introduction of forward-looking elements affect design limits? To understand these effects, we consider

$$A_0 x_t = \beta E_t x_{t+1} + A(L) x_{t-1} + B(L) u_t + \varepsilon_t . \tag{24}$$

This system is identical to (2) except for the addition of the forward-looking term $\beta E_t x_{t+1}$. Expectations are assumed to be rational. We are interested in characterizing the equilibrium moving average representation of the state vector given a control,

$$x_t = F^C(L) w_t = \begin{pmatrix} f_{11}^C(L) & f_{12}^C(L) \\ f_{21}^C(L) & f_{22}^C(L) \end{pmatrix} w_t , \tag{25}$$

where w_t are fundamental innovations. It is convenient to work with innovations that are contemporaneously uncorrelated. Let $v_t = V w_t$ denote any orthogonalization of the fundamental errors. Then,

$$x_t = F^C(L) V^{-1} v_t = G^C(L) v_t = \begin{pmatrix} g_{11}^C(L) & g_{12}^C(L) \\ g_{21}^C(L) & g_{22}^C(L) \end{pmatrix} v_t . \tag{26}$$

None of our results depend on the choice of orthogonalization.

As is well known, systems with forward-looking elements can exhibit multiple solutions as well as fail to have any solution at all. We will assume existence and uniqueness of solutions in our analysis since we have nothing to contribute to that well-studied subject; Appendix 2 discusses a set of sufficient conditions for existence and uniqueness of solutions.

The rational expectations assumption of course places structure on the individual $g_{ij}^{C}\left(L\right)$ elements. For our purposes, what matters is that each $g_{ij}^{C}\left(L\right)$ may be written as a ratio of finite polynomials with common denominator up to the denominator polynomials of $V^{-1}W\left(L\right)$ denoted by $v_{d,ij}\left(z\right)$, that are exogenous and do not depend on the control applied to the system; see the Appendix 1 for a proof that the $z-$transform of $G^{C}\left(L\right)$ in (26) may be written as

$$G^{C}(z)=\frac{1}{g_{d}^{C}\left(z\right)}\begin{pmatrix}\frac{g_{n,11}^{C}\left(z\right)}{v_{d,11}\left(z\right)v_{d,21}\left(z\right)} & \frac{g_{n,12}^{C}\left(z\right)}{v_{d,12}\left(z\right)v_{d,22}\left(z\right)}\\ \frac{g_{n,21}^{C}\left(z\right)}{v_{d,11}\left(z\right)v_{d,21}\left(z\right)} & \frac{g_{n,22}^{C}\left(z\right)}{v_{d,12}\left(z\right)v_{d,22}\left(z\right)}\end{pmatrix}. \tag{27}$$

Here, the subscripts n and d refer to numerator and denominator. The denominator polynomial $g_{d}^{C}(L)$ is the characteristic polynomial of the system; define g_{d} as its zero degree coefficient; this will prove useful. Similarly, define g_{n} as the coefficient on the zero degree of the polynomial $g_{n,11}^{C}\left(L\right)g_{n,22}^{C}\left(L\right)-g_{n,12}^{C}\left(L\right)g_{n,21}^{C}\left(L\right)$. The form (27) together with the above definitions is useful because it allows us to prove

Theorem 3. Design limits on the MA polynomial in a forwards-looking MIMO model

The orthogonalized moving average coefficients of a controlled system (26) must obey

$$\int_{-\pi}^{\pi} \log\left(\left|\det G^{C}\left(e^{-i\omega}\right)\right|^{2}\right) d\omega = K_{w,H}\,, \tag{28}$$

where

$$K_{w,H} = 4\pi\left(\log\left|g_{n}^{C}\right| - 2\log\left|g_{d}\right| + \sum_{u_{ij}} \log\left|g_{n,u_{ij}}^{C}\right|\right),\ ij \in \{u_{ij}\} \text{ if } \left|g_{n,ij}\right| > 1. \tag{29}$$

Pf. See Appendix 1.

In identifying restrictions on the sensitivity function for this system, we once again define a system with no control, i.e.

$$A_0 x_t = \beta E_t x_{t+1} + A(L)\,x_{t-1} + \varepsilon_t \tag{30}$$

and model the associated law of motion as

$$x_t = G^{NC}\left(L\right)v_t = \frac{1}{g_d^{NC}\left(L\right)}\begin{pmatrix} \dfrac{g_{n,11}^{NC}\left(L\right)}{v_{d,11}\left(L\right)v_{d,21}\left(L\right)} & \dfrac{g_{n,12}^{NC}\left(L\right)}{v_{d,12}\left(L\right)v_{d,22}\left(L\right)} \\ \dfrac{g_{n,21}^{NC}\left(L\right)}{v_{d,11}\left(L\right)v_{d,21}\left(L\right)} & \dfrac{g_{n,22}^{NC}\left(L\right)}{v_{d,12}\left(L\right)v_{d,22}\left(L\right)} \end{pmatrix} v_t\,. \tag{31}$$

In writing (31) it is assumed that a unique solution to (24) exists and that the solution can be expressed in MA form. This is excessively restrictive, in light of our earlier argument that no control systems may be unstable. However, in parallel to the backwards case, one can relax this requirement when formulating design function limits; details may be found in Appendix 3. In the subsequent discussion, we will work with $G^{NC}\left(z\right)$.

In parallel to the backwards-looking model, the sensitivity function for the hybrid model is

$$S(z) = G^C(z) G^{NC}(z)^{-1} \tag{32}$$

In turn, $\det S(z)$ may be expressed as

$$\begin{gathered}
\det S(z) = \det G^C(z) \det G^{NC}(z)^{-1} = \\
\left(\frac{g_{n,11}^C(z) g_{n,22}^C(z) - g_{n,12}^C(z) g_{n,21}^C(z)}{g_d^C(z)^2} \right) \left(\frac{g_d^{NC}(z)^2}{g_{n,11}^{NC}(z) g_{n,22}^{NC}(z) - g_{n,12}^{NC}(z) g_{n,21}^{NC}(z)} \right) = \\
\left(\frac{g_d^{NC}(z)^2}{g_d^C(z)^2} \right) \left(\frac{g_{n,11}^C(z) g_{n,22}^C(z) - g_{n,12}^C(z) g_{n,21}^C(z)}{g_{n,11}^{NC}(z) g_{n,22}^{NC}(z) - g_{n,12}^{NC}(z) g_{n,21}^{NC}(z)} \right) = \\
\left(\frac{g_d^2 \left(\Pi_{i=1}^{d^{NC}} \left(1 - g_{d,i}^{NC} z\right) \right)^2}{g_d^2 \left(\Pi_{i=1}^{d^C} \left(1 - g_{d,i}^C z\right) \right)^2} \right) \left(\frac{g_n^C \Pi_{i=1}^{n^C} \left(1 - g_{n,i}^C z\right)}{g_n^{NC} \Pi_{i=1}^{n^{NC}} \left(1 - g_{n,i}^{NC} z\right)} \right)
\end{gathered} \tag{33}$$

To understand the final equality in (35), observe that $\frac{g_d^2 \left(\Pi_{i=1}^{d^{NC}} \left(1 - g_{d,i}^{NC} z\right) \right)^2}{g_d^2 \left(\Pi_{i=1}^{d^C} \left(1 - g_{d,i}^C z\right) \right)^2}$ also appears in the calculation of the constraints for the sensitivity function of the backwards system as it is a ratio of simple polynomials based on the poles of the controlled and the uncontrolled system. In contrast, the second ratio $\frac{g_n^C \Pi_{i=1}^{n^C} \left(1 - g_{n,i}^C z\right)}{g_n^{NC} \Pi_{i=1}^{n^{NC}} \left(1 - g_{n,i}^{NC} z\right)}$ incorporates elements of the law of motion that did not affect the sensitivity function for the backwards-looking case. The application of a control can affect the value of g_n^C as well as the location of the zeros $g_{n,i}^C$ so that the second ratio does not collapse to 1. Notice that the eigenvalues $g_{n,i}^C$ and $g_{n,i}^{NC}$ are not restricted to be inside the unit circle. If they are, the corresponding system is said to be fundamental, as the innovations to the vector of expectational errors are the same as the innovations to the vector of disturbances w_t. If at least one of the eigenvalues is outside the unit circle, the corresponding system is said to be non-fundamental; in this case the variance of the vector of expectational errors is higher than the variance of the vector of disturbances w_t. Theorem 4 reveals that the latter case imposes design limits to the control of the system.

Some general differences exist in optimal policy between the hybrid and backwards cases when a policymaker seeks to minimize variances of the state variables. Variance minimization for backwards-looking systems is achieved by choosing controls that reduce the system dynamics to white noise; see Brock and Durlauf (2005) for general analysis. The case $\beta \neq 0$ is more complicated. In the scalar case Brock, Durlauf, and Rondina (2008) demonstrate that, when the shocks $\{\varepsilon_t\}$ are second order white noise and $\beta = 0$, then the best control reduces $\{x_t\}$ to second-order white noise. They find that when $\beta > 0$ and not too large (so that a solution to (24) exists) an AR(1) process for x_t, i.e. $A(L)x_{t-1} = A_1 x_{t-1}$, yields a solution for (24) that is more (less) persistent for $0 < A_1 < 1,\ (-1 < A_1 < 0)$. These results are reversed when $\beta < 0$ (absolute value not too large so that we have existence of a solution to (24)). Brock, Durlauf, and Rondina (2008) show that a positively persistent AR(1) process is turned into an AR(1) with negative persistence by variance minimizing optimal control when $\beta > 0$. Intuitively optimal control cancels the magnification effect on volatility of $\beta > 0$ by going "beyond reduction to white noise" by exploiting the ability of $\beta > 0$ to shrink the effect of $-1 < A_1 < 0$ on volatility. This same intuition applies to diagonal matrix versions of (24) although matters are more complicated for general matrix versions. We shall see that new and interesting differences emerge when one considers frequency-specific effects.

In parallel to the derivation of Theorem 2 from Theorem 1, Theorem 3 leads immediately to Theorem 4.

Theorem 4. Design limits on the sensitivity function in a forwards-looking MIMO model

The sensitivity function of a controlled system (24) must obey

$$\int_{-\pi}^{\pi} \log\left(\left|\det S\left(e^{-i\omega}\right)\right|^2\right) d\omega = K_H , \tag{34}$$

where

$$K_H = 4\pi\left(\log\left|g_n^C\right| - \log\left|g_n^{NC}\right| + \sum_{v_i}\log\left|g_{d,v_i}^{NC}\right| + \sum_{u_i^C}\log\left|g_{n,u_i}^C\right| - \sum_{u_i^{NC}}\log\left|g_{n,u_i}^{NC}\right|\right),$$
$$i \in \{v_i\} \text{ if } \left|g_{d,i}^{NC}\right| > 1,\ i \in \{u_i^C\} \text{ if } \left|g_{n,i}^C\right| > 1 \text{ and } i \in \{u_i^{NC}\} \text{ if } \left|g_{n,i}^{NC}\right| > 1. \tag{35}$$

From the perspective of design limits, there are several important differences between this case and the backwards-looking case.

First, in the presence of an expectations-based component, the sensitivity function constraint K_H can be negative. This means that it is possible for a control rule to reduce variance contributions at *all* frequencies relative to an uncontrolled system. Brock, Durlauf and Rondina (2008) provide a univariate example of this property. Their example illustrates the fact that distinct variance minimizing and uniform variance reduction controls can exist for a given system, which matters if a policymaker is concerned about loss function uncertainty, i.e. the policymaker is not sure whether or not all frequency-specific variances should be weighted equally.

Second, expectations also affect the nature of the constraint value K_H as the terms associated with $\log\left|g_n^C\right| - \log\left|g_n^{NC}\right|$ do not have an analog in the backwards-looking case. Recall from the Wiener-Kolmogorov prediction formula that if $x_t = G(L)v_t$ then $E_t x_{t+1} = L^{-1}(G(L) - G_0)v_t$. Thus when the term $\beta E_t x_{t+1}$ is added to the dynamics as in (24) above then "extra" terms should be expected to appear in the constraint value. These terms vanish when $\beta = 0$. More important, the value of the "constant" G_0 changes as the control choice changes. Metaphorically, for the forwards-looking case, the "budget constraint" defined by K_H shifts across feedback rules, so that a purchase of lower variance at one frequency band does not have to be paid for by an increase in variance at another band.

Finally, the term $\sum_{u_i^C} \log\left|g_{n,u_i}^C\right| - \sum_{u_i^{NC}} \log\left|g_{n,u_i}^{NC}\right|$ captures the possibility that both the controlled and the uncontrolled systems may exhibit a nonfundamental moving average representation (at least one of the eigenvalues of the numerator polynomial bigger than one). Theorem 4 shows, for instance, that a control $U(z)$ that turns a fundamental representation ($\left|g_{n,i}^{NC}\right| < 1 \;\; \forall i$) into a nonfundamental one ($\left|g_{n,i}^{C}\right| > 1$ for at least one i) is subject to stronger design limits. Intuitively, when a stabilization policy depends on past states, the policymaker under the new policy will be responding to fundamental innovations that do not correspond to the underlying orthogonal innovations v_t, thereby reducing the performance of the policy in terms of frequency-specific tradeoffs.

3. Unpacking frequency-specific tradeoffs: the design transformation matrix

i. general ideas

In most macroeconomic applications the object of interest in control problems is a function of the variances for the state variables. It follows that, when comparing a system with no control and a system with control, the relationship between the two can be described by those functions that transform an underlying set of orthogonal components v of the uncontrolled process into the associated components of the controlled process. In order to identify frequency bands where robustness is increased by the control (a good thing) and to identify frequency bands where fragility is increased (a bad thing) we introduce the concept of a design transformation matrix, denoted as

$$M(z) = \begin{pmatrix} \dfrac{f_{x_1,v_1|C}(z)}{f_{x_1,v_1|NC}(z)} & \dfrac{f_{x_1,v_2|C}(z)}{f_{x_1,v_2|NC}(z)} \\ \dfrac{f_{x_2,v_1|C}(z)}{f_{x_2,v_1|NC}(z)} & \dfrac{f_{x_2,v_2|C}(z)}{f_{x_2,v_2|NC}(z)} \end{pmatrix} \tag{36}$$

It is immediate that

$$f_{x_1|C}(z) = M_{11}(z) f_{x_1,v_1|NC}(z) + M_{12}(z) f_{x_1,v_2|NC}(z) \tag{37}$$

and

$$f_{x_2|C}(z) = M_{21}(z) f_{x_2,v_1|NC}(z) + M_{22}(z) f_{x_2,v_2|NC}(z). \tag{38}$$

These imply that the elements of the design matrix $M(z)$ are functions of the elements of the sensitivity matrix since

$$f_{x|C}(z) = D^{C}(z)\Sigma_w D^{C}(z)' = \left(S(z)D^{NC}(z)\right)\Sigma_w \left(S(z)D^{NC}(z)\right)' = S(z) f_{x|NC}(z) S(z)' \tag{39}$$

When spectral densities exist, the design transformation matrix provides a characterization of the frequency by frequency changes in the effects of shocks on the spectral density matrix of the state variables as one moves from an uncontrolled to a controlled system. Regardless of whether spectral densities exist, the design matrix describes how the frequency-specific effects of the innovations v on the state variable are affected by the choice of control. Note that for univariate systems the sensitivity matrix and the design matrix coincide.

The design transformation matrix is useful because it gives us a way of displaying the relative allocation of power between an uncontrolled system and a controlled system. This will be illustrated in graphical displays in Section 4 on Taylor Rules below (see especially Figures 3, 4, 5.B, 5.C, 6.B, and 6.C below). It is also useful in terms of understanding how design limits change when the number of states exceeds the number of controls.

ii. design transformation matrix for backwards-looking systems

For our backwards-looking system, the properties of the design transformation matrix can be derived from

$$f_{x_1}^C(z) = f_{x_1,v_1}^C(z) + f_{x_1,v_2}^C(z) = \frac{1}{d_d^C(z)\, d_d^C(z^{-1})}\left(d_{n,11}^C(z)\, d_{n,11}^C(z^{-1})\sigma_{v_1}^2 + d_{n,12}^C(z)\, d_{n,12}^C(z^{-1})\sigma_{v_2}^2\right) \tag{40}$$

and

$$f_{x_2}^C(z) = f_{x_2,v_1}^C(z) + f_{x_2,v_2}^C(z) = \frac{1}{d_d^C(z)\, d_d^C(z^{-1})}\left(d_{n,21}^C(z)\, d_{n,21}^C(z^{-1})\sigma_{v_1}^2 + d_{n,22}^C(z)\, d_{n,22}^C(z^{-1})\sigma_{v_2}^2\right) \tag{41}$$

where the terms $d_{n,ij}^C(z)$ are the numerator polynomials of the matrix $D^C(z)$ introduced in eq. (8) while $d_d^C(z)$ is the common denominator polynomial. For this system $M(z)$ can be written as

$$M(z) = \frac{d_d^{NC}(z)\, d_d^{NC}(z^{-1})}{d_d^C(z)\, d_d^C(z^{-1})}\begin{pmatrix} \frac{d_{n,11}^C(z)\, d_{n,11}^C(z^{-1})}{d_{n,11}^{NC}(z)\, d_{n,11}^{NC}(z^{-1})} & \frac{d_{n,12}^C(z)\, d_{n,12}^C(z^{-1})}{d_{n,12}^{NC}(z)\, d_{n,12}^{NC}(z^{-1})} \\ \frac{d_{n,21}^C(z)\, d_{n,21}^C(z^{-1})}{d_{n,21}^{NC}(z)\, d_{n,21}^{NC}(z^{-1})} & \frac{d_{n,22}^C(z)\, d_{n,22}^C(z^{-1})}{d_{n,22}^{NC}(z)\, d_{n,22}^{NC}(z^{-1})} \end{pmatrix} \tag{42}$$

How do constraints on $S(z)$ impinge on the freedom to design the elements in $M(z)$? Eq. (16) implies that

$$\det S(z) S(z)' = \frac{\det D^C(z)}{\det D^{NC}(z)} \frac{\det D^C(z^{-1})}{\det D^{NC}(z^{-1})} = \frac{d_d^{NC}(z)\, d_d^{NC}(z^{-1})}{d_d^C(z)\, d_d^C(z^{-1})}. \tag{43}$$

Combining (42) and (43) it is immediate that each term of the design transformation matrix is related to the sensitivity matrix according to

$$\log M_{ij}\left(z\right) = \log\det S\left(z\right) S\left(z\right)' + \log \frac{d_{n,ij}^{C}\left(z\right) d_{n,ij}^{C}\left(z^{-1}\right)}{d_{n,ij}^{NC}\left(z\right) d_{n,ij}^{NC}\left(z^{-1}\right)}. \tag{44}$$

Theorem 2 describes how each term of the design matrix is restricted by the Bode constraint's effect on $\log\det S\left(z\right) S\left(z\right)'$. However, each term in the design matrix also contains an additional component $\log \frac{d_{n,ij}^{C}\left(z\right) d_{n,ij}^{C}\left(z^{-1}\right)}{d_{n,ij}^{NC}\left(z\right) d_{n,ij}^{NC}\left(z^{-1}\right)}$ that depends on the control.

iii. design matrix for hybrid systems

For a hybrid system,

$$\begin{gathered} f_{x_1}\left(z\right) = f_{x_1,v_1}\left(z\right) + f_{x_1,v_2}\left(z\right) = \\ \frac{1}{g_{d}^{C}\left(z\right) g_{d}^{C}\left(z^{-1}\right)} \left(g_{n,11}^{C}\left(z\right) g_{n,11}^{C}\left(z^{-1}\right) f_{v,1}\left(z\right) + g_{n,12}^{C}\left(z\right) g_{n,12}^{C}\left(z^{-1}\right) f_{v,2}\left(z\right) \right) \end{gathered} \tag{45}$$

and

$$\begin{gathered} f_{x_2}\left(z\right) = f_{x_2,v_1}\left(z\right) + f_{x_2,v_2}\left(z\right) = \\ \frac{1}{g_{d}^{C}\left(z\right) g_{d}^{C}\left(z^{-1}\right)} \left(g_{n,21}^{C}\left(z\right) g_{n,21}^{C}\left(z^{-1}\right) f_{v,1}\left(z\right) + g_{n,22}^{C}\left(z\right) g_{n,22}^{C}\left(z^{-1}\right) f_{v,2}\left(z\right) \right) \end{gathered} \tag{46}$$

where

$$f_{v,1}(z) = \frac{\sigma_{v_1}^2}{v_{d,11}(z)v_{d,21}(z)v_{d,11}(z^{-1})v_{d,21}(z^{-1})}, \quad f_{v,2}(z) = \frac{\sigma_{v_2}^2}{v_{d,21}(z)v_{d,22}(z)v_{d,21}(z^{-1})v_{d,22}(z^{-1})}$$

and $g_d^C(z)$ and the $g_{n,ij}^C(z)$ polynomials are the elements of the matrix polynomial $G^C(L)$ in (27). The design transformation matrix $M(z)$ for this system can be written as

$$M(z) = \frac{g_d^{NC}(z)g_d^{NC}(z^{-1})}{g_d^C(z)g_d^C(z^{-1})} \begin{pmatrix} \frac{g_{n,11}^C(z)g_{n,11}^C(z^{-1})}{g_{n,11}^{NC}(z)g_{n,11}^{NC}(z^{-1})} & \frac{g_{n,12}^C(z)g_{n,12}^C(z^{-1})}{g_{n,12}^{NC}(z)g_{n,12}^{NC}(z^{-1})} \\ \frac{g_{n,21}^C(z)g_{n,21}^C(z^{-1})}{g_{n,21}^{NC}(z)g_{n,21}^{NC}(z^{-1})} & \frac{g_{n,22}^C(z)g_{n,22}^C(z^{-1})}{g_{n,22}^{NC}(z)g_{n,22}^{NC}(z^{-1})} \end{pmatrix} \tag{47}$$

The relationship between $S(z)$ and $M(z)$ is given by

$$\log M_{ij}(z) = \log\det S(z)S(z)' - \log\frac{g_n^C(z)}{g_n^{NC}(z)}\frac{g_n^C(z^{-1})}{g_n^{NC}(z^{-1})} + \log\frac{g_{n,ij}^C(z)g_{n,ij}^C(z^{-1})}{g_{n,ij}^{NC}(z)g_{n,ij}^{NC}(z^{-1})} \tag{48}$$

where $g_n^C(z) = g_{n,11}^C(z)g_{n,22}^C(z) - g_{n,21}^C(z)g_{n,12}^C(z)$; corresponding terms may be defined for the uncontrolled system. Notice the similarity in structure between (48) for hybrids and (44) for backwards-looking models even though the values of the elements are different.

iv. MIMO versus SIMO

Recall that a 2×2 MIMO system is a system where there is a single control instrument for each state variable whereas a SIMO system is one where there is only one control instrument. 2×2 SIMO systems are common in macroeconomics; a leading example is the use of the interest rate to simultaneously affect output and inflation. A main way of discriminating between MIMO and SIMO systems is the difference in their controllability subspaces and their stabilizability. See Kwakernaak and Sivan (1972, Chapter 1) and Zhou, Doyle, and Glover (1996, Chapter 3) for a treatment of controllability and stabilizability of linear control systems. Formally, a 2×2 SIMO system is defined as a 2×2 MIMO in which either row 1 or row 2 of the matrix $B(z)$ introduced in Section 2 is restricted to be zero. If row 1 is restricted to be zero we say that the control can be applied only to equation 2, and vice versa. In this section we develop a 2×2 MIMO and 2×2 SIMO comparison in terms of differences of the integral constraints that characterize their design limitations.

We first state a basic result.

Theorem 5. Tradeoffs in MIMO and SIMO Systems

i. For a MIMO system, if the design matrix is given by (36) then

$$\int_{-\pi}^{\pi} \log M_{ij}(\omega)\, d\omega = K_B + \int_{-\pi}^{\pi} \log\left|d_{n,ij}^{C}\left(e^{-i\omega}\right)\right|^2 d\omega - \int_{-\pi}^{\pi} \log\left|d_{n,ij}^{NC}\left(e^{-i\omega}\right)\right|^2 d\omega \qquad (49)$$

for the backwards-looking case and

$$\int_{-\pi}^{\pi} \log M_{ij}(\omega)\, d\omega = \sum_{v_i} \log\left|g_{d,v_i}^{NC}\right| + \int_{-\pi}^{\pi} \log\left|g_{n,ij}^{C}\left(e^{-i\omega}\right)\right|^2 d\omega - \int_{-\pi}^{\pi} \log\left|g_{n,ij}^{NC}\left(e^{-i\omega}\right)\right|^2 d\omega \qquad (50)$$

where $i \in \{v_i\}$ if $\left|g_{d,i}^{NC}\right| > 1$, for the hybrid case.

ii. For a SIMO system, suppose that the matrix $W(z)$ is diagonal and suppose that the control can be applied only to equation j. Then, for the backwards-looking system

$$\int_{-\pi}^{\pi} \log M_{ij}(\omega)\, d\omega = K_B \quad \forall i \qquad (51)$$

where $i, j \in \{1,2\}$. For the hybrid case the constraint is still (50).

Pf: See Appendix 1.

Part (i) of Theorem 5 formalizes the idea that, when considering each element of the design transformation matrix, the design limits that apply are generally different from the design limits on the sensitivity function described in Theorems 2 and 4. Notice that (50) can be also written in terms of K_H; we do not do this because the associated expression is too cumbersome to be useful. In principle, a control can achieve reduction of volatility at all frequencies for a given design transformation matrix element for both the backwards and the hybrid cases. This is obviously not possible for all the elements of the matrix, but the policymaker in the MIMO case has the flexibility of choosing any element on which to impose an overall variance reduction at all frequencies. Part (ii) of the Theorem shows that such flexibility is lost in the backwards-looking case for the SIMO case. Interestingly, the integral constraints for the MIMO and SIMO cases are the same for the hybrid model.

In order to illustrate Theorem 5 we consider simple backwards-looking $AR(1)$ system. The comparable hybrid system provides less clean results without additional insight and is therefore omitted. The system is

$$\begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1t-1} \\ x_{2t-1} \end{pmatrix} + \begin{pmatrix} b_1 u_{1t} + v_{1t} \\ b_2 u_{2t} + v_{2t} \end{pmatrix} \quad (52)$$

.

For the MIMO model both b_1 and b_2 are nonzero whereas, according to our definition, for the SIMO model $b_1 = 0$ and b_2 is nonzero. Control rules are restricted to

$$u_{1t} = g_{11}x_{1,t-1} + g_{12}x_{2,t-1}, \quad u_{2t} = g_{21}x_{1,t-1} + g_{22}x_{2,t-1} \quad (53)$$

The form of the polynomial matrix for the uncontrolled system, $D^{NC}(z)$, is

$$D^{NC}\left(z\right)=\frac{\begin{pmatrix}1-a_{22}z & -a_{12}z\\ -a_{21}z & 1-a_{11}z\end{pmatrix}}{\left(1-a_{11}z\right)\left(1-a_{22}z\right)-\left(a_{12}z\right)\left(a_{21}z\right)} \tag{54}$$

The controlled system $D^{C}\left(z\right)$ takes the same form and we denote the controlled coefficients by a_{ij}^{*}. These coefficients equal

$$a_{11}^{*}=a_{11}+b_{1}g_{11},\; a_{12}^{*}=a_{12}+b_{1}g_{11},\; a_{21}^{*}=a_{21}+b_{2}g_{21},\; a_{22}^{*}=a_{22}+b_{2}g_{22} \tag{55}$$

Using Theorem 5 we can express the restrictions on the elements of the design matrix as

$$\int_{-\pi}^{\pi}\log M_{11}\left(\omega\right)d\omega=K_{B}+\int_{-\pi}^{\pi}\log\left|1-a_{22}^{*}e^{-i\omega}\right|^{2}d\omega-\int_{-\pi}^{\pi}\log\left|1-a_{22}e^{-i\omega}\right|^{2}d\omega, \tag{56}$$

$$\int_{-\pi}^{\pi}\log M_{12}\left(\omega\right)d\omega=K_{B}+4\pi\left(\log\left|a_{12}^{*}\right|-\log\left|a_{12}\right|\right) \tag{57}$$

$$\int_{-\pi}^{\pi}\log M_{21}\left(\omega\right)d\omega=K_{B}+4\pi\left(\log\left|a_{21}^{*}\right|-\log\left|a_{21}\right|\right), \tag{58}$$

$$\int_{-\pi}^{\pi}\log M_{22}\left(\omega\right)d\omega=K_{B}+\int_{-\pi}^{\pi}\log\left|1-a_{11}^{*}e^{-i\omega}\right|^{2}d\omega-\int_{-\pi}^{\pi}\log\left|1-a_{11}e^{-i\omega}\right|^{2}d\omega, \tag{59}$$

and the overall integral constraint is

$$K_{B}=-\int_{-\pi}^{\pi}\log\left(\left|\det\left(I-A^{*}e^{-i\omega}\right)\right|^{2}\right)d\omega+\int_{-\pi}^{\pi}\log\left(\left|\det\left(I-Ae^{-i\omega}\right)\right|^{2}\right)d\omega \tag{60}$$

In the SIMO case, $b_{1}=0$ so that, for this example, $a_{1j}^{*}=a_{1j}, j=1,2,\; a_{21}^{*}=a_{21}+b_{2}g_{21}$, and $a_{22}^{*}=a_{22}+b_{2}g_{22}$.

One can use this simple example to elucidate how the difference between MIMO and SIMO is reflected in the integral constraints (56)-(59). To do this, we employ some basic elements of linear control theory specialized to our 2×2 system. A 2×2 system $x_t = Ax_t + Bu_t$ is *completely controllable* if and only if the column vectors of the matrix (B, AB) span 2-dimensional space (Kwakernaak and Sivan (1972, Theorem 6.6)). The set of MIMO systems we consider are all completely controllable except for nongeneric [7]cases. It is easy to check (by checking the spanning condition for (B, AB)) that complete controllability for the class of SIMO systems we consider holds if and only if $b_2^2 a_{12} \neq 0$. By Kwakernaak and Sivan (1972, page 462)), if a system is completely controllable, then controls may be found that stabilize it. We shall always assume that controls are picked to stabilize the system if it is possible to do so. We are now in the position to exposit the main difference between 2×2 MIMO systems and 2×2 SIMO systems for this example. We shall take the main difference to be the lack of potential controllability (stabilizability) for a generic class of SIMO systems.

Since our class of 2×2 MIMO systems are stabilizable (Kwakernaak and Sivan (1972, page 462, Definition 6.5)) except for nongeneric cases, we shall assume all eigenvalues are inside the unit circle for the 2×2 matrix A^* for our controlled MIMO systems. Thus an essential difference between MIMO and SIMO that is reflected in the set of integral constraints (56)-(59) above lies in the difference

$$K_{B,MIMO} - K_{B,SIMO} = \\ -\int_{-\pi}^{\pi} \log\left(\left|\det\left(I - A_{MIMO}^* e^{-i\omega}\right)\right|^2\right) d\omega + \int_{-\pi}^{\pi} \log\left(\left|\det\left(I - A_{SIMO}^* e^{-i\omega}\right)\right|^2\right) d\omega \qquad (61)$$

If both MIMO and SIMO were controllable we assume control choices are made to stabilize both. Thus the difference above would be zero. However, unless $b_2^2 a_{12} \neq 0$, our class of SIMO systems is *not* completely controllable. Hence the difference above is always nonnegative and may be positive. The inequality faced by a planner which always

[7]By nongeneric, we mean that the set of parameters under which controllability fails has measure zero in the set of possible parameter values.

does the best it can to stabilize the system it faces i.e. $K_{B,MIMO} - K_{B,SIMO} \geq 0$ thus represents an additional fundamental limitation faced in SIMO environments.

This inequality finding also holds for more general environments. Since b_2 is assumed to be nonzero, lack of controllability for the example arises only when $a_{12} = 0$. If $a_{12} = 0$, since $b_1 = 0$ it follows that $x_{1t} = a_{11}x_{1,t-1}$ which is obviously not stabilizable by any choice of control. However, if A_1, b_1 and b_2 in the example are replaced by $A(L)$ $b_1(L)$ and $b_2(L)$ where all lag operators are finite degree polynomials, then for MIMO systems one must obtain the transfer function and obtain the state space realization in order to evaluate the controllability matrix for it (Zhou, Doyle, and Glover (1996, Section 3.7)) for $A(L)$, $b_1(L)$ and $b_2(L)$; a similar calculation is needed for SIMO systems under the restriction $b_1(L) = 0$. Since the lag polynomials can be of any finite degree the state space realization can have very high dimension. Thus the four numerical restrictions in (56)-(59) above can *not* characterize the difference between MIMO and SIMO. But the result $K_{B,MIMO} - K_{B,SIMO} \geq 0$ still holds for planners who do the best they can to choose controls to stabilize the system they face.

4. Application: monetary policy rules

In this section we explore the limits encountered by a policymaker trying to design the response of output and inflation at different frequencies conditional on the now standard two-equation new Keynesian class of inflation/output models. The monetary policy rule literature contains both backwards-looking and hybrid models of the type we have analyzed and so is a natural environment for considering design limits. The system consists first of a Phillips curve equation

$$\pi_t = \mu E_t \pi_{t+1} + (1-\mu)\sum_{i=1}^{4} \alpha_i \pi_{t-i} + \gamma y_t + \varepsilon_t. \quad (62)$$

The error term is assumed to be AR(1), $\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + v_{1t}$. The second equation is a IS curve,

$$y_t = \delta_f E_t y_{t+1} + \left(1-\delta_f\right)\sum_{i=1}^{4}\delta_i y_{t-i} - \sigma r_t + \eta_t . \tag{63}$$

The error term is also assumed to be AR(1), $\eta_t = \rho_\eta \eta_{t-1} + v_{2t}$.

We focus on two forms of this model. The first specification we consider is the backwards-looking model elaborated by Rudebusch and Svensson (1999) which sets $\mu = 0$, imposes $\sum_{i=1}^{4}\alpha_i = 1$ to ensure a long run vertical Phillips curve, and measures the real interest rate as $r_t^B = .25\sum_{i=1}^{4}\left(i_{t-i} - \pi_{t-i}\right)$. We employ their parameter estimates. The second specification, comprehensively studied in Woodford (2003), assumes $\mu > 0$, $\alpha_i = 0\ \forall i$ (which essentially rules out any exogenous persistence to the inflation rate), and $r_t^H = i_t - E_t\pi_{t+1}$. For this model specification, we take parameter estimates for the Phillips curve from Gali, Gertler and Lopez-Salido (2005. Table 1) and parameter estimates of the IS equation from Linde (2005, Table 5). Table 1 reports the parameter values for the two cases.

We consider policies that are simple variants of linear feedback rules of the form.

$$i_t = g_\pi\left(L\right)\pi_{t-1} + g_y\left(L\right)y_{t-1} + g_i\left(L\right)i_{t-1} . \tag{64}$$

All polynomials are of course one-sided, following the restrictions on (5). Rules in this class have of course been extensively studied. We focus on simple variants given their importance in current monetary policy debates.

i. inflation-output volatility tradeoffs across frequencies

The original Phillips hypothesis of a long-run negative tradeoff between the level of inflation and the level of output has been fundamentally modified by theoretical and empirical advances since Phillips' time. Contemporary research focuses on the existence of a tradeoff between variance of inflation and the variance of output deviations from its natural level. As policies are computed to minimize different linear combinations of variance for the output gap and inflation, a negatively sloped frontier emerges. From the perspective of design limits, it is natural to ask how one can understand variance tradeoffs as they are manifested at different frequencies.

In order to understand frequency-specific tradeoffs, we perform several exercises. First, we compute the frequency-specific losses that are implicit in the tradeoffs associated with the variance-based Phillips curve. To do this, we compute variance tradeoff frontiers for inflation and output. For each point on the frontier, parameters are chosen for the interest rate rule

$$i_t = g_\pi \pi_{t-1} + g_y y_{t-1} + g_i i_{t-1} \tag{65}$$

so that feedbacks are restricted to $t-1$ levels of output, inflation, and the interest rate. Points on the frontier are chosen to minimize

$$J = \lambda \operatorname{var}(\pi_t) + (1-\lambda)\operatorname{var}(y_t) + \phi \operatorname{var}(\Delta i_t). \tag{66}$$

By varying λ between 0 and 1, one traces out the efficient frontier of inflation/output variance pairs from which a policymaker may choose. The position of the frontier depends of course on the value of ϕ. For expositional purposes we report the frontiers for the case of free control $\phi = 0$ and for the case of costly control $\phi = 0.1$.

For each point on the frontier we report an associated decomposition of the variance values into components corresponding to the same division between low frequencies (cycles of 8 years or more), business cycle frequencies (cycles of 2 to 8 years), and high frequencies (cycles of less than 2 years). This division follows the NBER classifications of minor and major business cycles. The frequency-specific tradeoffs in these

Figures indicate how the unconditional variance frontier contains additional frontiers where optimality no longer applies. The shape of the frontier is obviously related to the structural model acting as a constraint on the optimization problem of the policy maker. The existence of design limits shapes the frontiers at different frequencies.

Results of this exercise are reported for the backwards model in Figure 1.A for costless control and Figure 1.B for costly control. The Figures are qualitatively very similar and each indicates how the tradeoffs associated with overall variance mask very different behaviors across frequencies. The general shape of the overall variance tradeoff found for the backwards model is replicated for the variance at the low frequency bands, but not for the others.

The frequency interval tradeoffs indicate some unpleasant implied tradeoffs at the business cycle frequencies and high frequencies. Suppose that the policy rule is initially optimally set by a policymaker D (for dove) who possesses a relative distaste for output variance over inflation variance, so that $\lambda = 0.05$. Suppose that a new policymaker H (hawk) replaces the first policymaker and that H possesses a relative distaste for inflation variance over output variance, so that $\lambda = 0.95$. As one would expect, the transition from D to H moves along the frontier as indicated in the upper left panel of Figure 1.A or 1.B as lower inflation is substituted for higher output variance. This overall tradeoff masks interesting frequency-specific effects. For low frequencies, the qualitative finding of an inflation/output variance tradeoff is preserved, although a substantially larger reduction in inflation variance may be obtained from a given increase in output variance when the low frequencies are considered in isolation. Tradeoffs are very different for the business cycle frequencies, as shown in the lower left panels of Figures 1.A and 1.B. Both the variance of inflation and output *increase* as the policy shifts from D to H. While it is relatively cheap to reduce inflation variance at low frequencies (measured in terms of low frequency output variance), a price is paid at the business cycle frequencies, where the variance of inflation is increased. At high frequencies, on the other hand, both inflation and output variances decline when the policy shifts from D to H, although the magnitude is very small compared to the rest of the spectrum.

Figures 2.A and 2.B report the same exercise when a policymaker faces a hybrid model. The qualitative messages of the Figures are similar, as occurred with Figures 1.A.

and 1.B, although the shape of the high frequency tradeoffs are quite different in magnitude. With respect to overall variance, the qualitative difference between the backwards and hybrid models is that the marginal rate of substitution between output and inflation variance is considerably smaller than the backwards-looking case. In other words, moving along the variance frontier entails a smaller cost under the hybrid model. The upper right and lower left panels of Figures 2.A and 2.B show that this difference in costs is a consequence of differences in the tradeoffs associated with the business cycle frequencies. For this case, as the variance of inflation is reduced at low frequencies, a similar reduction happens at business cycle frequencies. The cost of reducing the variance for inflation is higher at high frequencies but the relative importance of those frequencies in terms of overall variance remains small.

ii. original Taylor rule redux

Our second exercise considers the frequency-specific effects of the original Taylor (1993) rule (OTR): $g_\pi = 1.5, g_y = 0.5, g_i = 0.0$ and draws comparisons with a class of modifications that has been proposed. For a policymaker with the loss function (66) and associated parameters $\lambda = \frac{1}{2}$ and $\phi = 0.1$, the original Taylor rule produces losses of 9.20 and 6.47 for the backwards and hybrid models respectively. In terms of loss components, for the backwards model the volatility of inflation and output under OTR are 12.2 and 5.7 respectively, while for the hybrid they are 3.1 and 9.4. These are the sorts of calculations that are conventionally reported in the monetary policy rules literature. In unpacking variance calculations of this type to understand frequency-specific losses, we first consider the spectral density components of inflation and output associated with the innovation to inflation v_{1t} and the innovation to output v_{2t}, i.e. $f_{\pi,v_1}(\omega)$, $f_{\pi,v_2}(\omega)$, $f_{y,v_1}(\omega)$ and $f_{y,v_2}(\omega)$. The values of these functions under the OTR are reported in the left hand side panels of Figure 3 for the backwards model and Figure 4 for the hybrid model. For the backwards model, the volatility consequences of inflation innovations on inflation are concentrated at low frequencies, i.e. those associated with cycles of 8 years or longer. The volatility consequences on output of inflation innovations follow the same pattern. On the other hand, the frequency-specific effects on output due to output shocks is associated with a peak around business cycles of 8-16 years whereas the frequency specific effects of output shocks on inflation are concentrated at very low frequencies. For the hybrid model, much of the variance of both inflation and output is concentrated at low frequencies with no peaks prior to the zero frequency. That said, a substantial portion of the variance is also concentrated in the business cycle frequencies of 2-8 years, typically considered to be the primary business cycles; this is especially noticeable with respect to the spectral density effect of output shocks on output.

How should a policymaker proceed who wishes to improve performance relative to the OTR? We present two alternative rules to highlight the relevance of frequency-specific tradeoffs in the design of good policies. The first alternative we consider to the OTR is the optimal policy rule conditional on the model, OPR, defined as the choice of parameters in (65) that minimizes (66) with $\lambda = \frac{1}{2}$ and $\phi = 0.1$. Second, we contrast the OTR (and the OPR) with a "modified Taylor rule" (MTR) in which 1) the reaction coefficients to inflation and output are increased by 1 unit, so that $g_{\pi} = 2.5$ and $g_{y} = 1.5$ and 2) a non-trivial degree of persistence is added to the OTR control rule by specifying $g_{i} = 0.5$. These modifications capture some intuitions that have appeared in the monetary rules literature. First, our reading of the monetary policy literature, specifically McCallum and Nelson (2004) is that an overall variance performance improvement should occur if the policymaker is slightly more aggressive in response to changes in either inflation or output than occurs in the OTR. Second, persistence in the interest rate rule via a lagged term has been shown to be valuable in reducing the overall variance of macroeconomic aggregates because of the effect on expectations, see for example Woodford (1999) and Giannoni and Woodford (2003). These considerations lead us to propose the MTR as an example of an alternative "rule of thumb" to the OTR.

Figure 3 reports the spectral densities and the design transformation matrix components of the backwards-looking model under the three rules, OTR, OPR and MTR. The optimal rule OPR in this case is $g_{\pi} = 1.9, g_{y} = 1.2, g_{i} = 0.3$. The overall loss under

OPR is reduced to 6.06 (as compared to 9.20 for OTR), while under MTR the overall loss is reduced to 8.37. In terms of variable-specific volatility, the variances of inflation and output under OPR are 5.1 and 5.7 respectively, while under MTR they are 3.5 and 9.7, compared to 12.2 and 5.6 for OTR. OPR reduces inflation variance while keeping output variance essentially unchanged while MTR strongly reduces inflation variance while increasing output variance. However, as Theorem 2 and 5 inform us, reductions in variance cannot happen across all frequency ranges, so that these overall performance improvements are masking a nontrivial set of gains and losses. Both the OPR and the MTR reduce the contribution to the inflation variance at low frequencies (8 years and longer) from shocks to inflation and output[8]. This can be seen from the top right panels of Figure 3 where both $M_{11}(\omega)$ and $M_{12}(\omega)$ for each alterative rule to OTR are below 1 for cycles of 8 years or more. However, both rules increase the variance of inflation at cycles between 2 and 4 years, this is especially so under the MTR. This is not the only tradeoff entailed when the MTR and the OPR reduce the overall variance. Both rules increase the variance of output from shocks to inflation at all frequencies, as the panels in the third row of Figure 3 show. Finally, both the OTR and the MTR reduce the variance of output from output shocks at frequencies of 8-16 years, but, as a consequence, they increase the variance at cycles between 2 and 4 years. Summarizing, even though both rules improve the performance of the policymaker with respect to the Taylor rule, such an improvement is paid for by increases in the variance of fluctuations at business cycle frequencies.

Figure 4 reports the performance of the OPR and the MTR under the hybrid model. The optimal rule the OPR for this model is $g_\pi = 0.1, g_y = 1.9, g_i = 0.4$. The overall loss under the OPR is reduced to 2.44, while under the MTR is reduced to 5.26. In terms of specific variables, the variance of inflation and output under the OPR are 3.5 and 0.8 respectively, while under the MTR they are 3.1 and 5.9, compared to 3.1 and 9.4 for the OTR. On the one hand, the OPR increases inflation variance while suppressing most of the output volatility. On the other hand, the MTR reduces both inflation variance

[8]Rondina (2008) shows generally that the first-order conditions for variance minimization in hybrid models instruct the policy maker to completely annihilate the contribution to the variance at low frequencies in order to correctly “manage” the expectations of forward-looking agents. This is consistent with the effects we find for the OPR and MTR alternatives to the original Taylor rule.

and output variance, although the latter effect is far less dramatic than occurs under the OPR. It is evident from the Figure that both rules perform virtually identically with respect to fluctuations in inflation and output that are due to shocks to output. This can be seen from rows 2 and 4 in Figure 4. In terms of the fluctuations in inflation and output coming from shocks to inflation, the two rules differ markedly. Under the MTR the component of the variance of inflation due to shocks to inflation is relatively unchanged when compared to the OTR baseline; there is a slight increase at low frequencies and a slight reduction of the same magnitude at business cycle frequencies whereas the OPR generates a large increase in variance at cycles of 2 years or greater. In contrast, the MTR increases the contribution of shocks to inflation to the variance of output for cycles between 1 and 4 years. In this respect, the OPR does exceptionally well as it systematically compresses the component of the spectral density of output generated by shocks to inflation. The cost of this outstanding performance is the increase at business cycle and lower frequencies of the effect of inflation shocks on inflation.

Figures 3 and 4 offer a clear illustration of the powerful tradeoffs that operate in the frequency domain when a control rule is applied to a dynamic economic system. Interestingly, one can find cases where power is pushed towards the business cycle frequencies. This is most evident for both the OPR and the MTR for the backwards-looking model. One also sees this in the effect of the OPR on inflation shocks for the hybrid model; although, consistent with our theoretical results, the tradeoffs are generally less stark for the hybrid case. Whether or not such peaks are an acceptable price to pay for variance reduction obviously depends on the objective function of the policymaker, but the knowledge of the existence of such severe tradeoffs may be of value in the design of good policies and in understanding the implications of deviating from the Taylor rule towards more complicated monetary policy rules.

iii. monetary policy regimes and design limits

Our final illustration of the value of design limits analysis concerns the interpretation of changes in monetary policy. The last 40 years of monetary policy can to some extent be understood as consisting of three periods: the pre-1979 or Burns period, the 1979-1987 or Volcker period and the post-1987 or Greenspan period[9]. In this exercise, we compare the performances of the three regimes at different frequency bands to expose the "hidden tradeoffs" forced by the conservation laws developed in Section 2.

In order to operationalize the comparison of the regimes, we employ estimates due to Judd and Rudebusch (1998) that describe these different monetary policy regimes in terms of changes in the parameters of interest rate rules. Judd and Rudebusch (1998) consider two specifications of the monetary policy rule for each regime. One is a generalized Taylor rule

$$i_t^* = g_\pi \pi_{t-1} + g_{y1} y_{t-1} + g_{y2} y_{t-2} \tag{67}$$

which does not contain any persistence of the policy instrument. They interpret this as a "recommended rule" for interest rates. They consider both the case where the Federal Reserve can implement its recommended rule as well as a second "measured" rule of the form

$$i_t = \tilde{g}_\pi \pi_{t-1} + \tilde{g}_{y1} y_{t-1} + \tilde{g}_{y2} y_{t-2} + g_{i1} i_{t-1} + g_{i2} i_{t-2}. \tag{68}$$

The use of 2 lags in interest rates, following Judd and Rudebusch, is done to allow for the possibility that the observed interest rate does not coincide with the policymaker's preferred interest rate, but rather adjusts towards this preferred interest rate via an error correction model. (Of course, as previously discussed, interest rate inertia may have desirable stabilization properties.) The values of the coefficients for (67) and (68) for the three regimes are reported in Table 2. We evaluate the regimes using the backwards and hybrid models parameterized as in Section 4.i. We follow Judd and Rudebusch (1998, p.

[9]We follow Judd and Rudebusch (1998) and Sims and Zha (2006) in working with distinct Volcker and Greenspan regimes rather than Clarida, Gali and Gertler (2000) or Taylor (1999) who combine them into a common one. There is no consensus on the number of monetary policy regimes for the post-war US. Sargent, Williams, and Zha (2006) provide evidence that changes in government beliefs about the nature of the Phillips curve explain changes in monetary policy; their evidence on time series of these beliefs suggests that it is sensible to distinguish between the Volcker and Greenspan years

4) and omit any discussion of the G William Miller's time as FRB chairman (1978.Q2-1979.Q2) because of his short tenure.

Tables 3 and 4 report the behaviors of output, inflation and changes in interest rates under the three regimes. Note that the variances under the Burns reaction function are infinite as the model evaluated at the Burns reaction function is nonstationary. This finding is consistent with Judd and Rudebusch (1998, Table 2, page 12) where they observe that the model did not converge for their estimated Burns reaction function. Convergence does occur for the hybrid case in Table 4. Contrasts are also drawn with the original Taylor rule.

As indicated by Table 3, for the backwards model both Greenspan and Volcker perform better than OTR. However, most of the difference in Volcker's performance is due to lower inflation volatility – 9.6 against 12.2 – while output volatility is essentially the same as OTR – 5.4 against 5.6. On the other hand, Greenspan's better performance is split between lower inflation volatility and lower output volatility, 11.3 against 12.2 and 4.6 against 5.6, respectively. The recommended interest rate rule for each regime produces larger volatility in the backwards model than the measured version of each, especially for inflation; this indicates that the stabilization possibilities generated by interest rate inertia were not being exploited. Turning to the hybrid model, for both the preferred and recommended cases, one finds that the Volcker regime performs slightly better in terms of inflation volatility but much worse in terms of output volatility than the Burns and Greenspan regimes. The OTR performs very similarly to Volcker's rules. We note that that for the hybrid model the distinction between the preferred and measured rules is second-order, in particular in terms of inflation variance implications.

How do the different monetary regimes compare when frequency-specific effects are considered? To facilitate comparisons, we employ a modified design matrix defined as the ratio of the spectral density components for a variable/shock pair under a given rule (note that we omit Burns when spectral densities do not exist) to the corresponding spectral density components of the variable/shock pair under the original Taylor rule[10]. This allows for visual representations of the different effects of each rule relative to the OTR for

[10]While it was natural, in developing our Theorems to compare controlled and uncontrolled systems, for applications, there is no need to choose a no control system as a baseline. In particular we let the OTR play the role of the "uncontrolled" system here.

different types of shocks for each frequency. Figure 5 reports spectral densities and design transformation matrix terms for the backwards model; the corresponding objects for the hybrid model appear in Figure 6.

For the backwards model, Theorem 1 implies that the integral of the logarithm of the determinant of the spectral density matrix is a constant whose value is independent of the choice of control rule.[11] This suggests that Volcker and Greenspan must move undesired power somewhere in the frequency domain relative to the OTR baseline. This kind of conservation law should appear in the spectral density plots. The left column plots in Figure 5.A reveal that (assuming the economy's dynamics followed the estimated backwards model) Volcker, under the recommended rule specification, reduced spectral power for inflation at the lower frequencies relative to Greenspan and to the Taylor baseline at the cost of more variance at the higher ones. Interestingly the Volcker rule does the opposite for output in the sense that it increases spectral power at lower frequencies relative to Greenspan and the Taylor standard. Figure 5.A also suggests that the spectral density effects of the recommended and measured rules on output and inflation are relatively similar, although the effects on interest behavior are quite different.

Figures 5.B and 5.C, illustrate how these overall effects are associated with distinct spectral density effects of inflation and output shocks. Starting with Figure 5.B, compared with Greenspan, the Volcker regime reduced the effects of inflation shocks on inflation variance at the very lowest frequencies; in contrast the Volcker regime performs slightly less well than others in attenuating the variance effects of output shocks on inflation for cycles of 8 year or more. Differences between the regimes with respect to the effects of output shocks on inflation are harder to summarize, as indicated by the multiple intersections of the modified design matrix elements. However, one can say that, for cycles of 8 years or greater, variance is substantially higher under Volcker than for the OTR whereas Greenspan outperforms the OTR. Similar conclusions hold for effects of output shocks on output. As indicated by a comparison of Figures 5.B and 5.C, there do not appear to be interesting qualitative differences in comparative regime performance with respect to

[11]This claim follows from Theorem 1 if one integrates the logarithm of the determinant of $f_{x|C}(\omega)$ in (17).

inflation and output as one moves from recommended to measured rules, which mirrors the results for overall spectral densities.

Figures 6.A-6.C illustrate the behavior of the regimes under the hybrid model. In interpreting these figures, recall that effects of conservation laws are not so sharp for the hybrid model because, as shown by Theorem 3, the sensitivity function constraint depends upon the rule. But since the constraint is one-dimensional and rules are multidimensional, Theorem 3 does suggest a tendency for multidimensional rules to be somewhat constrained in their freedom to diminish spectral power at a frequency band without a tendency to cause spectral power to rise at some other frequency band.

As indicated in Figure 6.A, under both the recommended and measured interest rate specifications, there is a clear ranking of the rules for spectral power of inflation. It appears that Volcker is successful at reducing spectral power for inflation across all frequencies, when compared to the other rules. The relatively superior performance of the Volcker rule is especially dramatic for cycles of 8 years or greater, which contrasts with the backwards case where the large performance improvements occur at cycles of 32 years or longer. An examination of the spectral density plots for output suggests that the conservation law imposed costs along this dimension. In fact, the spectral density of output under the Volcker regime exceeds others by a magnitude of 2 to 4 times, for cycles of 8 years or longer. It is interesting to note that the Volcker recommended rule generates substantial high frequency variance in interest rate changes compared to the other rules, but performs relatively well when a measured version is considered.

Figures 6.B and 6.C provide additional insights into the performance of the different regimes for the hybrid model. As indicated by the Figures, the Volcker regime (for both the recommended and measured specifications) begins to outperform the other regimes with respect to the effects of inflation shocks on inflation once the cycle length equals or exceeds four years. For cycles slightly shorter than 4 years, the Volcker regime is slightly outperformed by the OTR; this is most evident when one considers the design matrix. When one considers the effects of output shocks on inflation, the recommended form of the Volcker rule, remarkably, is outperformed at all frequencies by Greenspan (with the exception of very high frequencies). Similar tradeoffs are evident when one considers output. Compared to the others, Volcker (under either rule specification) amplifies the low frequency effects of both inflation and output shocks on output. Surprisingly, in terms of the effects of inflation shocks on output, Burns outperforms Volcker at all frequencies. This is a dramatic example of the tradeoffs with which we have been concerned. For higher frequencies, the various design functions intersect so there are no general comparisons to be drawn.

How do the various monetary policy regimes perform relative to the inflation/output variance frontiers we have described in Section 4.i? Figures 1 and 2 include the locations of outcomes under the Burns (when Burns converges), Volcker and Greenspan rules relative to the inflation/output frontiers. In terms of overall variance there are no surprises except possibly the domination of Burns by Greenspan in the hybrid model. For the hybrid model the performance of all three regimes is about the same for the implied frontier at high frequencies. But the "conservation law of the logarithm of spectral power" suggests that the volatility must end up somewhere at the business cycle frequencies and the lower frequencies. For the hybrid model the important difference shows up at the low frequencies. Burns squashes output volatility in return for a high price in terms of inflation volatility at low frequencies while Volcker does almost the exact opposite; from this perspective Greenspan may be regarded as a compromiser between the two. Note that, at business cycle frequencies for the hybrid model, the three chairmen are much closer together. These important contrasts and similarities are completely masked by the standard frontier.

Let us sum up the conclusions we draw. The conservation law/design limits perspective developed in this paper motivates a detailed analysis of relative performance of different rule regimes at low, business cycle, and high frequency bands. This is so because our Theorems (even for hybrid models) show a tendency for some measure of volatility to be conserved across different frequencies. This suggests several additions to the standard way in which analysts report the effects of alternative monetary policy rules. In particular the standard efficiency frontier analysis of regimes should be supplemented by implied frontiers evaluated at the low, business cycle, and high frequencies. Further, plots should be prepared that spotlight frequency bands where a rule is robust to shocks at those frequencies and to spotlight frequency bands where the rule is fragile to shocks at such frequencies. These additional calculations are valuable because we know that increased robustness at one set of frequencies must be offset by increased fragility at some other set of frequencies for backwards models. There is still a tendency for this to be true for hybrids although the effect is more subtle.

5. Summary and conclusions

This paper has argued the case for introducing macroeconomics to the theory of design limits in control theory. The general theory of design limits (e.g. Skogestad and Postlewaite (1996)) stresses limitations on the ability of control design to move variance across frequencies (expressed in the form of various "conservation laws") as well as limitations on the ability of control design to cope with measurement error and robustification against various forms of model uncertainty. We have only touched on one feature of the general theory of design limits in this paper in that we have focused all attention on the basic conservation laws which give precise content to the intuitive idea that attempts to reduce variance down at one frequency band can cause variance to increase at some other frequency band.

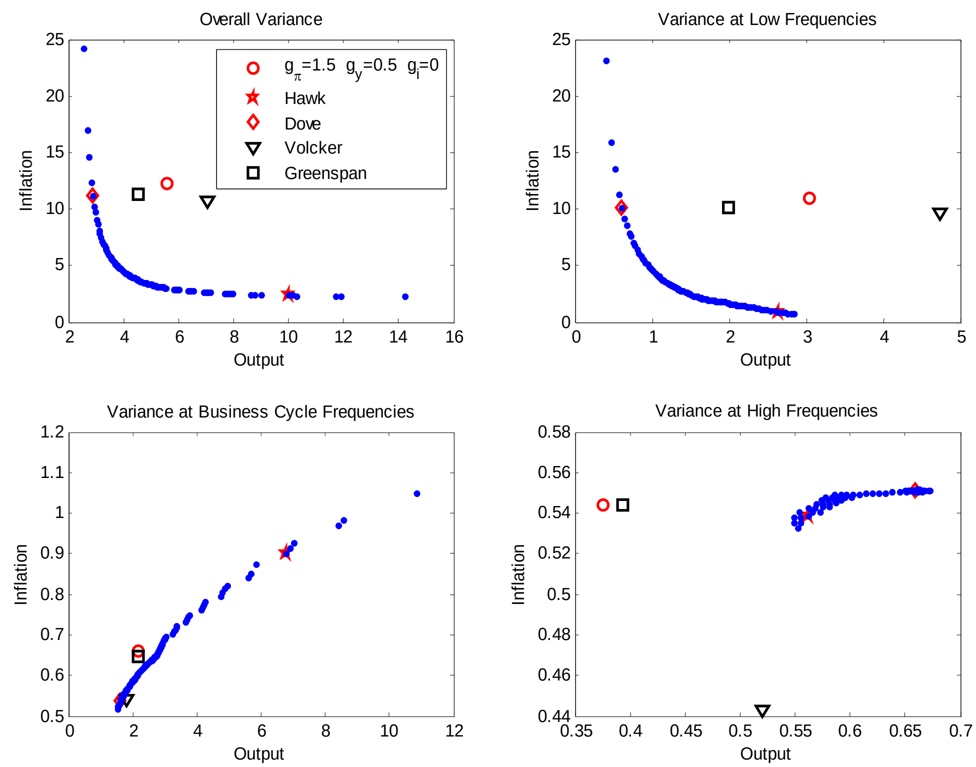
Many outstanding questions exist. For example, we have said nothing about good designs to cope with measurement error. While the sensitivity function, $S(z)$, is the function from which one can design good policies to cope with outside shocks to the dynamics, the complementary sensitivity function, $T(z)$, is relevant in coping with separate issues that arise in the presence of measurement error. See Skogestad and Postlethwaite (1996, Section 2.2.2 and Section 6.2) for the definition of $T(z)$ as well as the design limits constraint, $S(z)+T(z)=I$ and its use in uncovering design limits constraints in the presence of measurement error. The constraint $S(z)+T(z)=I$, plays a key role in showing that measurement error results in another type of conservation law that constrains placement of volatility across different frequency bands. We are developing this line of research in a sequel to the current paper. Further, there is a close connection between the robust control literature (e.g. Hansen and Sargent (2007)) and the theory of design limits.

Design limits theory focuses on control design to robustify against (i.e. moderate) outside shocks. Robust control theory focuses on control design to robustify against a lack of confidence in analyst's ability to specify the dynamics of the system under study. Design limits theory should be useful to robust control theorists because it uncovers frequency bands where model uncertainty can do the most damage to the designer's goal. Thus, using this information, the designer can design a control to mitigate damage at the most vulnerable frequency bands; Brock and Durlauf (2004) provide an example. Similar considerations exist if one wants to consider model uncertainty for spaces comprised of distinct models as done in Brock, Durlauf, and West (2003, 2007) or Levin and Williams (2003). Yet another important set of questions concern the generalization of design limits theory to nonlinear systems; Pataracchia (2008) provides an analysis of this type for switching regime models. For these reasons we believe that design limits theory is an unusually rich area for future research.

Table 1. Model Parameter Values

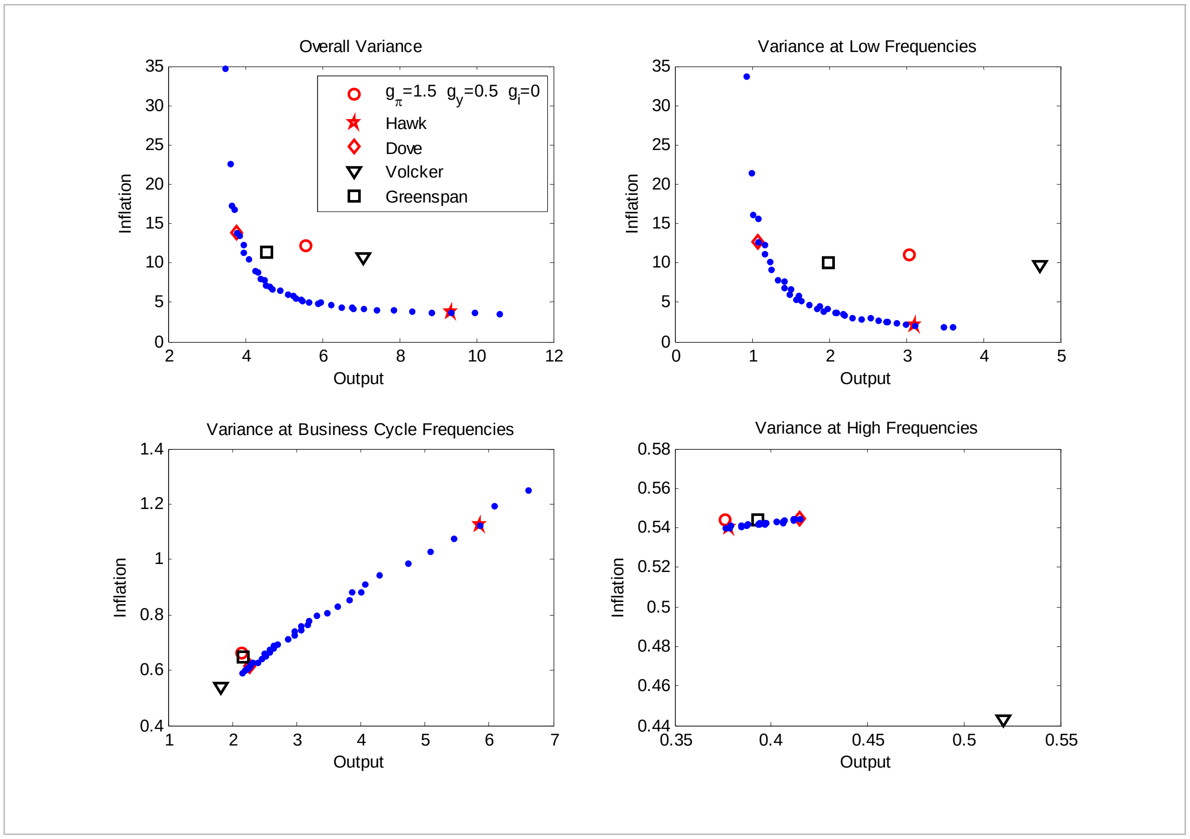
	Phillips Curve			Output Equation	
	Hybrid	*Backwards*		*Hybrid*	*Backwards*
μ	0.635	0	δ_f	0.430	0
γ	0.013	0.14	δ_1	1.275	1.16
α_1	0	0.70	δ_2	–0.253	–0.25
α_2	0	–0.10	δ_3	0.012	0
α_3	0	0.28	δ_4	0.012	0
α_4	0	0.12	σ	0.087	0.10
ρ_ε	0.75	0	ρ_η	0.35	0
σ_v^2	0.7957	1.009	σ_v^2	0.4006	0.819

Figure 1.A. Tradeoff Frontiers: Backwards Model, Costless Control



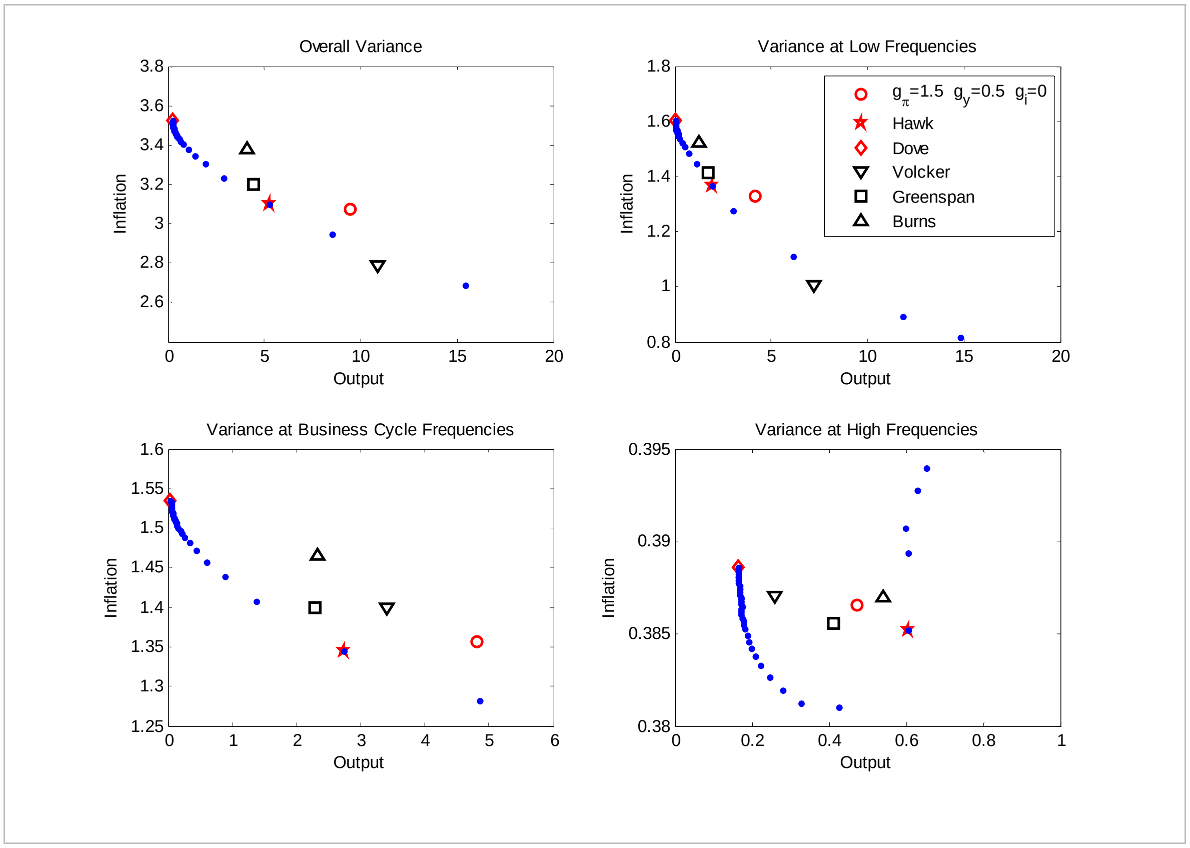
Note: The four panels report aspects of the inflation and output processes that correspond to the minimization of the loss function (66) as λ is varied between 0 and 1, $\phi = 0$ under the backwards-looking model. The upper left panel reports the frontier for the overall variance of inflation and output. The upper right panel reports the implied tradeoffs for the variance of inflation and output at frequency of 8 years or more for the different pairs in the variance frontier. The bottom panels report the implied tradeoffs for the variance of inflation and output at business cycle frequencies (2-8 years) and at higher frequencies (less than 2 years). Each panel also locates the variances of output and inflation for the relative frequency range that result under five policy rules: (i) the Original Taylor Rule (circle), (ii) the "Dove" Optimal Policy (diamond), (iii) the "Hawk" Optimal Policy (star), (iv) the Volcker regime, (v) the Greenspan Regime. The Burns regime results in a non-stationary system and therefore is not reported. The optimal policies correspond to rules of the form (66) with coefficients chosen to minimize (66) with $\lambda = 0.05$ (D) and $\lambda = 0.95$ (H). The coefficients are derived using a grid search over the space $g_\pi \in [0.0, 10.0], g_y \in [0.0, 10.0]$ and $g_i \in [-0.9, 0.9]$. The two policies are $g_\pi = 4,\ g_y = 8.2,\ g_i = -0.9$ (D) and $g_\pi = 10.0, g_y = 4.0, g_i = -0.3$ (H).

Figure 1B. Tradeoff Frontiers: Backwards Model, Costly Control



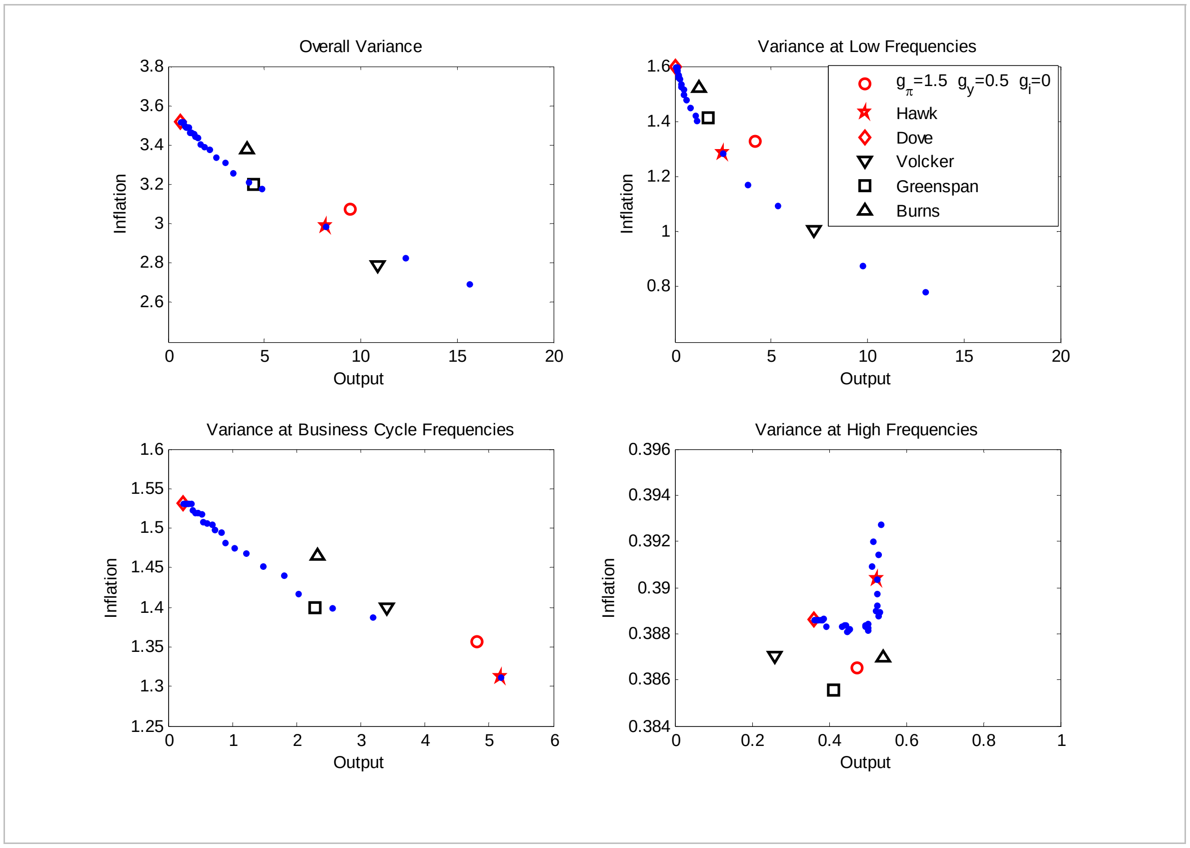
Note: The four panels report aspects of the inflation and output processes that correspond to the minimization of the loss function (66) as λ is varied between 0 and 1, $\phi = 0.1$ under backwards-looking model. The upper left panel reports the frontier for the overall variance of inflation and output. The upper right panel reports the implied tradeoffs for the variance of inflation and output at frequency of 8 years or more. The bottom panels report the implied for the variance of inflation and output at business cycle frequencies (2-8 years) and at higher frequencies (less than 2 years). Each panel also locates the variances of output and inflation for the relative frequency range that result under five policy rules: (i) the Original Taylor Rule (circle), (ii) the "Dove" Optimal Policy (diamond), (iii) the "Hawk" Optimal Policy (star), (iv) the Volcker regime, (v) the Greenspan Regime. The Burns regime results in a non-stationary system and therefore is not reported. The optimal policies correspond to rules of the form (66) with coefficients chosen to minimize (66) with $\lambda = 0.05$ (D) and $\lambda = 0.95$ (H). The coefficients are derived using a grid search over the space $g_\pi \in [0.0, 10.0]$, $g_y \in [0.0, 10.0]$ and $g_i \in [-0.9, 0.9]$. The optimal policies are $g_\pi = 1.4, g_y = 1.7, g_i = 0.1$ (D) and $g_\pi = 2.1, g_y = 0.9, g_i = 0.5$ (H).

Figure 2.A. Tradeoff Frontiers: Hybrid Model, Costless Control



Note: The four panels report aspects of the inflation and output processes that correspond to the minimization of the loss function (66) as λ is varied between 0 and 1, $\phi = 0$ under the hybrids model. The upper left panel reports the frontier for the overall variance of inflation and output. The upper right panel reports the implied tradeoffs for the variance of inflation and output at frequency of 8 years or more. The bottom panels report the implied tradeoffs for the variance of inflation and output at business cycle frequencies (2-8 years) and at higher frequencies (less than 2 years). Each panel also locates the variances of output and inflation for the relative frequency range that result under five policy rules: (i) the Original Taylor Rule (circle), (ii) the "Dove" Optimal Policy (diamond), (iii) the "Hawk" Optimal Policy (star), (iv) the Volcker regime, (v) the Greenspan Regime. The Burns regime results in a non-stationary system and therefore is not reported. The optimal policies correspond to rules of the form (65) with coefficients chosen to minimize (66) with $\lambda = 0.05$ (D) and $\lambda = 0.95$ (H). The coefficients are derived using a grid search over the space $g_\pi \in [0.0, 10.0], g_y \in [0.0, 10.0]$ and $g_i \in [-0.9, 0.9]$. The optimal policies are $g_\pi = 0.1, g_y = 10.0, g_i = 0.0$ (D) and $g_\pi = 1.6, g_y = 1.0, g_i = 0.5$ (H).

Figure 2.B. Tradeoff Frontiers: Hybrid Model, Costly Control



Note: The four panels aspects of the inflation and output processes that correspond to the minimization of the loss function (66) as λ is varied between 0 and 1, $\phi = 0.1$ under the hybrid model. The upper left panel reports the frontier for the overall variance of inflation and output. The upper right panel reports the implied tradeoffs for the variance of inflation and output at frequency of 8 years or more. The bottom panels report the implied tradeoffs for the variance of inflation and output at business cycle frequencies (2-8 years) and at higher frequencies (less than 2 years). Each panel also locates the variances of output and inflation for the relative frequency range that result under five policy rules: (i) the Original Taylor Rule (circle), (ii) the "Dove" Optimal Policy (diamond), (iii) the "Hawk" Optimal Policy (star), (iv) the Volcker regime, (v) the Greenspan Regime. The Burns regime results in a non-stationary system and therefore is not reported. The optimal policies correspond to rules of the form (65) with coefficients chosen to minimize (66) with $\lambda = 0.05$ (D) and $\lambda = 0.95$ (H). The coefficients are derived using a grid search over the space $g_\pi \in [0.0, 10.0], g_y \in [0.0, 10.0]$ and $g_i \in [-0.9, 0.9]$. The optimal policies are $g_\pi = 0.1, g_y = 2.4, g_i = 0.4$ (D) and $g_\pi = 0.5, g_y = 0.3, g_i = 0.8$ (H).

Figure 3. Spectral Densities and Design Transformation Matrix
Backwards Model

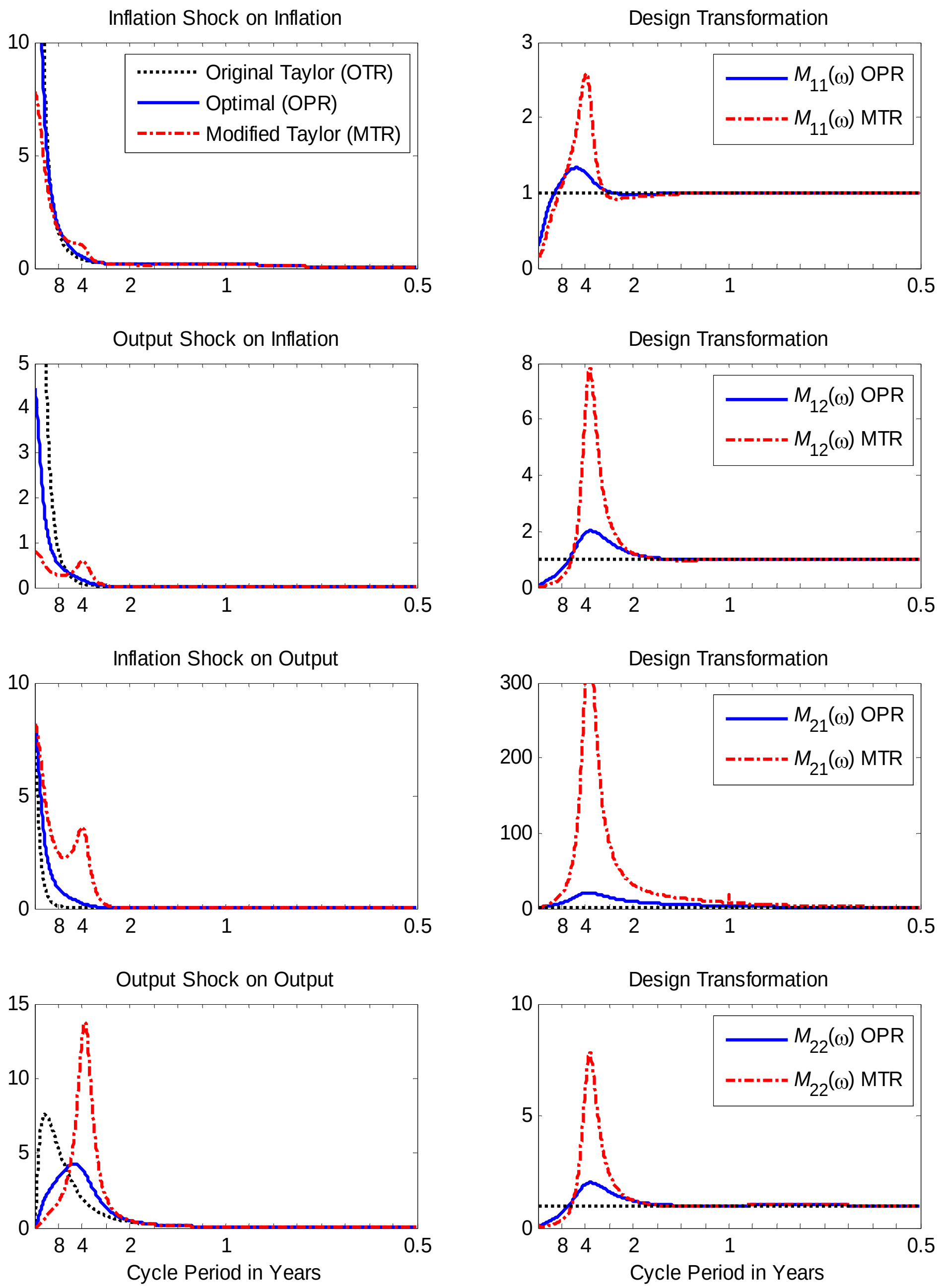


Figure 4. Spectral Densities and Design Transformation Matrix

Hybrid Model

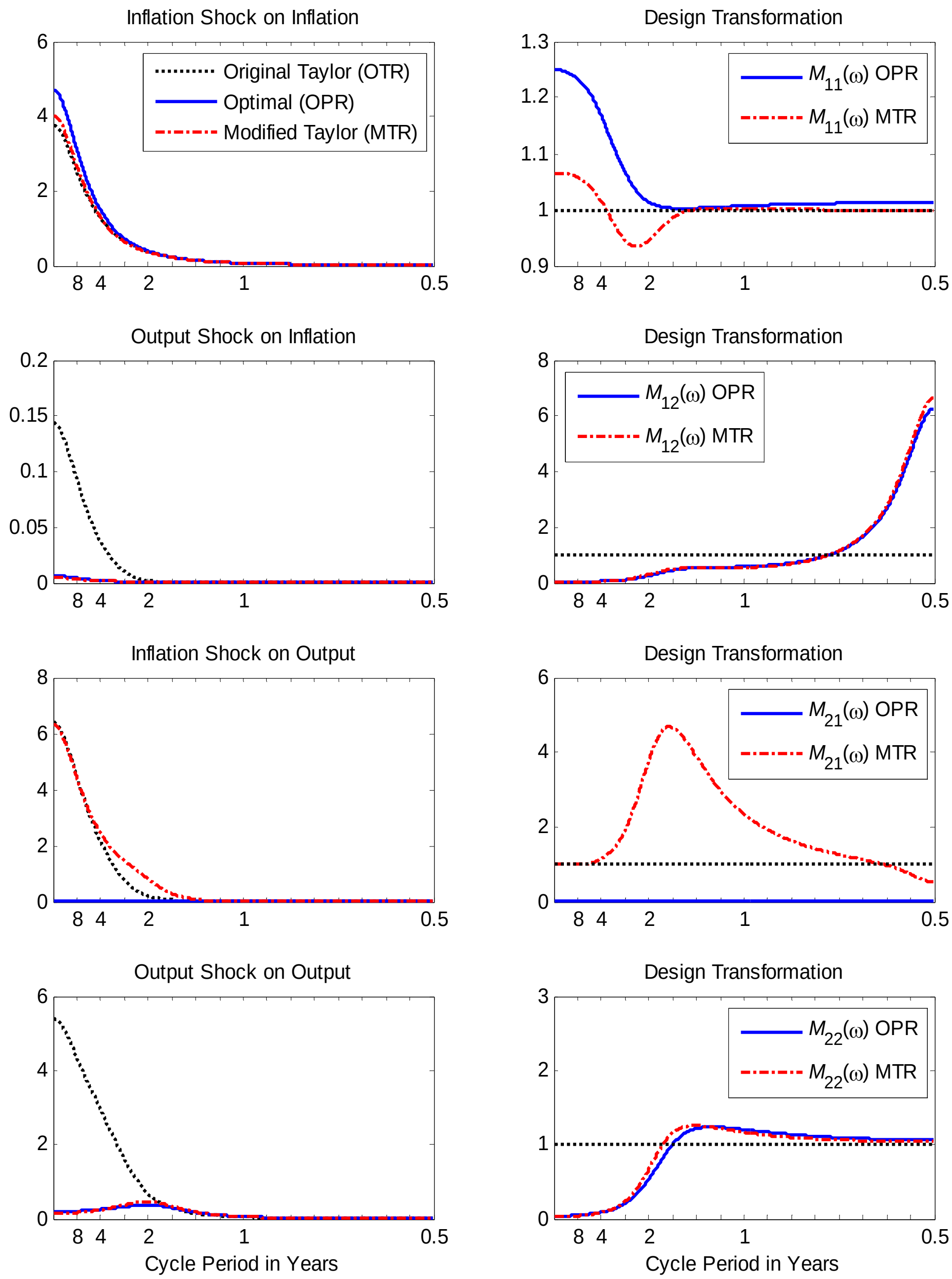


Table 2. Monetary Policy Regimes

	RR			MR				
	g_π	g_{1y}	g_{2y}	$\tilde{g}_\pi$	$\tilde{g}_{y1}$	$\tilde{g}_{y2}$	g_{i1}	g_{i2}
Burns	0.85	0.16	0.72	0.16	0.09	0.40	0.69	-0.25
Volcker	1.69	2.40	-2.04	2.40	0.86	-0.73	0.56	0.08
Greenspan	1.57	1.10	-0.12	1.10	0.30	-0.03	1.16	-0.43

Table 2 reports the measures of the Burns (1970Q1-1978Q1), Volcker (1979Q3-1987Q2) and Greenspan (1987Q3-1997Q4) regimes of Judd and Rudebush (1998). The left side panel reports the coefficients of the recommended rule **(RR)**

$$i_t^* = g_\pi \pi_{t-1} + g_{y1} y_{t-1} + g_{y2} y_{t-2} .$$

The right side panel reports the coefficients of the measured rule **(MR)**

$$i_t = \tilde{g}_\pi \pi_{t-1} + \tilde{g}_{y1} y_{t-1} + \tilde{g}_{y2} y_{t-2} + g_{i1} i_{t-1} + g_{i2} i_{t-2} .$$

Table 3. Regime Performance: Backwards Model

	$v(\pi_t)$	$v(y_t)$	$v(\Delta i_t)$
Taylor	12.2	5.6	3.1
	Recommended Rule (RR)		
Burns	∞	∞	∞
Volcker	9.6	5.4	10.5
Greenspan	11.3	4.6	4.2
	Measured Rule (MR)		
Burns	∞	∞	∞
Volcker	11.7	6. 8	1.8
Greenspan	12.1	5.5	0.9

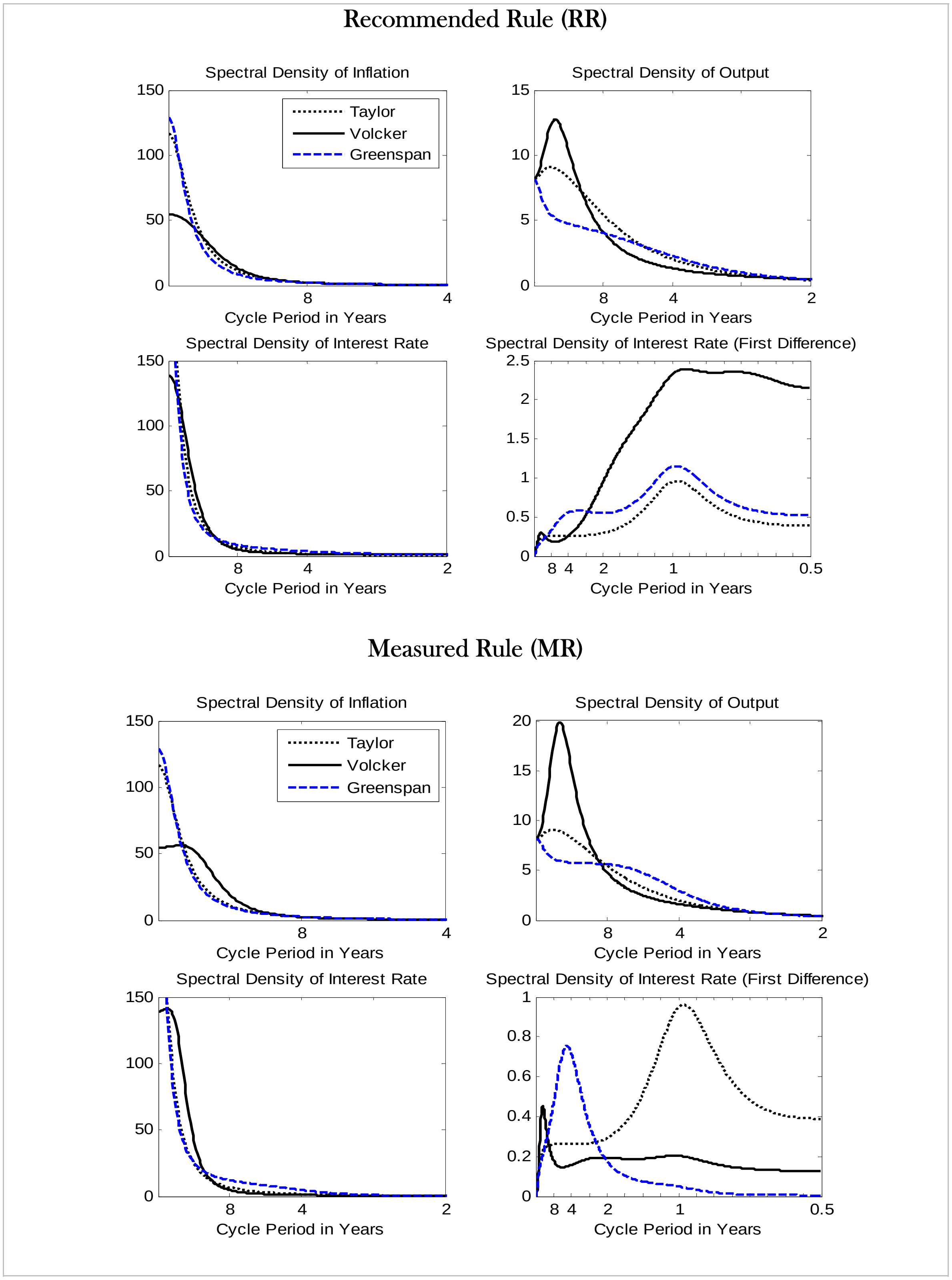
Note: Table 3 reports the unconditional variances for inflation, output and the interest rate computed using the backwards model and 4 alternative policy rules. The first row reports the results for the Original Taylor Rule. Rows 2-4 report the results for the 3 regimes - Burns, Volcker and Greenspan - under the specification RR for the policy rule (see Table 2). Rows 5-7 report the results for the same 3 regimes now in the form of specification MR. The value ∞ signals that the unconditional variance of at least one of the variables of the system is unbounded.

Table 4. Regime Performance: Hybrid Model

	$v(\pi_t)$	$v(y_t)$	$v(\Delta i_t)$
Taylor	3.1	9.4	2.2
	Recommended Rule (RR)		
Burns	3.4	4.1	1.4
Volcker	2.8	10.9	6.1
Greenspan	3.2	4.5	2.9
	Measured Rule (MR)		
Burns	3.4	4.7	1.2
Volcker	2.6	14.4	1.0
Greenspan	3.1	6.6	1.1

Note: Table 4 reports the unconditional variances for inflation, output and the interest rate computed using the hybrid model and 4 alternative policy rules. The first row reports the results for the Original Taylor Rule. Rows 2-4 report the results for the 3 regimes - Burns, Volcker and Greenspan - under the specification RR for the policy rule (see Table 2). Rows 5-7 report the results for the same 3 regimes now in the form of specification MR.

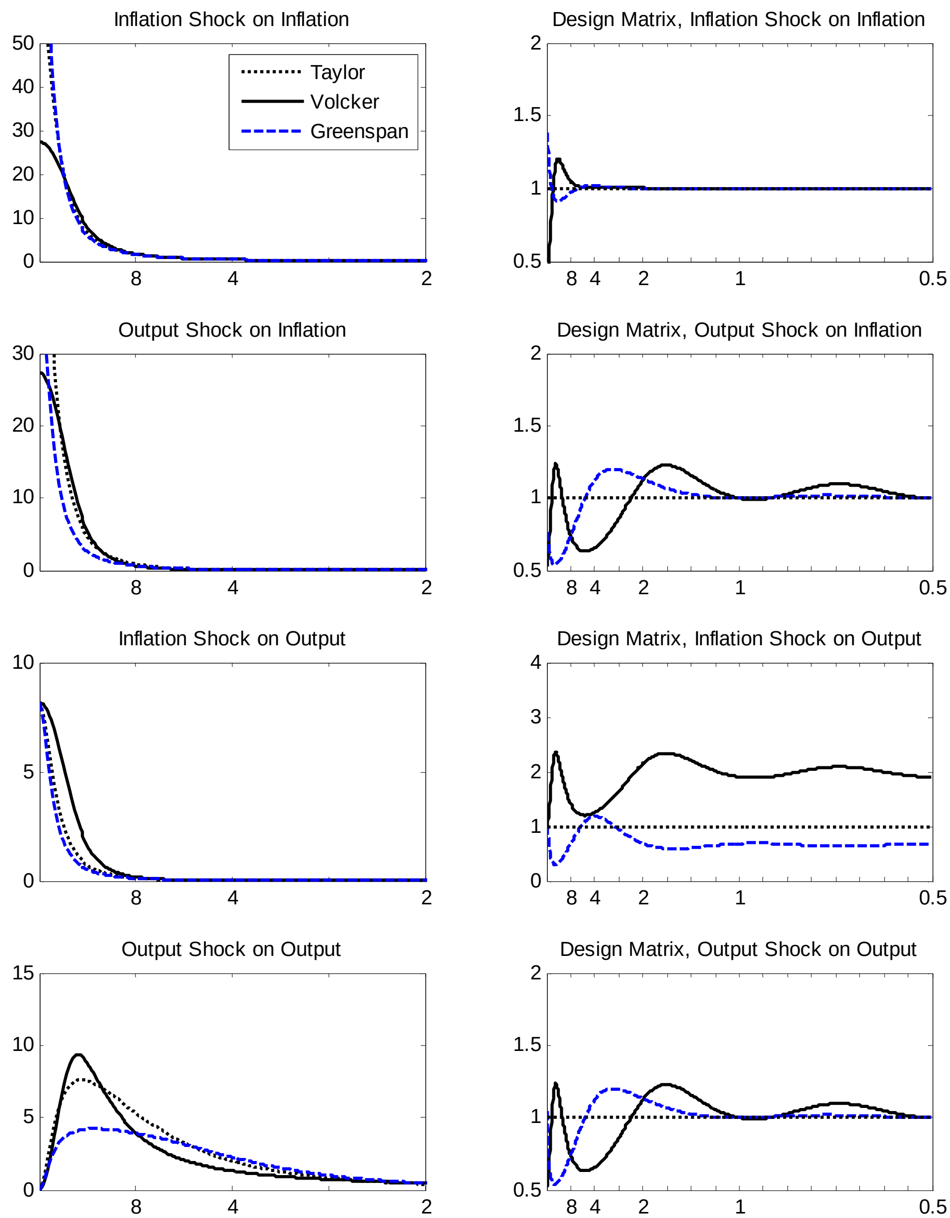
Figure 5.A. Spectral Densities for Inflation, Output, Interest Rates: Backwards-Looking Model



Note: Figure 5.A reports the spectral densities of inflation, output, and the interest rate in levels and in first differences under three different policy regimes: (i) Original Taylor Rule (dotted line); (ii) Volcker Regime (black line); (iii) Greenspan Regime (dashed line). The spectral densities under the Burns regime are not well defined. The backwards model is used to evaluate the performance of the three regimes. The upper four panels are generated using the Recommended Rule as a measure of the regimes, the lower four panels use the Measured Rule as a measure of the regime. The coefficients are those reported in Table 2. When necessary for a better appreciation of the spectral densities, the frequency ranges reported in the plots are restricted to cycles of longer duration.

Figure 5.B. Spectral Densities and Design Transformation Matrix for Inflation and Output: Backwards-Looking Model

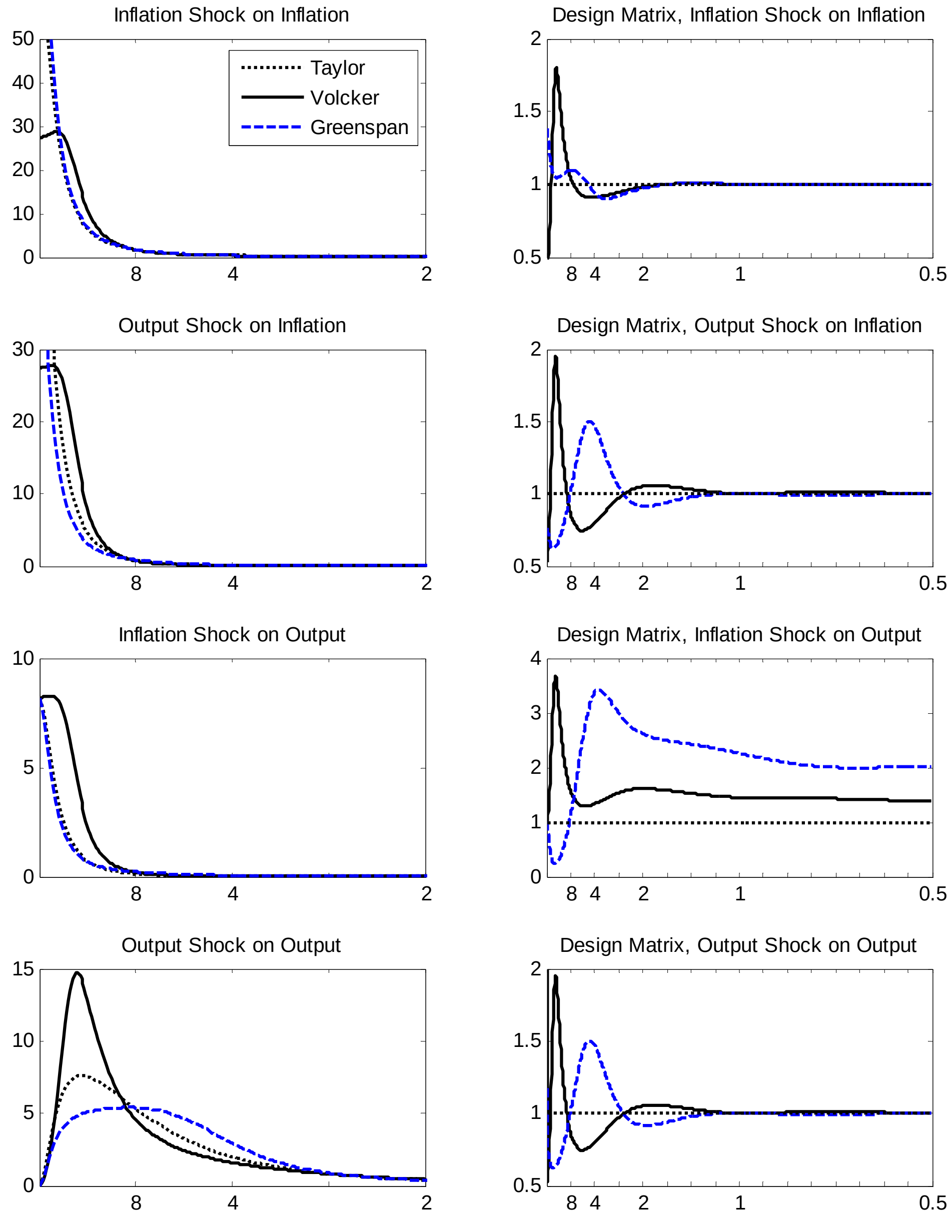
Recommended Rule



Note: The left side panels of Figure 5.B report the spectral densities of inflation ($f_{\pi,v_1}(\omega), f_{\pi,v_2}(\omega)$) and output ($f_{y,v_1}(\omega), f_{y,v_2}(\omega)$) decomposed according to the nature of the disturbance that generates them (shock to inflation (v_1) and shock to output (v_2)). The spectral densities are derived using the backwards-looking model under three different policy regimes: (i) Original Taylor Rule (dotted line); (ii) Volcker Regime (plain line); (iii) Greenspan Regime (dashed line). The spectral densities under the Burns regime are not well defined. The regimes are expressed in the form of the Recommended Rule, the coefficients are those reported in Table 2. For readability, the frequency ranges reported in the plots are restricted to cycles of longer duration. The right side panels report the Design Transformation Matrix under the Volcker (plain line) and Greenspan (dashed line) with the Taylor regime acting as the no-control specification. A value of a Design Transformation Matrix component above 1 signals that a given regime increases the contribution to the variance at that frequency with respect to the Taylor regime. A value below 1 signals a reduction of the contribution to the variance at that frequency.

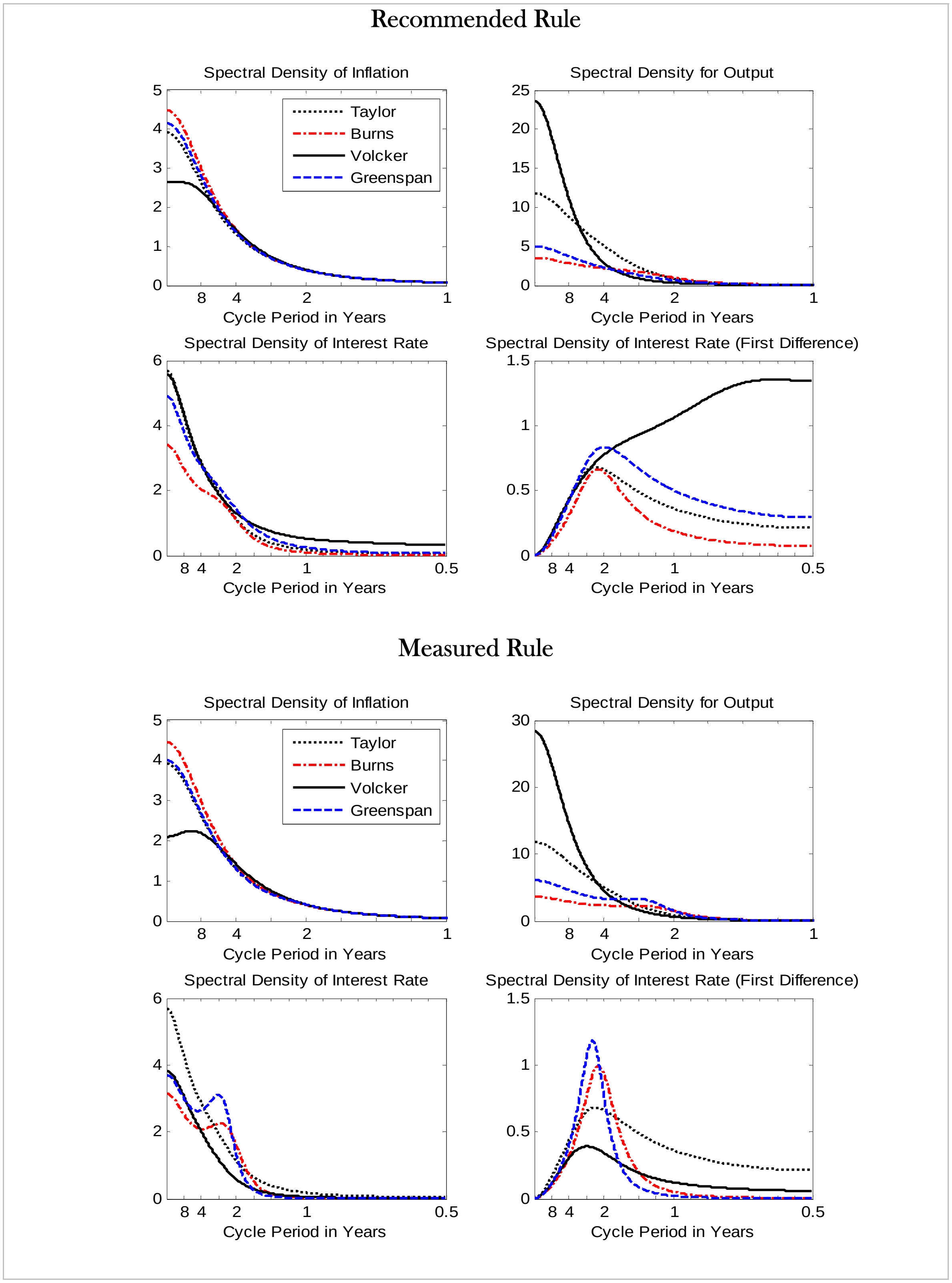
Figure 5.C. Spectral Densities and Design Transformation Matrix for Inflation and Output: Backwards Model

Measured Rule



Note: The left side panels of Figure 5.C report the spectral densities of Inflation ($f_{\pi,v_1}(\omega), f_{\pi,v_2}(\omega)$) and Output ($f_{y,v_1}(\omega), f_{y,v_2}(\omega)$) decomposed according to the nature of the disturbance that generates them (shock to inflation (v_1) and shock to output (v_2)). The spectral densities are derived using the backwards-looking model under three different policy regimes: (i) Original Taylor Rule (dotted line); (ii) Volcker Regime (plain line); (iii) Greenspan Regime (dashed line). The spectral densities under the Burns regime are not well defined. The regimes are expressed in the form of the Measured Rule, the coefficients are those reported in Table 2. For readability, the frequency ranges reported in the plots are restricted to cycles of longer duration. The right side panels report the Design Transformation Matrix under the Volcker (plain line) and Greenspan (dashed line) with the Taylor regime acting as the no control specification. A value of a Design Transformation Matrix component above 1 signals that a given regime increases the contribution to the variance at that frequency with respect to the Taylor regime. A value below 1 signals a reduction of the contribution to the variance at that frequency.

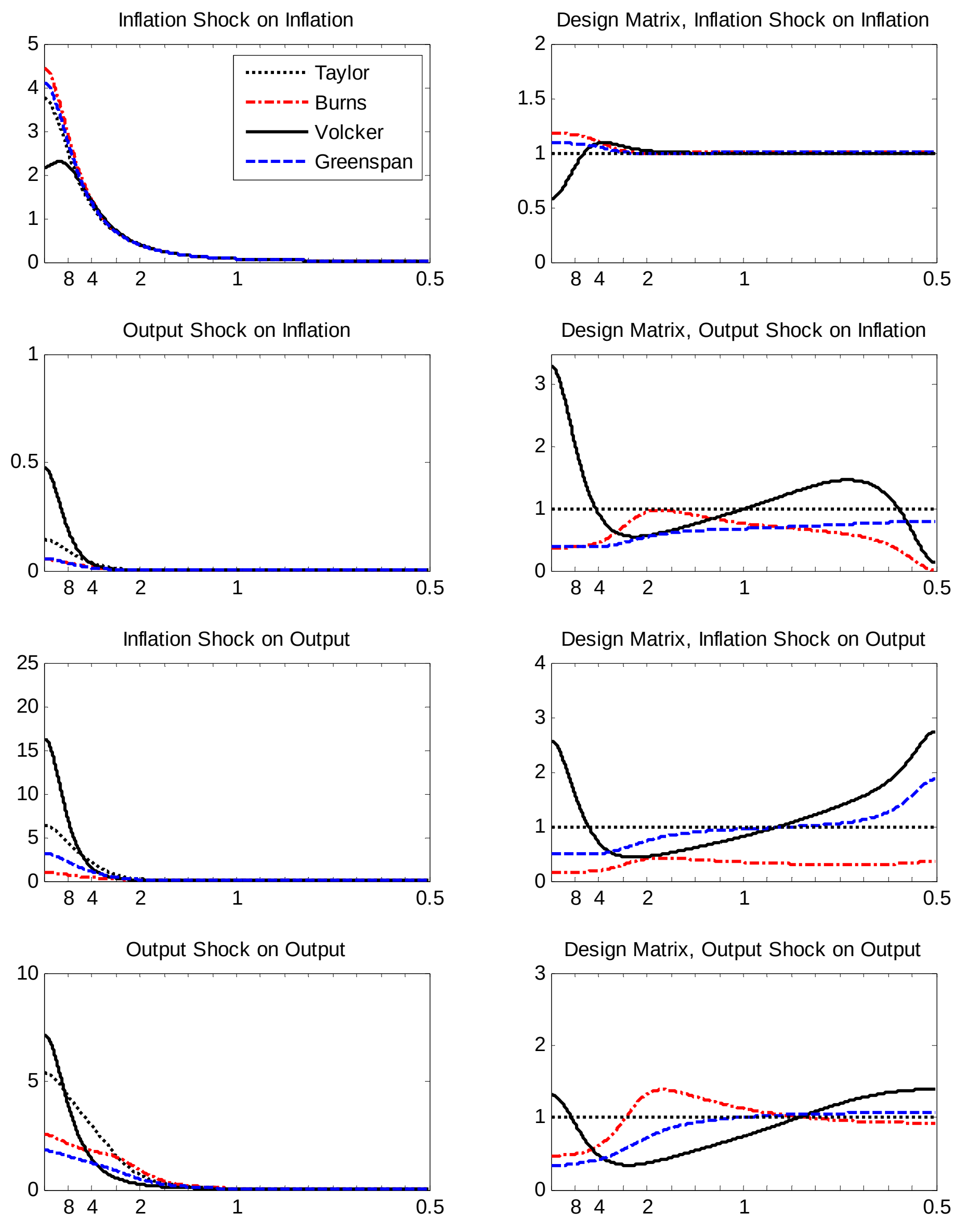
Figure 6.A. Spectral Densities for Inflation, Output, and Interest Rates:
Hybrid Model



Note: Figure 6.A reports the spectral densities of inflation, output, and the interest rate in levels and in first differences under four different policy regimes: (i) Original Taylor Rule (dotted line); (ii) Volcker Regime (plain line); (iii) Greenspan Regime (dashed line); (iv) Burns Regime (dotted-dashed line). The hybrid model is used to evaluate the performance of the three regimes. The upper four panels are generated using the Recommended Rule as a measure of the regimes; the lower four panels use the Measured Rule as a measure of the regime. The coefficients are those reported in Table 2. For readability, the frequency ranges reported in the plots are restricted to cycles of longer duration.

Figure 6.B. Spectral Densities and Design Transformation Matrix for Inflation and Output: Hybrid Model

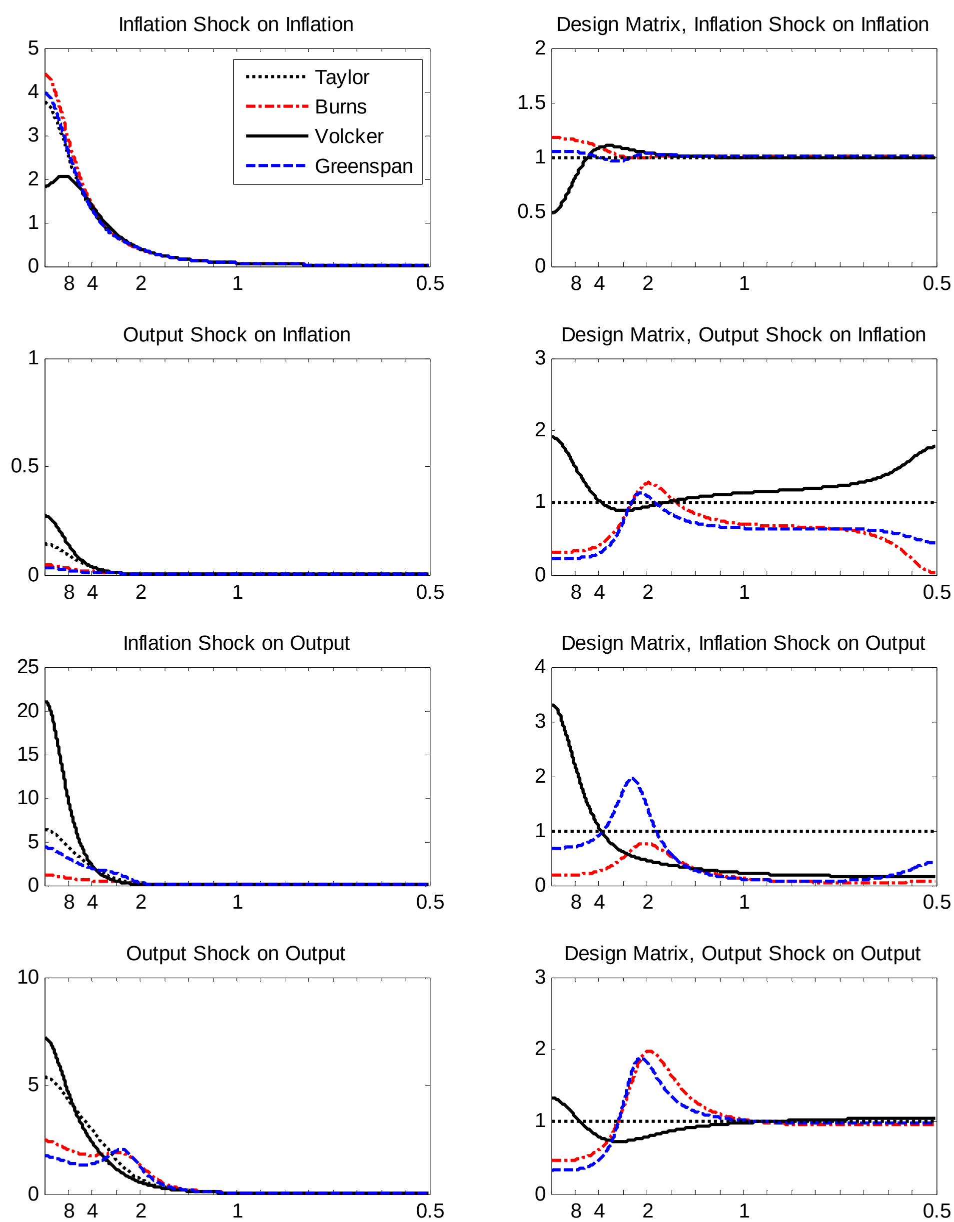
Recommended Rule



Note: The left side panels of Figure 6.B report the spectral densities of inflation ($f_{\pi,v_1}(\omega), f_{\pi,v_2}(\omega)$) and output ($f_{y,v_1}(\omega), f_{y,v_2}(\omega)$) decomposed according to the nature of the disturbance that generates them (shock to inflation (v_1) and shock to output (v_2)). The spectral densities are derived using the hybrid model under four different policy regimes: (i) Original Taylor Rule (dotted line); (ii) Volcker Regime (plain line); (iii) Greenspan Regime (dashed line); (iv) Burns Regime (dotted-dashed line). The regimes are expressed in the form of the Recommended Rule, the coefficients are those reported in Table 2. For readability, the frequency ranges reported in the plots are restricted to cycles of longer duration. The right side panels report the Design Transformation Matrix under the Volcker (plain line), Greenspan (dashed line) and Burns (dotted-dashed line) with the Taylor regime acting as the no control specification. A value of a Design Transformation Matrix component above 1 signals that a given regime increases the contribution to the variance at that frequency with respect to the Taylor regime. A value below 1 signals a reduction of the contribution to the variance at that frequency.

Figure 6.C. Spectral Densities and Design Transformation Matrix for Inflation and Output Shocks, Hybrid Model

Measured Rule



Note: The left side panels of Figure 6.C report the spectral densities of inflation ($f_{\pi,v_1}(\omega), f_{\pi,v_2}(\omega)$) and output ($f_{y,v_1}(\omega), f_{y,v_2}(\omega)$) decomposed according to the nature of the disturbance that generates them (shock to inflation (v_1) and shock to output (v_2)). The spectral densities are derived using the hybrid model under four different policy regimes: (i) Original Taylor Rule (dotted line); (ii) Volcker Regime (plain line); (iii) Greenspan Regime (dashed line); (iv) Burns Regime (dotted-dashed line). The regimes are expressed in the form of the Measured Rule, the coefficients are those reported in Table 2. For readability, the frequency ranges reported in the plots are restricted to cycles of longer duration. The right side panels report the Design Transformation Matrix under the Volcker (plain line), Greenspan (dashed line) and Burns (dotted-dashed line) with the Taylor regime acting as the no control specification. A value of a Design Transformation Matrix component above 1 signals that a given regime increases the contribution to the variance at that frequency with respect to the Taylor regime. A value below 1 implies a reduction of the contribution to the variance at that frequency.

Appendix 1. Proofs and Derivations

Many of our derivations employ the following lemma, due to Wu and Jonckheere (1992), which we report for convenience.

Lemma 1.

$$\int_{-\pi}^{\pi} \left|e^{i\omega} - r\right|^2 d\omega = 0 \text{ if } \left|r\right| \leq 1; = 2\pi \log\left|r\right|^2 \text{ if } \left|r\right| > 1 \tag{A.1}$$

Proof of Theorem 1

From (8) and (10) we can write,

$$\det D^C\left(e^{-i\omega}\right) = \frac{\det W\left(e^{-i\omega}\right)}{\det\left(A_0 - \left(A\left(e^{-i\omega}\right) + B\left(e^{-i\omega}\right)U\left(e^{-i\omega}\right)\right)e^{-i\omega}\right)} = \frac{1}{\det\left(A_0\right)\Pi_{i=1}^{m}\left(1-\lambda_i^C e^{-i\omega}\right)}\bar{w}\frac{\Pi_{i=1}^{w_{MA}}\left(1-w_i e^{-i\omega}\right)}{\Pi_{i=1}^{w_{AR}}\left(1-\rho_i e^{-i\omega}\right)}. \tag{A.2}$$

Therefore,

$$
\begin{aligned}
&\int_{-\pi}^{\pi} \log\left(\left|\det D^C\left(e^{-i\omega}\right)\right|^2\right) d\omega = \\
&\int_{-\pi}^{\pi} \log\left|\frac{1}{\det\left(A_0\right)\Pi_{i=1}^{m}\left(1-\lambda_i^C e^{-i\omega}\right)}\,\overline{w}\,\frac{\Pi_{i=1}^{w_{MA}}\left(1-w_i e^{-i\omega}\right)}{\Pi_{i=1}^{w_{AR}}\left(1-\rho_i e^{-i\omega}\right)}\right|^2 d\omega = \\
&\int_{-\pi}^{\pi} \log\left(\frac{1}{\det\left(A_0\right)^2 \Pi_{i=1}^{m}\left|e^{i\omega}-\lambda_i^C\right|^2}\,\overline{w}^2\,\frac{\Pi_{i=1}^{w_{MA}}\left|e^{i\omega}-w_i\right|^2}{\Pi_{i=1}^{w_{AR}}\left|e^{i\omega}-\rho_i\right|^2}\right) d\omega = \\
&-\sum_{i=1}^{m}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i^C\right|^2 d\omega - 4\pi\log\det\left(A_0\right) + 4\pi\log\overline{w} \\
&+\sum_{i=1}^{w_{MA}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega - \sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\rho_i\right|^2 d\omega
\end{aligned}
\tag{A.3}
$$

From Lemma 1, $\int_{-\pi}^{\pi}\left|e^{i\omega}-r\right|^2 d\omega = 0$ if $|r|<1$. We have assumed that the driving process is second-order stationary which means that $\left|\rho_i\right|<1$. Hence the last terms in (A.3) are 0. The terms of interest are $\sum_{i=1}^{m}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i^C\right|^2 d\omega$ and $\sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega$. Concerning the former, the λ_i^C's are the eigenvalues of the controlled system. When a control is applied to a system it seems desirable to eliminate any unstable eigenvalues, which means that $\left|\lambda_i^C\right|<1\ \forall i$. From Lemma 1 this means $\sum_{i=1}^{m}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-\lambda_i^C\right|^2 d\omega = 0$.

Consider the term. $\sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega$. Since it is possible that $\left|w_i\right|>1$,

$$
\sum_{i=1}^{w_{AR}}\int_{-\pi}^{\pi}\log\left|e^{i\omega}-w_i\right|^2 d\omega = 4\pi\sum_{u_i}\log\left|w_{u_i}\right| \quad i\in\{u_i\} \text{ if } \left|w_i\right|>1 \tag{A.4}
$$

which verifies the Theorem.

Proof of Theorem 2

Using the definition in (20) and the expression (21) we can apply Lemma 1 and get

$$\int_{-\pi}^{\pi} \log\left(\left|\det D^{NC}\left(e^{-i\omega}\right)\right|^2\right) d\omega = -4\pi \log\det\left(A_0\right) + 4\pi \log \overline{w}$$
$$+4\pi \sum_{v_i} \log\left|\lambda_{v_i}\right| + 4\pi \sum_{u_j} \log\left|w_{u_j}\right| \text{ , } i \in \{v_i\} \text{ if } \left|\lambda_i\right| > 1, j \in \{u_j\} \text{ if } \left|w_j\right| > 1 \qquad \text{(A.5)}$$

Subtracting the result of Theorem 1 to (A.4), Theorem 2 immediately follows.

Construction of Unique Solution for Hybrid System

Assuming that the hybrid system has a unique solution, we use Whiteman (1983) to derive the solution in the space of z-transforms. Let the moving average representation for the solution be $x_t = G\left(L\right)v_t$. Define $V\left(z\right) = V^{-1}W\left(z\right)$. Applying the Wiener-Kolmogorov formula and letting $A^C\left(z\right) = A(z) + B(z)U(z)$, the equilibrium MA solution must follow,

$$A_0 G\left(z\right) = \beta\left(G\left(z\right) - G\left(0\right)\right)z^{-1} - A^C\left(z\right)zG\left(z\right) + V\left(z\right). \qquad \text{(A.6)}$$

Multiplying both sides by z and rearranging

$$\left(A_0 z - \beta - A^C\left(z\right)z^2\right)G\left(z\right) = -\beta G\left(0\right) + V\left(z\right)z \qquad \text{(A.7)}$$

Let $J\left(z\right) = Adj\left(\beta - A_0 z + A^C\left(z\right)z^2\right)$ and $\tilde{g}_d\left(z\right) = \det J\left(z\right)$; these imply

$$G\left(z\right) = \frac{1}{\tilde{g}_d\left(z\right)} J\left(z\right)\left(\beta G\left(0\right) - V\left(z\right)z\right). \qquad \text{(A.8)}$$

Without any additional restrictions, (A.8) expresses a solution to (A.6) that holds for any $J\left(z\right)$ and $V\left(z\right)$.

Before we proceed further it is useful to make an observation about the structure of (A.8). First notice that if the 2×2 matrix β is the zero matrix, then analysis entirely centers around the roots of the modulus of the determinant of the 2×2 polynomial matrix $A_0 z - A^C\left(z\right)z^2$. Assume that the determinant of the 2×2 matrix A_0 is nonzero so that we can multiply both sides of (A.7) by A_0^{-1}. Analysis now centers around the roots of the polynomial equation $\varphi(z,\beta) = 0$, where $\varphi(z,\beta)$ is defined as $\varphi(z,\beta) = \det(zI - A_0^{-1}\beta - A_0^{-1}A^C\left(z\right)z^2)$. In the case of no control one defines the corresponding polynomial by replacing $A^C\left(z\right)$ with $A(z)$.

As our general analysis of the construction of a solution to (A.7) is complicated, it is useful to first consider a simple special case in order to develop intuition. Since any polynomial matrix may be reduced to diagonal Smith form by pre-multiplication (post-multiplication) by an appropriate unimodular matrix $U_L(z),(U_R(z))$ (Zhou, Doyle, and Glover (1996, page 80, Lemma 3.25)) it is useful to examine diagonal cases. Recall that unimodular polynomial matrices have nonzero determinants that are constant in z. For example there exist unimodular polynomial matrices $U_L(z), U_R(z)$ such that $Q(z) = U_L(z)(A_0 z - \beta - A^C\left(z\right)z^2)U_R(z)$ where $Q(z)$ is diagonal. Thus the set of roots to the polynomial equation $\det Q(z) = 0$ is the same as for $\det(A_0 z - \beta - A^C\left(z\right)z^2) = 0$. In view of this result we use diagonal systems below in order to generate intuition about general systems.

Assume that $A^C\left(z\right)$ and $A(z)$ are diagonal polynomial matrices and $V(z) = I$ for all z. Furthermore suppose that β is a diagonal matrix with common diagonal element μ, i.e. $\beta = \mu I$. It is evident that a solution to the MA form is defined by

$$(z - \mu - A_{ii}^C\left(z\right)z^2)G_{ii}(z) = z - \mu G_{ii}(0),. \tag{A.9}$$

for $i = 1,2$. Recall that $A_{ii}^C\left(z\right)$ are polynomials of finite degree. Factoring out $-\mu$ from both sides of the above expression,

$$(1-\mu^{-1}z+\mu^{-1}A_{ii}^{C}\left(z\right)z^{2})G_{ii}(z)= \\ (\Pi_{k=1}^{n_i+2}(1-g_{d,k}^{C}z))G_{ii}(z)=G_{ii}(0)(1-(bG_{ii}(0))^{-1}z) \tag{A.10}$$

for $i=1,2$. Now let g_{d,n_i+2}^{C} be the root that will exist if μ is small enough and which goes to infinity as μ goes to zero. Choose $(\mu G_{ii}(0))^{-1}=g_{d,n_i+2}^{C}$, and cancel the term off both sides of the above expression for $i=1,2$. This operation is analogous to the procedure in Appendix 3. If we now take the modulus of both sides, take the natural log and integrate for z over the unit circle, i.e. for $z=e^{-i\omega},\omega\in[-\pi,\pi]$, we can use the formula of Wu and Jonckheere from (A.1), to obtain

$$\int_{-\pi}^{\pi}\ln\left|G_{ii}(e^{-i\omega})\right|d\omega=-2\pi\sum_{\{k:|\lambda_k|>1\}}\ln|\,g_{d,k}^{C}\,|+2\pi\ln\left|G_{ii}(0)\right|. \tag{A.11}$$

Notice that this equation makes sense even if some of the roots are unstable. We always maintain the assumption that when control is applied, that the controlled system is stable. In that case the first term on the RHS of is zero.

We next consider a general 2×2 matrix case. For $A^{C}\left(z\right)\equiv A(z)+B(z)U(z)$ and with $V\left(z\right)=1$ we have

$$(-\beta)[I-\beta^{-1}A_0z+\beta^{-1}A^{C}\left(z\right)z^{2}]G(z)=(-\beta G(0))[I-(\beta G(0))^{-1}z]$$

Taking determinants on both sides, $\det(\beta)$ cancels off so that

$$\Pi_{k=1}^{k=m}(1-g_{d,k}^{C}z)\det G(z)=\det(G(0))\det(I-(\beta G(0))^{-1}z) \tag{A.12}$$

Notice that $\det(I-(\beta G(0))^{-1}z)$ can be written in the form

$$\det(I-(\beta G(0))^{-1}z)=(1-\eta_1 z)(1-\eta_2 z)\,. \tag{A.13}$$

Thus we perform a root cancelling exercise like that above for the diagonal case where we cancel the two roots that play the role of the twice repeated solution $z = 0$. Label these two roots as $m, m-1$. Then we have

$$\Pi_{k=1}^{k=m-2}(1 - g_{d,k}^{C} z) \det G(z) = \det(G(0)) \,. \tag{A.14}$$

One may repeat the above analysis to compute $\int_{-\pi}^{\pi} \ln\left|\det G\left(e^{-i\omega}\right)\right| d\omega$. While this decomposition sheds light on the forces that determine Bode-like integral contraints for hybrid systems, it is important to note that the "constants" depend upon the choice of control $A^{C}\left(z\right)$ in the hybrid case. In the backwards-looking case, $\beta = 0$, the constants are independent of the choice of control so long as control is chosen to stabilize the system.

We next consider the fully general case. The stability of $G\left(z\right)$ depends on the location of the roots of the characteristic polynomial $\tilde{g}_{d}\left(z\right)$. Notice that $G\left(0\right) = \begin{pmatrix} g_{0,11} & g_{0,12} \\ g_{0,21} & g_{0,22} \end{pmatrix}$ appears on the RHS of (A.8). In principle, we could specify any arbitrary $G\left(0\right)$ and the solution (A.8) would still be valid. Suppose, for example, that the determinant $\tilde{g}_{d}\left(z\right)$ has no roots inside the unit circle. Since $G\left(0\right)$ does not affect the characteristic roots of the system, (A.8) is a stationary solution for any bounded $G\left(0\right)$. In this case there exist multiple stationary solutions to (A.8). To obtain uniqueness one needs additional conditions to restrict $G\left(0\right)$. These conditions are provided by the requirement that $\tilde{g}_{d}\left(z\right)$ contains unstable roots. If this is the case, the elements in $G\left(0\right)$ can be chosen in order to exactly cancel those unstable roots. How is this condition related to the proposition on uniqueness we present in Appendix 2? The connection lies in the fact is that we are searching for a solution in the space of one-sided moving averages in v_{t} that are square summable. For any solution $G\left(z\right) = \sum_{i=0}^{\infty} g_{i} z^{i}$ belonging to that space, equation (A.8) must hold. In addition to consistency with (A.6), square-summability is the only additional requirement for a solution. The conditions for uniqueness that we assume ensure that the elements of this matrix are chosen in order to ensure that the system is stable. Stability of the system is entirely determined by the roots of the polynomial $\tilde{g}_d(z)$. The choice of the elements is made in order to ensure that any unstable root at the denominator (a pole inside the unit circle of $\tilde{g}_d(z)$) is cancelled with an unstable root at the numerator (a zero inside the unit circle of each element of $J(z)(\beta G(0)-V(z)z)$. The conditions for uniqueness of the rational expectations solution corresponds to each numerator term in this matrix to become zero at the unstable poles of the denominator. Let $\overline{p}_i$ denote an unstable pole, then each $\overline{p}_i$ provides the associated set of equations

$$J(\overline{p}_i)(\beta G(0)-V(\overline{p}_i)\overline{p}_i)=0 \tag{A.15}$$

whose rank is zero, since $J(\overline{p}_i)$ is by construction not invertible. Generally, the condition for uniqueness in a 2-equation system corresponds to requiring two roots of the polynomial $\tilde{g}_d(z)$ being unstable. In that case, the system that solves for the constant is

$$\begin{aligned}
J_{11}(\overline{p}_1)(\beta_{11}g_{0,11}+\beta_{12}g_{0,21}-\overline{p}_1v_{11}(\overline{p}_1))+J_{12}(\overline{p}_1)(\beta_{21}g_{0,11}+\beta_{22}g_{0,21}-\overline{p}_1v_{21}(\overline{p}_1))=0\\
J_{11}(\overline{p}_2)(\beta_{11}g_{0,11}+\beta_{12}g_{0,21}-\overline{p}_2v_{11}(\overline{p}_2))+J_{12}(\overline{p}_2)(\beta_{21}g_{0,11}+\beta_{22}g_{0,21}-\overline{p}_2v_{21}(\overline{p}_2))=0\\
J_{11}(\overline{p}_1)(\beta_{11}g_{0,12}+\beta_{12}g_{0,22}-\overline{p}_1v_{12}(\overline{p}_{\mathbf{1}}))+J_{12}(\overline{p}_1)(\beta_{21}g_{0,12}+\beta_{22}g_{0,22}-\overline{p}_1v_{22}(\overline{p}_1))=0\\
J_{11}(\overline{p}_2)(\beta_{11}g_{0,12}+\beta_{12}g_{0,22}-\overline{p}_2v_{12}(\overline{p}_2))+J_{12}(\overline{p}_2)(\beta_{21}g_{0,12}+\beta_{22}g_{0,22}-\overline{p}_2v_{22}(\overline{p}_2))=0
\end{aligned} \tag{A.16}$$

To simply notation, define

$$\begin{aligned}
\alpha(z)&=J_{11}(z)\beta_{11}+J_{12}(z)\beta_{21}\\
\delta(z)&=J_{11}(z)\beta_{12}+J_{12}(z)\beta_{22}\\
\kappa_1(z)&=z(J_{11}(z)v_{11}(z)+J_{12}(z)v_{21}(z))\\
\kappa_2(z)&=z(J_{11}(z)v_{12}(z)+J_{12}(z)v_{22}(z))
\end{aligned} \tag{A.17}$$

The explicit expressions for the constants are

$$g_{0,11}\left(\overline{p}_1,\overline{p}_2\right)=\frac{\delta\left(\overline{p}_2\right)\kappa_1\left(\overline{p}_1\right)-\delta\left(\overline{p}_1\right)\kappa_1\left(\overline{p}_2\right)}{\alpha\left(\overline{p}_1\right)\delta\left(\overline{p}_2\right)-\alpha\left(\overline{p}_2\right)\delta\left(\overline{p}_1\right)} \tag{A.18}$$

$$g_{0,21}\left(\overline{p}_1,\overline{p}_2\right)=\frac{\alpha\left(\overline{p}_1\right)\kappa_1\left(\overline{p}_2\right)-\alpha\left(\overline{p}_2\right)\kappa_1\left(\overline{p}_1\right)}{\alpha\left(\overline{p}_1\right)\delta\left(\overline{p}_2\right)-\alpha\left(\overline{p}_2\right)\delta\left(\overline{p}_1\right)} \tag{A.19}$$

$$g_{0,12}\left(\overline{p}_1,\overline{p}_2\right)=\frac{\delta\left(\overline{p}_2\right)\kappa_2\left(\overline{p}_1\right)-\delta\left(\overline{p}_1\right)\kappa_2\left(\overline{p}_2\right)}{\alpha\left(\overline{p}_1\right)\delta\left(\overline{p}_2\right)-\alpha\left(\overline{p}_2\right)\delta\left(\overline{p}_1\right)} \tag{A.20}$$

$$g_{0,22}\left(\overline{p}_1,\overline{p}_2\right)=\frac{\alpha\left(\overline{p}_1\right)\kappa_2\left(\overline{p}_2\right)-\alpha\left(\overline{p}_2\right)\kappa_2\left(\overline{p}_1\right)}{\alpha\left(\overline{p}_1\right)\delta\left(\overline{p}_2\right)-\alpha\left(\overline{p}_2\right)\delta\left(\overline{p}_1\right)}. \tag{A.21}$$

We can now derive (29) in the text. Recall that

$$\tilde{g}_d\left(z\right)\equiv\det\left(zA_0-\beta-\left(A\left(z\right)-B\left(z\right)U\left(z\right)\right)z^2\right). \tag{A.22}$$

We denote the determinant by $\tilde{g}_d\left(z\right)$ to stress that the cancellation of unstable roots that allows the uniqueness of the solution has not yet been considered. The solution matrix can thus be written as

$$G\left(z\right)=\frac{1}{\tilde{g}_d\left(z\right)}\begin{pmatrix}J_{11}\left(z\right) & J_{12}\left(z\right)\\ J_{21}\left(z\right) & J_{22}\left(z\right)\end{pmatrix}\begin{pmatrix}\beta_{11}g_{0,11}+\beta_{12}g_{0,21}-zv_{11}\left(z\right) & \beta_{11}g_{0,12}+\beta_{12}g_{0,22}-zv_{12}\left(z\right)\\ \beta_{21}g_{0,11}+\beta_{22}g_{0,21}-zv_{21}\left(z\right) & \beta_{21}g_{0,12}+\beta_{22}g_{0,22}-zv_{22}\left(z\right)\end{pmatrix} \tag{A.23}$$

Once the constants in $G\left(0\right)$ are properly specified it is possible to write each term of $G\left(L\right)$ as having a common denominator whose zeros are all outside the unit circle, we denote such a denominator by $g_d\left(z\right)$; as noticed, this denominator is common across terms modulo additional terms that are functions of the denominator polynomials in $V^{-1}W\left(L\right)$. This property is important since the last term does not depend on the control applied to the system. We make this statement formal in what follows.

Recalling that $W(L)$ has the form (4) in the text it must be the case that for $V(L)$

$$V\left(L\right)=V^{-1}W\left(L\right)=\begin{pmatrix} v_{11}\left(L\right) & v_{12}\left(L\right) \\ v_{21}\left(L\right) & v_{22}\left(L\right) \end{pmatrix}=\begin{pmatrix} \frac{v_{n,11}\left(L\right)}{v_{d,11}\left(L\right)} & \frac{v_{n,12}\left(L\right)}{v_{d,12}\left(L\right)} \\ \frac{v_{n,21}\left(L\right)}{v_{d,21}\left(L\right)} & \frac{v_{n,22}\left(L\right)}{v_{d,22}\left(L\right)} \end{pmatrix}, \tag{A.24}$$

where the numerator polynomials are defined so that:

$$v_{ij}\left(L\right)=\frac{v_{n,ij}\left(L\right)}{v_{d,ij}\left(L\right)}=\frac{v_{n,ij}\left(L\right)}{\prod_{h=1}^{k_{ij}}\left(1-v_{d,h}L\right)}. \tag{A.25}$$

The terms in the second matrix of the solution matrix will take the form

$$\beta_{11}g_{0,11}+\beta_{12}g_{0,21}-z\frac{v_{n,11}\left(L\right)}{v_{d,11}\left(L\right)}=\frac{v_{d,11}\left(L\right)\left(\beta_{11}g_{0,11}+\beta_{12}g_{0,21}\right)-zv_{n,11}\left(L\right)}{v_{d,11}\left(L\right)}\equiv\frac{\tilde{v}_{n,11}\left(L\right)}{v_{d,11}\left(L\right)}. \tag{A.26}$$

It follows that the form of the solution matrix is

$$
\begin{aligned}
G(z) &= \frac{1}{\tilde{g}_d(z)} \begin{pmatrix} J_{11}(z) & J_{12}(z) \\ J_{21}(z) & J_{22}(z) \end{pmatrix} \begin{pmatrix} \frac{\tilde{v}_{n,11}(z)}{v_{d,11}(z)} & \frac{\tilde{v}_{n,12}(z)}{v_{d,12}(z)} \\ \frac{\tilde{v}_{n,21}(z)}{v_{d,21}(z)} & \frac{\tilde{v}_{n,22}(z)}{v_{d,22}(z)} \end{pmatrix} \\
&= \frac{1}{\tilde{g}_d(z)} \begin{pmatrix} J_{11}(z)\frac{\tilde{v}_{n,11}(z)}{v_{d,11}(z)} + J_{12}(z)\frac{\tilde{v}_{n,21}(z)}{v_{d,21}(z)} & J_{11}(z)\frac{\tilde{v}_{n,21}(z)}{v_{d,21}(z)} + J_{12}(z)\frac{\tilde{v}_{n,22}(z)}{v_{d,22}(z)} \\ J_{21}(z)\frac{\tilde{v}_{n,11}(z)}{v_{d,11}(z)} + J_{22}(z)\frac{\tilde{v}_{n,21}(z)}{v_{d,21}(z)} & J_{21}(z)\frac{\tilde{v}_{n,12}(z)}{v_{d,12}(z)} + J_{22}(z)\frac{\tilde{v}_{n,22}(z)}{v_{d,22}(z)} \end{pmatrix} \\
&= \frac{1}{\tilde{g}_d(z)} \begin{pmatrix} \frac{\tilde{g}_{n,11}(z)}{v_{d,11}(z)v_{d,21}(z)} & \frac{\tilde{g}_{n,12}(z)}{v_{d,21}(z)v_{d,22}(z)} \\ \frac{\tilde{g}_{n,21}(z)}{v_{d,11}(z)v_{d,21}(z)} & \frac{\tilde{g}_{n,22}(z)}{v_{d,12}(z)v_{d,22}(z)} \end{pmatrix} \\
&= \frac{1}{g_d(z)} \begin{pmatrix} \frac{g_{n,11}(z)}{v_{d,11}(z)v_{d,21}(z)} & \frac{g_{n,12}(z)}{v_{d,12}(z)v_{d,22}(z)} \\ \frac{g_{n,21}(z)}{v_{d,11}(z)v_{d,21}(z)} & \frac{g_{n,22}(z)}{v_{d,12}(z)v_{d,22}(z)} \end{pmatrix}
\end{aligned}
\qquad \text{(A.27)}
$$

The first step in this expression is matrix multiplication, the second step defines the numerators of each term as $\tilde{g}_{n,ij}(z)$ since the free parameters have not been pinned down, and the step applies the requirements for the uniqueness of a solution by choosing the free parameters so as to cancel the zeros of the common denominator $\tilde{g}_d(z)$ inside the unit circle with the zeros of each numerator $\tilde{g}_{n,ij}(z)$. Equation (A.27) shows that the solution of the hybrid model can be written in matrix moving average form where each element is a ratio of a denominator term and a numerator term. While the numerator terms are potentially different from one another, the numerator term has an endogenous component that is common across elements and an exogenous component that differs across elements and which cannot be affected by the control policy. This form is very convenient in the proof of Theorems 3 and 4.

Proof of Theorem 3

We need to evaluate the integral of the expression $\log\left(|\det D^C\left(e^{-i\omega}\right)|^2\right)$. According to our definitions, $G^C(z) \equiv G\left(z\right)$ for $U(z) \neq 0$, therefore we can use (A.27), which implies

$$\begin{aligned}\det G^C(z) &= \det\left[\frac{1}{g_d\left(z\right)}\begin{pmatrix}\frac{g_{n,11}\left(z\right)}{v_{d,11}\left(z\right)v_{d,21}\left(z\right)} & \frac{g_{n,12}\left(z\right)}{v_{d,12}\left(z\right)v_{d,22}\left(z\right)} \\ \frac{g_{n,21}\left(z\right)}{v_{d,11}\left(z\right)v_{d,21}\left(z\right)} & \frac{g_{n,22}\left(z\right)}{v_{d,12}\left(z\right)v_{d,22}\left(z\right)}\end{pmatrix}\right] \\ &= \frac{1}{g_d\left(z\right)^2}\frac{g_{n,11}\left(z\right)g_{n,22}\left(z\right)-g_{n,21}\left(z\right)g_{n,12}\left(z\right)}{v_{d,11}\left(z\right)v_{d,21}\left(z\right)v_{d,12}\left(z\right)v_{d,22}\left(z\right)}\end{aligned} \tag{A.28}$$

It therefore follows that

$$\begin{gathered}\int_{-\pi}^{\pi}\log\left(\left|\det G^C\left(e^{-i\omega}\right)\right|^2\right)d\omega = \\ \int_{-\pi}^{\pi}\log\left(\left|g_{n,11}\left(e^{-i\omega}\right)g_{n,22}\left(e^{-i\omega}\right)-g_{n,21}\left(e^{-i\omega}\right)g_{n,12}\left(e^{-i\omega}\right)\right|^2\right)d\omega\,. \\ -2\int_{-\pi}^{\pi}\log\left(\left|g_d\left(e^{-i\omega}\right)\right|^2\right)d\omega-\sum_{i,j=1,2}\int_{-\pi}^{\pi}\log\left(\left|v_{d,ij}\left(e^{-i\omega}\right)\right|^2\right)d\omega\end{gathered} \tag{A.29}$$

The interesting features of this expression derive from the first component, i.e.

$$\int_{-\pi}^{\pi}\log\left(\left|g_{n,11}\left(e^{-i\omega}\right)g_{n,22}\left(e^{-i\omega}\right)-g_{n,21}\left(e^{-i\omega}\right)g_{n,12}\left(e^{-i\omega}\right)\right|^2\right)d\omega\,.$$

Before dealing with this component, we evaluate the others. For the term $\int_{-\pi}^{\pi}\log\left(\left|g_d\left(e^{-i\omega}\right)\right|^2\right)$, first write $g_d\left(z\right)$ as $g_d\prod_{i=1}^{d^C}\left(1-g_{d,i}^C z\right)$, where, because of the stability requirement, $\left|g_{d,i}^C\right|<1$ for every $i=1,..,d^C$. Notice that the zero degree coefficient in this polynomial does not depend on the control applied to the system; it can be shown to equal the determinant of the matrix coefficients for the forward looking elements, β , i.e. $g_d = \det\beta = \left(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}\right)$. As for the second term, the elements $v_{d,ij}\left(z\right)$ have been constructed so that

$$v_{d,11}\left(z\right)v_{d,21}\left(z\right)v_{d,12}\left(z\right)v_{d,22}\left(z\right) = \prod_{k=1}^{h}\left(1 - v_{d,k}L\right) \tag{A.30}$$

where h is the sum of the degrees of each denominator term and the $v_{d,k}$'s are the eigenvalues associated to the zeros of each denominator term, all of which are assumed to lie inside the unit circle, the contribution of $\sum_{i,j=1,2}\int_{-\pi}^{\pi}\log\left(\left|v_{d,ij}\left(e^{-i\omega}\right)\right|^2\right)d\omega$ in (A.29) is zero.

With respect to the first term, first rewrite the polynomial expression as

$$g_{n,11}\left(z\right)g_{n,22}\left(z\right) - g_{n,21}\left(z\right)g_{n,12}\left(z\right) = g_n^C\prod_{i,j=1,2}^{N^C}\left(1 - g_{n,ij}^C z\right) \tag{A.31}$$

Neither the zero degree coefficient g_n^C nor the roots of the individual polynomials in (A.25) are bounded by any stability requirement; rather they depend on the interaction between the structural parameters of the model and the properties of the processes of the exogenous disturbances entering the system. This can be clearly see from the expressions above for the determination of the constant matrix $G\left(0\right)$. Once this matrix is substituted into the solution and the desired roots canceled, both the common term and the zeros are reallocated in ways that are, loosely speaking, unrestricted. It follows that there are no basis upon which we can a priori rule out their contribution to the Bode constraint. It follows that

$$\int_{-\pi}^{\pi} \log\left(\left| g_{n,11}\left(e^{-i\omega}\right) g_{n,22}\left(e^{-i\omega}\right) - g_{n,21}\left(e^{-i\omega}\right) g_{n,12}\left(e^{-i\omega}\right)\right|^2\right) d\omega =$$

$$\int_{-\pi}^{\pi} \log\left(\left(g_n^C\right)^2 \left| \prod_{i,j=1,2}^{N^C} \left(1 - g_{n,ij}^C e^{-i\omega}\right)\right|^2\right) d\omega. \tag{A.32}$$

Using Lemma 1 as in the proof of Theorem 1 the result of Theorem 3 follows immediately.

An essential distinction between a system containing forward looking expectations and a system that is purely backwards-looking lies in the fact that the coefficient on the zero degree term g_n^C is *always* affected by the control through the commitment of the policymaker to a rule that forces agents to respond in a particular way to shocks in the time period they are realized and observed.

Proof of Theorem 5

Part i. The result for the backwards-looking case is a direct consequence of Lemma 1 applied to eq. (46) in the text. For the hybrid case the design transformation matrix obeys the sequences of equalities

$$\int_{-\pi}^{\pi} \log M_{ij}\left(\omega\right) d\omega =$$

$$\int_{-\pi}^{\pi} \log\left|\det S\left(\omega\right)\right|^2 d\omega - \int_{-\pi}^{\pi} \log \frac{\left|g_n^C\left(e^{-i\omega}\right)\right|^2}{\left|g_n^{NC}\left(e^{-i\omega}\right)\right|^2} d\omega + \int_{-\pi}^{\pi} \log \frac{\left|g_{n,ij}^C\left(e^{-i\omega}\right)\right|^2}{\left|g_{n,ij}^{NC}\left(e^{-i\omega}\right)\right|^2} d\omega =$$

$$\int_{-\pi}^{\pi} \log \frac{\left|g_d^{NC}\left(e^{-i\omega}\right)\right|^2}{\left|g_d^C\left(e^{-i\omega}\right)\right|^2} d\omega + \int_{-\pi}^{\pi} \log \frac{\left|g_n^C\left(e^{-i\omega}\right)\right|^2}{\left|g_n^{NC}\left(e^{-i\omega}\right)\right|^2} d\omega \tag{A.33}$$

$$-\int_{-\pi}^{\pi} \log \frac{\left|g_n^C\left(e^{-i\omega}\right)\right|^2}{\left|g_n^{NC}\left(e^{-i\omega}\right)\right|^2} d\omega + \int_{-\pi}^{\pi} \log \frac{\left|g_{n,ij}^C\left(e^{-i\omega}\right)\right|^2}{\left|g_{n,ij}^{NC}\left(e^{-i\omega}\right)\right|^2} d\omega =$$

$$\sum_{v_i} \log\left|g_{d,v_i}^{NC}\right| + \int_{-\pi}^{\pi} \log\left|g_{n,ij}^C\left(e^{-i\omega}\right)\right|^2 d\omega - \int_{-\pi}^{\pi} \log\left|g_{n,ij}^{NC}\left(e^{-i\omega}\right)\right|^2 d\omega,$$

where $i \in \{v_i\}$ if $\left|g_{d,i}^{NC}\right| > 1$.

Part ii. Write the backwards-looking system as

$$A_0 x_t = A^C(L) x_{t-1} + W(L) w_t \tag{A.34}$$

where $A^C(z) = A(z) + B(z)U(z)$ and let $A^{NC}(z) = A(z)$. Since we have assumed that $W(L)$ is diagonal, which results also in $w_t = v_t$, the controlled system is

$$x_t = \frac{1}{\det\left(A_0 - A^C(L)L\right)} \begin{pmatrix} a_{0,22} - a_{22}^C(L)L & -a_{0,12} + a_{12}^C(L)L \\ -a_{0,21} + a_{21}^C(L)L & a_{0,11} - a_{11}^C(L)L \end{pmatrix} \begin{pmatrix} w_{11}(L) & 0 \\ 0 & w_{22}(L) \end{pmatrix} v_t = \frac{1}{\det\left(A_0 - A^C(L)L\right)} \begin{pmatrix} \left(a_{0,22} - a_{22}^C(L)L\right) w_{11}(L) & \left(-a_{0,12} + a_{12}^C(L)L\right) w_{22}(L) \\ \left(-a_{0,21} + a_{21}^C(L)L\right) w_{11}(L) & \left(a_{0,11} - a_{11}^C(L)L\right) w_{22}(L) \end{pmatrix} v_t \tag{A.35}$$

which means that

$$D^C(z) = \frac{1}{d_d^C(z)} \begin{pmatrix} d_{n,11}^C(z) & d_{n,12}^C(z) \\ d_{n,21}^C(z) & d_{n,22}^C(z) \end{pmatrix} = \frac{1}{\det\left(A_0 - A^C(L)L\right)} \begin{pmatrix} \left(a_{0,22} - a_{22}^C(L)L\right) w_{11}(L) & \left(-a_{0,12} + a_{12}^C(L)L\right) w_{22}(L) \\ \left(-a_{0,21} + a_{21}^C(L)L\right) w_{11}(L) & \left(a_{0,11} - a_{11}^C(L)L\right) w_{22}(L) \end{pmatrix}. \tag{A.36}$$

Notice that

$$\det D^C(z) = \det\left(A_0 - A^C(L)L\right) \det W(L). \tag{A.37}$$

The uncontrolled case has a similar form. The design transformation matrix can be written as

$$
\begin{gathered}
M(z) = \\
\frac{\det\left(A_0 - A^{NC}(z)z\right)\det\left(A_0 - A^{NC}\left(z^{-1}\right)z^{-1}\right)}{\det\left(A_0 - A^{C}(z)z\right)\det\left(A_0 - A^{C}\left(z^{-1}\right)z^{-1}\right)} \times \\
\begin{pmatrix}
\frac{\left(a_{0,22} - a_{22}^{C}(z)z\right)}{\left(a_{0,22} - a_{22}^{NC}(z)z\right)}\frac{\left(a_{0,22} - a_{22}^{C}\left(z^{-1}\right)z^{-1}\right)}{\left(a_{0,22} - a_{22}^{NC}\left(z^{-1}\right)z^{-1}\right)} & \frac{\left(a_{0,12} - a_{12}^{C}(z)z\right)\left(a_{0,12} - a_{12}^{C}\left(z^{-1}\right)z^{-1}\right)}{\left(a_{0,12} - a_{12}^{NC}(z)z\right)\left(a_{0,12} - a_{12}^{NC}\left(z^{-1}\right)z^{-1}\right)} \\
\frac{\left(a_{0,21} - a_{21}^{C}(z)z\right)\left(a_{0,21} - a_{21}^{C}\left(z^{-1}\right)z^{-1}\right)}{\left(a_{0,21} - a_{21}^{NC}(z)z\right)\left(a_{0,21} - a_{21}^{NC}\left(z^{-1}\right)z^{-1}\right)} & \frac{\left(a_{0,11} - a_{11}^{C}(z)z\right)}{\left(a_{0,11} - a_{11}^{NC}(z)z\right)}\frac{\left(a_{0,11} - a_{11}^{C}\left(z^{-1}\right)z^{-1}\right)}{\left(a_{0,11} - a_{11}^{NC}\left(z^{-1}\right)z^{-1}\right)}
\end{pmatrix}.
\end{gathered} \quad \textbf{(A.38)}
$$

For a **SIMO** model the control can be applied only to one equation. Without loss of generality, we focus on $j = 1$, which corresponds to the control being applicable only to the first equation in (A.34). For this case, $a_{21}^{C}\left(z^{-1}\right) = a_{21}^{NC}\left(z^{-1}\right)$ and $a_{22}^{C}\left(z^{-1}\right) = a_{22}^{NC}\left(z^{-1}\right)$, which imply that

$$
\begin{gathered}
M(z) = \\
\frac{\det\left(A_0 - A^{NC}(z)z\right)\det\left(A_0 - A^{NC}\left(z^{-1}\right)z^{-1}\right)}{\det\left(A_0 - A^{C}(z)z\right)\det\left(A_0 - A^{C}\left(z^{-1}\right)z^{-1}\right)} \times \\
\begin{pmatrix}
1 & \frac{\left(a_{0,12} - a_{12}^{C}(z)z\right)\left(a_{0,12} - a_{12}^{C}\left(z^{-1}\right)z^{-1}\right)}{\left(a_{0,12} - a_{12}^{NC}(z)z\right)\left(a_{0,12} - a_{12}^{NC}\left(z^{-1}\right)z^{-1}\right)} \\
1 & \frac{\left(a_{0,11} - a_{11}^{C}(z)z\right)}{\left(a_{0,11} - a_{11}^{NC}(z)z\right)}\frac{\left(a_{0,11} - a_{11}^{C}\left(z^{-1}\right)z^{-1}\right)}{\left(a_{0,11} - a_{11}^{NC}\left(z^{-1}\right)z^{-1}\right)}
\end{pmatrix},
\end{gathered} \quad \textbf{(A.39)}
$$

.

so that

$$
\log M_{11}(z) = \log M_{21}(z) = \log\frac{\det\left(A_0 - A^{NC}(z)z\right)\det\left(A_0 - A^{NC}\left(z^{-1}\right)z^{-1}\right)}{\det\left(A_0 - A^{C}(z)z\right)\det\left(A_0 - A^{C}\left(z^{-1}\right)z^{-1}\right)} \quad \textbf{(A.40)}
$$

and the result follows from Lemma 1. When $j=2$, $a_{11}^{C}\left(z^{-1}\right)=a_{11}^{NC}\left(z^{-1}\right)$ and $a_{12}^{C}\left(z^{-1}\right)=a_{12}^{NC}\left(z^{-1}\right)$ one has

$$\log M_{12}\left(z\right)=\log M_{22}\left(z\right)=\log\frac{\det\left(A_0-A^{NC}\left(z\right)z\right)\det\left(A_0-A^{NC}\left(z^{-1}\right)z^{-1}\right)}{\det\left(A_0-A^{C}\left(z\right)z\right)\det\left(A_0-A^{C}\left(z^{-1}\right)z^{-1}\right)} \quad \textbf{(A.41)}$$

which completes the proof for the backwards-looking case.

Appendix 2. Conditions for Unique Solution to Hybrid Model

Several methods have been suggested to state the conditions for the existence and the uniqueness of a solution for linear rational expectations models with forward looking components. Despite a continuing ongoing effort, a truly general method encompassing any generic form of multivariate rational expectations models is still unavailable. Onatski (2006) appears to be, at the time of this writing, the most promising attempt for an analytical method that is both straightforward to interpret and is applicable to a large family of rational expectations models[12]. Sims (2007) refers to computer programs available on his website that resolve the issue for models with a finite number of leads and lags. For our purpose, as long as a unique equilibrium exists, the results presented in this paper hold. For completeness we report the conditions for uniqueness that apply to the models of this paper. The hybrid model we employ takes the form

$$A_0 x_t = \beta E_t x_{t+1} + \left(A\left(L\right) + B\left(L\right) F\left(L\right)\right) x_{t-1} + \varepsilon_t \qquad \textbf{(A.42)}$$

where $\varepsilon_t = W\left(L\right) w_t$. We work under the assumption that $W\left(L\right)$ is a rational function and is invertible inside the unit circle. Invertibility is not a major issue as one can always rotate the space of disturbances to obtain an invertible representation. The assumption of rationality ensures that the process ε_t has a rational spectral density matrix. We look for a solution in the space of the square-summable linear combinations of current and past realizations of the driving processes. It is useful for us to work in the space of orthogonal innovations to the driving process, so we employ ν_t with orthogonal elements such that $\varepsilon_t = W\left(L\right) V v_t$. For the following results the scaling by a constant matrix is irrelevant therefore we abstract from the orthogonalization issue.

The hybrid model of this paper belongs to a family of multivariate rational expectations models that can be represented as

[12] Sims (2007) expresses some concerns about Onatski (2006).

$$\sum_{j=-\infty}^{\infty} \sum_{s=0}^{r} \tilde{A}_{j,s} E_{t-s} x_{t-j} = Hf_t \tag{A.43}$$

The mathematical expectations are taken conditional on the information available at time $t-s$, including the structure of the model and all the current and past realizations of exogenous and endogenous variables. The vector f_t is a vector of current and possibly past realizations of the exogenous driving process w_t. The real matrix H can take any form and it will absorb, for instance, the constant matrix V. As an example, consider the simple case of $W(L) = I$, then (A.42) may be represented by (A.43) if

$$\tilde{A}_{-1,0} = \beta, \quad \tilde{A}_{0,0} = A_0, \quad \tilde{A}_{j,0} = A_{j-1} + B_{j-1} F_{j-1} \text{ for } j > 0 .$$

When $W(L)$ takes more complicated forms, the mapping between the two representations is convoluted but it is always well defined.

In order to state the conditions for a unique solution to (A.43) we define $\tilde{A}(\omega)$ as

$$\tilde{A}(\omega) \equiv \sum_{j=-\infty}^{\infty} \sum_{s=0}^{r} \tilde{A}_{j,s} e^{-i(j-r)\omega} \tag{A.44}$$

This is essentially the evaluation of the $z-$transforms of the expectational equation along the unit circle. The key result in Onatski states that the behavior of the solution to the linear rational expectations model depends on the behavior of the graph of the function $\det \tilde{A}(\omega)$ for $\omega \in [0, 2\pi]$. Since $e^{-i(j-r)\omega}$ are periodic functions with period $\frac{2\pi}{j-r}$ the graph of $\det \tilde{A}(\omega)$ designs a closed contour in the complex plane. The number of times that the graph of rotates clockwise around zero is called the *winding number* of $\tilde{A}(\omega)$. The winding number is negative if the rotation around zero is counterclockwise. The following proposition is adapted from Onatski (2006).

Proposition A.2.1. Necessary and sufficient conditions for unique rational expectations equilibrium

The rational expectations model of the type (A.43) has a unique solution if and only if the winding number of $\det \tilde{A}(\omega)$ is equal to zero.

The winding number of $\det \tilde{A}(\omega)$ is equal to the sum of the partial indices of $\tilde{A}(\omega)$. The partial indices are the exponents of the diagonal elements in the diagonal matrix of the Wiener-Hopf factorization of $\tilde{A}(\omega)$. As an example of the result in practice, for a simple hybrid model of the form

$$A_0 x_t = \beta E_t x_{t+1} + \left(A - BF\right) x_{t-1} + w_t \qquad \text{(A.45)}$$

One has

$$\tilde{A}(\omega) = -\beta e^{i\omega} + A_0 - \left(A - BF\right) e^{-i\omega} \qquad \text{(A.46)}$$

Applying the Wiener-Hopf factorization one can derive the partial indices and check whether the parameter values satisfy the condition for uniqueness.

Appendix 3. Justification of design limits for unstable no control proceses

In order to understand how design limits may be computed in the presence of the nature of the argument, we explicitly solve for a scalar AR(1) case

$$x_t = \beta E_t x_{t+1} + a x_{t-1} + b u_t + \varepsilon_t \qquad \text{(A.47)}$$

with associated scalar feedback rule

$$u_t = ux_{t-1}. \tag{A.48}$$

The $z-$transform of the MA coefficients for controlled system equilibrium may be written

$$G^C(z,\beta) = \frac{\left(g_0^C - \beta^{-1}z\right)}{\left(z - \beta - cz^2\right)} = \frac{g_0^C\left[1-\left(\beta g_0^C\right)^{-1} z\right]}{\left(1 - g_{d,1}^C z\right)\left(1 - g_{d,2}^C z\right)} \tag{A.49}$$

where $c = a + bu$. Let $g_{d,2}^C$ be the root with larger modulus and $g_{d,1}^C$ be the root with smaller modulus. It can be shown that $\lim_{\beta \Rightarrow 0} g_{d,1}^C = c$ and $\lim_{\beta \Rightarrow 0} \left|g_{d,2}^C\right| = \infty$. One can then choose g_0^C so that $\left(\beta g_0^C\right)^{-1} = g_{d,2}^C$ and thus cancel common terms from numerator and denominator of $G^C\left(z,\beta\right)$. We refer to this operation as root cancelling to achieve analyticity in z. It is straightforward to show that $\lim_{\beta \Rightarrow 0} D^C\left(z,\beta\right) = \frac{1}{1-cz}$ (using L'Hospital's rule). Now write an analogous expression for the "pseudo solution" $G^{NC}\left(z,\beta\right)$ and conduct root cancelling analogously to the above. One can then consider the analog to the sensitivity function,

$$S\left(z,\beta\right) = \frac{z - \beta g_0^C}{z - \beta g_0^{NC}} \frac{z-\beta-az^2}{z-\beta-cz^2} = \frac{g_0^C\left[1-\left(\beta g_0^C\right)^{-1} z\right]}{g_0^{NC}\left[1-\left(\beta g_0^{NC}\right)^{-1} z\right]} \frac{z-\beta-az^2}{z-\beta-cz^2} \tag{A.50}$$

Write

$$\frac{z-\beta-az^2}{z-\beta-cz^2} = \frac{1-\beta^{-1}z+\beta^{-1}az^2}{1-\beta^{-1}z+\beta^{-1}cz^2} = \frac{\left(1-g_{d,1}^{NC}z\right)\left(1-g_{d,2}^{NC}z\right)}{\left(1-g_{d,1}^{C}z\right)\left(1-g_{d,2}^{C}z\right)} \tag{A.51}$$

Cancelling roots for both the C and the NC part of the above expressions and collecting terms,

$$S(z,\beta) = \frac{g_{d,2}^{C}\left(1 - g_{d,1}^{C} z\right)}{g_{d,2}^{NC}\left(1 - g_{d,1}^{NC} z\right)} \tag{A.52}$$

We *define* $S\left(z,\beta\right)$ to be the sensitivity function for the hybrid model with parameter β. Note that every mathematical operation that needs to be done to form $S(z,\beta)$ in the definition of sensitivity function above is valid independent of the stability of the underlying stochastic processes just as in the definition of the sensitivity function in Wu and Jonckheere (1992, page 1801). One should think of the construction above as the analog to the open loop poles (the NC case) and the closed loop poles (the C case) of Wu and Jonckheere (1992, p. 1801). The same kind of construction as that above applies to general scalar cases so long as the resulting polynomials are of finite degree. Generalization of this argument to the general scalar case is straightforward. For the backwards case, the analysis hinged upon equation (21) in the text. Factoring the RHS polynomials in equation (21) in the text produces the closed loop poles (open loop poles) analogously to Wu and Jonckheere (1992, p. 1801). For the hybrid matrix case one can follow in an analogous fashion by replacing (A.47) with its matrix analog, generalizing (A.48) to allow for more complicated feedback rules. The same kind of cancellation of appropriate roots we did for the scalar case can be done in with case in order to define the determinant of the sensitivity function $\det\left(S\left(z,\beta\right)\right)$. Note that g_0^{NC}, g_0^{C} are now 2×2 matrices. One might think at first blush that four unknowns cannot be "cancelled" by two roots. But there are symmetries in the structure of this problem that induce dependencies in the matrices of unknowns g_0^{NC}, g_0^{C} so that there are effectively only two unknowns per each matrix. Appendix 2 discusses the construction of the equilibrium MA coefficients in more detail.

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