Intentional Vagueness*

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Abstract

This paper analyzes communication with a language that is vague in the sense that identical messages do not always result in identical interpretations. It is shown that strategic agents frequently add to this vagueness by being intentionally vague, i.e. they deliberately choose less precise messages than they have to among the ones available to them in equilibrium. Having to communicate with a vague language can be welfare enhancing because it mitigates conflict. In equilibria that satisfy a dynamic stability condition intentional vagueness increases with the degree of conflict between sender and receiver.

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1 Introduction

We frequently are intentionally vague, i.e. we make statements that are open to interpretation although more precise statements are available. A variety of possible explanations come to mind: Social norms may prevent us from being too blunt, more precision may require more effort, or vagueness may serve to control the flow of information. In this paper, we focus on the strategic use of vague messages to manipulate information.

We take it as given that language is an imperfect technology that leaves messages subject to interpretation. Language is vague in this sense because the receiver's interpretation of a message, while generally close to the sender's intent, need not perfectly coincide with it. A further source of vagueness, intentional vagueness, results when strategically-motivated senders choose messages with less definite interpretations when they could be more precise.

We ask when exogenous vagueness is compounded by strategic concerns and when, in contrast, it can help moderate strategic concerns and mitigate conflict. Importantly, we show that strategic players try to utilize exogenous vagueness rather than merely to minimize its effects. In our framework, there is a natural coexistence of messages with different endogenously generated degrees of vagueness. Privately informed speakers select from this menu of differentially vague messages and in many cases elect to be more vague rather than less vague.

Some situations and institutions are notorious for vague communication. For instance, in negotiating sexual relationships, interest is typically not expressed too overtly. Instead the parties involved resort to flirting and innuendo. Pinker [18] asserts that "It is in the arena of sexual relationships, however, that the linguistic dance can be its most elaborate." He continues by recounting an event from a Seinfeld episode: "George is asked by his date if he would like to come up for coffee. He declines, explaining that caffeine keeps him up at night. Later he slaps his forehead: '*Coffee* doesn't mean coffee! *Coffee* means sex!'" Similarly, legal statutes are widely perceived as subject to interpretation and there is an ongoing debate about which interpretive stance is appropriate for the judiciary. This is emphasized by Rizzo and Arnold [21] who state that "In reality, however, there are numerous sources of ambiguity and vagueness in any statute, ranging from disputes concerning the meaning of simple statutory language to uncertainties about overall legislative intent." A third example is that of central-bank announcements. According to Stein [26] (p. 32) "It is not that the Fed never makes any policy statements. Rather, the common complaint is that these statements are vague, or difficult to interpret." Alan Greenspan, who was famous for his inscrutability, stated that "Since I've become a central banker, I've learned to mumble with great incoherence. If I seem unduly clear to you, you must have misunderstood what I said." (quoted in Geraats [8]).

Common to each of these situations is interpretability of messages. A given message does not produce a single predictable response. Instead, the examples appear to be better described by messages that induce a distribution of interpretations, some close to the speaker's intent, and some that qualify as misunderstandings.

We model interpretability as noise in the communication process, where interpretations are centered at but generally deviate from the intended message. This noisy communication technology exogenous. We then investigate the strategic use of such interpretable messages. Our central findings are that (i) an important component of vagueness is endogenous, with speakers frequently choosing less precise messages than are available and (ii) this intentional vagueness increases with the conflict between sender and receiver. In addition, we confirm our finding from earlier work (Blume *et al.* [2]) that exogenous vagueness may be efficiency enhancing when there is sufficient conflict of interest between the parties.

The exogenous component of vagueness is represented by the noise in the language, formally measured by the variance of the distribution of interpretations conditional on the sent message. The endogenous component comes from the fact that similar messages are more likely to induce similar interpretations, and one type of speaker may send a message close to the preferred message of another type. This increases overlap in the distribution of interpretations and therefore compromises the integrity of both messages. We show that frequently speakers of some type will take advantage of this option and be intentionally vague by sending a message close to, although not identical with, the preferred message of another type, while they could more reliably identify their own type by sending a more distant message. A potential welfare benefit from increasing the overlap of distributions and thereby increasing vagueness is that it has a moderating effect on the listener's response to the message and therefore diminishes the scope for strategic manipulation by the speaker. We show that there are instances in which this effect enables communication that could not be achieved with a precise language.

The term 'vagueness' itself is subject to, sometimes contentious, interpretation. We are not interested in being drawn into this debate, but we should differentiate our usage from some related work in the literature. Specifically, there is a sense in which equilibrium messages in the model of strategic information transmission studied by Crawford and Sobel [3] (henceforth CS) are vague. In the CS model, whenever there is conflict of interest between the sender and receiver, any equilibrium message cannot be fully revealing but has as its referent a nontrivial subset of the type space. Unlike in our model, however, there is no role for interpretation in the CS model. Regardless of which message he sends, the sender knows exactly how the receiver will respond to that message. Likewise the receiver, while he may be uncertain about which type sent a given message, knows precisely which set of types the message refers to and which action he is expected to take.

Interestingly, there is another feature of equilibrium behavior in the CS model that has not received as much attention but comes to the fore in our model: Taking the receiver's equilibrium strategy as fixed, the receiver prefers for some subset of types that they deviate from their equilibrium strategies. These types engage in deception in the sense that they do not induce the action closest to their type. In the CS model this deceiver set is generally a non-convex subset of the type space. In our model the deceiver set is convex and includes all types that are not constrained by the message space. But in both models the sender is intentionally vague for a subset of the type space that has positive probability. The sender is less precise than she could be with the messages that are available to her in equilibrium, where by "less precise" we mean that she does not send the message that, given the receiver's equilibrium response, would maximize receiver utility.

The purpose of the present paper is to yield a deeper understanding of the phenomenon of intentional vagueness. Specifically, we show that it is not specific to concealment of information through pooling of types that is characteristic of CS equilibria. In our environment in equilibrium all types whose messages do not coincide with the boundaries of the message space send distinct messages and at the same time are intentionally vague.

The paper is structured as follows. The next section describes our model for the case of two sender types, which is our main focus. We provide necessary and sufficient conditions for the existence of a communicative equilibrium, and demonstrate that the low type of the sender will frequently be intentionally vague in such an equilibrium. We then define *excess demand for vagueness* and show how it can be used to study the structure of the equilibrium set. We also introduce a simple dynamic according to which the low-type sender increases vagueness by raising his message whenever his excess demand for vagueness is positive and *vice versa*. For equilibria that are hyperbolically stable under this dynamic, we show that equilibrium vagueness increases with the degree of conflict between sender and receiver. We also demonstrate the existence of a stable equilibrium and show that pooling is asymptotically stable if and only if the degree of conflict is high relative to the prior probability of a high type. Section 3 generalizes our model to an arbitrary finite number, n, of types, extends the concept of *intentional vagueness* to the n-type case, demonstrates that in any monotone equilibrium pooling can only occur at the top and the bottom of the type space and that the remaining (interior) types will all be intentionally vague. Section 4 discusses related literature and section 5 concludes.

2 The Model

2.1 Setup

There are two players, a sender, S, and a receiver, R. At the beginning of the game the sender observes a private signal, her *type*. The sender's type t takes one of two values, 0 with probability $(1-\theta)$ or 1 with probability $\theta \in (0,1)$.[1] After observing t, the sender sends a message m from the set $M = [0,1]$. If message m is sent, the receiver's *interpretation* $q \in \mathbb{R}$ is drawn from a normal distribution with mean m and variance σ^2. The receiver observes q but not the sender's type or the message she actually sent, and takes an action, $a \in A = \mathbb{R}$. Payoffs for the sender and receiver are given by $U^S(a,t,b) = -(t+b-a)^2$ and $U^R(a,t) = -(t-a)^2$. So, messages have no direct effect on payoffs and $b > 0$ (the *bias* of the sender) measures the degree to which the sender's and the receiver's preferences coincide.

The sender's strategy is a pair $\mathbf{m} = (m_0, m_1)$, where $m_t \in M$ is the message she sends when her type is t; we assume without loss of generality that $m_0 \leq m_1$.[2] The receiver's strategy is an action function $\mathbf{a} : \mathbb{R} \to \mathbb{R}$ describing which action he chooses for each message he might receive. In a (perfect Bayesian) equilibrium, each player's strategy is optimal given her opponent's strategy and her beliefs, and those beliefs are derived from Bayes' rule whenever possible. For the sender, this means that her chosen message must maximize her expected payoff given her type. Since the normal distribution has full support, the receiver's beliefs about t are uniquely determined by Bayes' rule from the sender's strategy; and with quadratic preferences, his expected utility is maximized when he chooses an action equal to his expectation of t, which is simply his belief that $t = 1$. To save notation, we suppress the receiver's beliefs in our formal definition of equilibrium.

Definition 1 *An* equilibrium *is a strategy profile* $(\mathbf{m}^*, \mathbf{a}^*)$ *where*

$$m_t^* \in \arg\max_{m\in[0,1]} \int_{-\infty}^{\infty} U^S(\mathbf{a}^*(q), t, b) \cdot \phi_{m,\sigma^2}(q)\, dq, \qquad \textit{for } t = 0,1 \tag{1}$$

and

$$\mathbf{a}^*(q) = \frac{\theta \cdot \phi_{m_1^*,\sigma^2}(q)}{(1-\theta)\cdot \phi_{m_0^*,\sigma^2}(q) + \theta \cdot \phi_{m_1^*,\sigma^2}(q)}, \qquad \textit{for all } q \in \mathbb{R}. \tag{2}$$

[1] The two-type model suffices to make our main point that strategic players add intentional vagueness to a vague language. Furthermore, it allows us to provide a nearly complete characterization of the conditions under which communicative equilibria exist for the entire range of prior type distributions. We will show later how to extend our analysis to an arbitrary finite number of types.

[2] A simple symmetry argument shows that if there is an equilibrium in which the sender chooses messages m_0 and m_1 with $m_0 > m_1$, then there is a corresponding equilibrium in which she chooses messages $1-m_0$ and $1-m_1$. Our focus on pure-strategy equilibria is partially justified by the fact that whenever a communicative equilibrium exists, it is strict (that is both players have unique best replies) and therefore forms a minimal *curb* set [1] while no mixed equilibrium belongs to a minimal *curb* set.

($\phi_{m,\sigma^2}(q) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(q-m)^2}{2\sigma^2}}$ is the probability density of the normal distribution with mean m and variance σ^2.)

Whatever the parameters of the model, for every $m \in [0,1]$, there is a (pooling) equilibrium with $m_0^* = m_1^* = m$. In this case, the receiver's observed interpretation conveys no information about the sender's type, so his expectation of t and hence his action are equal to θ (his prior expectation); this strategy does not depend on q, so whatever her type, the sender can do nothing better (or worse) than send message m.

We are more interested in the possible existence of *communicative* equilibria, where $m_0^* \neq m_1^*$. The next subsection investigates when a communicative equilibrium exists.

2.2 Existence of a communicative equilibrium

It is well known that in cheap talk models, communication is possible only if the interests of the sender and the receiver are sufficiently closely aligned. In the present context, it is easy to see that there cannot be a communicative equilibrium if $b \geq 1$; on the other hand, if $b = 0$, there is always a communicative equilibrium with *maximal differentiation*, where $m_0 = 0$ and $m_1 = 1$. Although it is not in general possible to provide an analytic solution for communicative equilibria when b takes on an intermediate value, we are able to do so for the special case where $b = \frac{1}{2}$.

Proposition 1 *Suppose the sender's bias b is equal to $\frac{1}{2}$. Then*

1. *if $\theta \leq \frac{1}{2}$, there is no communicative equilibrium;*

2. *if $\theta > \frac{1}{2}$, there is a unique communicative equilibrium with*

$$m_0^* = \max\left\{0, 1 - \sigma\sqrt{2\log\left(\frac{\theta}{1-\theta}\right)}\right\} \qquad \text{and} \qquad m_1^* = 1.$$

The proof of this and all other results can be found in the appendix. First notice that, when a communicative equilibrium exists, the type-1 sender always chooses $m_1^* = 1$. This is because she wants a higher action $(1 + b)$ than the receiver, regardless of his beliefs, and hence chooses the highest message available. This observation holds true in any communicative equilibrium, regardless of parameter values (see Lemma 3 below). It follows that a communicative equilibrium can be completely characterized by the message m_0^* chosen by the type-0 sender. In the equilibrium described above, m_0^* is a (weakly) decreasing function of θ and of σ. When θ is close to $\frac{1}{2}$, m_0^* is strictly between 0 and 1: the type-0 sender sends a different message from her type-1 counterpart, but does not identify herself as precisely as she could: she is *intentionally vague*. (Recall from the introduction that we characterized the precision of a message by comparing it to the message that, given his equilibrium response, the receiver would have wanted the sender to send. In any communicative equilibrium, the receiver's utility is maximized when the type-0 sender chooses $m_0 = 0$, so we have intentional vagueness whenever $m_0^* > 0$, and furthermore the value of m_0^* measures the degree of intentional vagueness, on a 0 to 1 scale.) For higher values of θ, the equilibrium exhibits *maximal differentiation*, with $m_0^* = 0$ and $m_1^* = 1$. Figure 1 below plots m_0^* as a function of θ when $\sigma = \frac{1}{2}$.

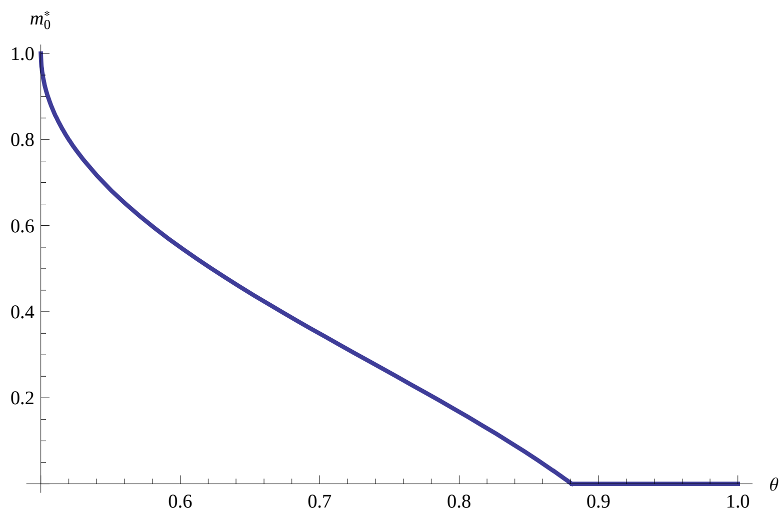

Figure 1: Equilibrium values of m_0 when $b = \frac{1}{2}$ and $\sigma = \frac{1}{2}$.

To understand better what happens in a communicative equilibrium, consider the case where $\theta = \frac{3}{4}$ (again, with $\sigma = \frac{1}{2}$). Figure 2 below plots the best-response action functions of the receiver when the type-1 sender chooses:

(i) $m_0' = 0$ (dotted red curve);

(ii) $m_0^* = 0.26$ (solid black curve); and

(iii) $m_0'' = 0.52$ (dotted blue curve).

In the diagram we consider three different messages the type-0 sender *could* send in a candidate equilibrium. In each case, the optimal message for her to *actually* send is where the resulting action function crosses $a = \frac{1}{2}$. This follows from two facts, both proved in the appendix: (i) the action function is rotationally symmetric about this point (Lemma 1); and (ii) the type-0 sender's utility is strictly quasiconcave in m_0 (Lemma 4). Since $b = \frac{1}{2}$, it follows from (i) that, for any given action function, the sender's utility has reflectional symmetry about the interpretation where $a = \frac{1}{2}$; given (ii), this must correspond with the optimal message for the sender to choose. At $m_0' = 0$, then, the sender wants to send a higher message than she currently is (indicated by the vertical red line), while at m_0'', the sender wants to send a lower message (given by the vertical blue line). When m_0^*, the message the sender wants to send and the message she is sending coincide and we have an equilibrium.

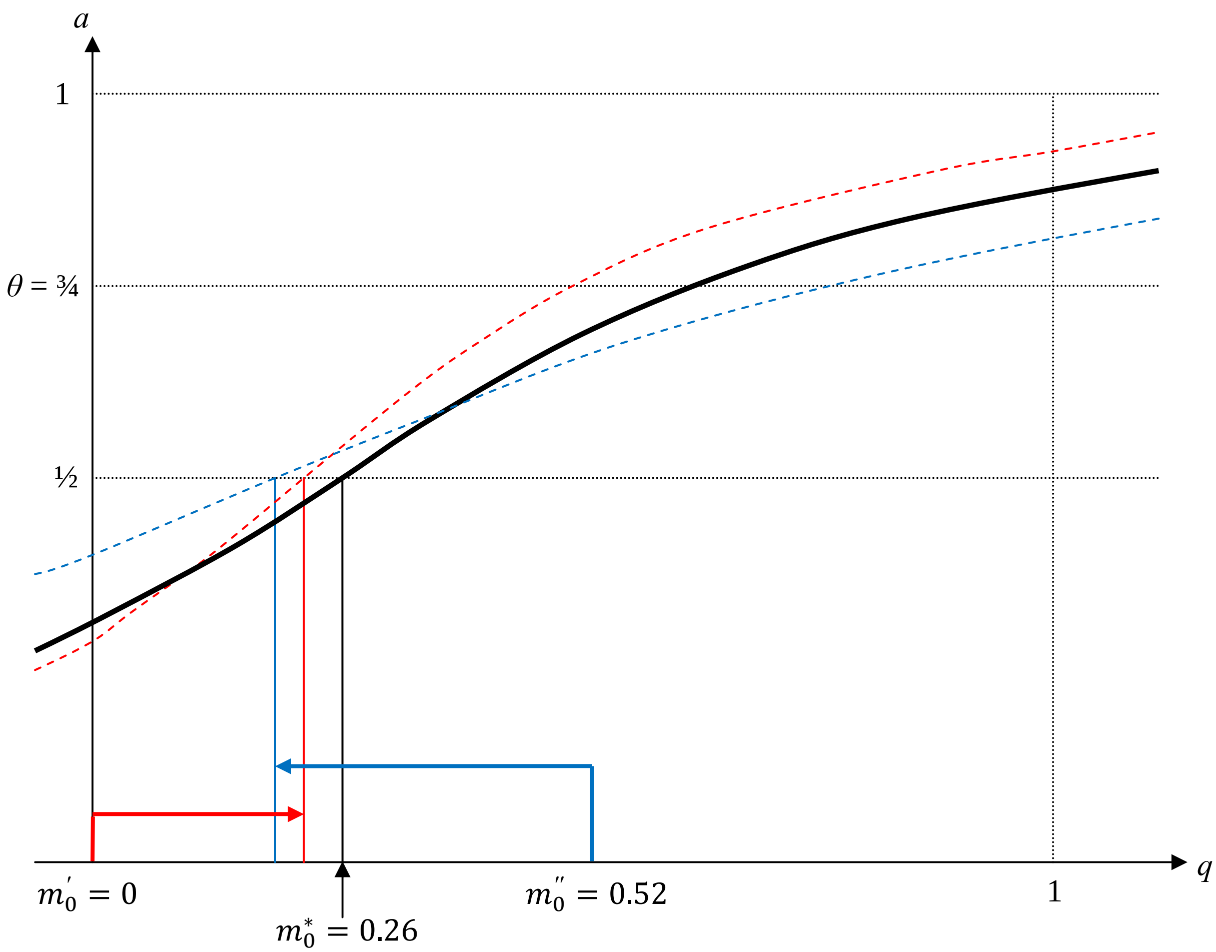

Figure 2: Communicative equilibrium, $b = \frac{1}{2}$, $\theta = \frac{3}{4}$, and $\sigma = \frac{1}{2}$.

Our next two results show, if we fix θ and σ, the degree of vagueness chosen by the type-0 sender in equilibrium depends on her bias. Proposition 2 states that, if b is low enough, the unique communicative equilibrium exhibits maximal differentiation, i.e. $m_0^* = 0$. This result is reminiscent of CS's finding that more communication is possible when the sender's and receiver's incentives are closely aligned.

Proposition 2 *There exists $\underline{b} \in (0, 1)$ such that for all $b \in [0, \underline{b}]$ there is a unique communicative equilibrium, in which $m_0^* = 0$.*

If the bias is higher than this value, on the other hand, the type-0 sender may choose to be vague. It might be tempting to conclude that the equilibrium degree of vagueness is a (weakly) increasing function of the sender's bias b. Under an additional stability condition, this turns out to be true, though we can find counterexamples when this condition does not hold. We explain why, and explore the comparative statics of the model in more detail in section 2.4. Here we present a weaker result, Proposition 3, which states that, for any message $m \in (0,1)$, we can find a b such that there is a communicative equilibrium with $m_0^* = m$.

Proposition 3 *For any message $m \in [0,1)$, there exists a bias $b \in (0,1)$ for which $m_0^* = m$ in a communicative equilibrium.*

We now address the issue of when a communicative equilibrium exists. Our next result establishes a sufficient condition for existence.

Proposition 4 *The condition $b < \theta$ is sufficient for the existence of a communicative equilibrium.*

To understand why the value of θ, the prior probability that the sender's type is 1, is important for the possibility of communication, observe that θ is also equal to the receiver's expectation of t in a pooling equilibrium. Hence whenever $b < \theta$, the type-0 sender has some incentive to separate herself from the higher type. In the absence of noise, separation is "all-or-nothing", and hence can be achieved in equilibrum only when $b \leq \frac{1}{2}$; when the bias is larger than this, the type-0 sender would prefer the action ($a = 1$) induced by the other type to the action ($a = 0$) she induces herself. In the present setting, however, noise prevents complete separation and by choosing a message arbitrarily close to $m_1 = 1$, the type-0 sender can obtain (in equilibrium) an expected action that is arbitrarily close to θ. As long as $b < \theta$, then, we can always find a message $m_0 < m_1$ which generates an equilibrium with some degree of separation, and therefore some communication.

When θ is larger than $\frac{1}{2}$, the condition $b < \theta$ is also necessary for the existence of an informative equilibrium.

Proposition 5 *If $\theta \geq \frac{1}{2}$, the condition $b < \theta$ is necessary for the existence of a communicative equilibrium.*

When θ is less than $\frac{1}{2}$, however, communication may be possible even if $b > \theta$. For example, if $\theta = 0.1$, $b = 0.13$ and $\sigma = 1$, there are in fact two communicative equilibria, with $m_0^* = 0$ and with $m_0^* = 0.27$ (we explain how to find such equilibria in the next section); with $\theta = 0.1$, $b = 0.3$ and $\sigma = 1$, however, no communicative equilibrium exists. Although we do not in general know how large b has to be before communication breaks down, the next results states that $b \geq \frac{1}{2}$ is too large.

Proposition 6 *If $\theta < \frac{1}{2}$, the condition $b < \frac{1}{2}$ is necessary for the existence of a communicative equilibrium.*

Figure 3 below summarizes Propositions 4 – 6, and shows when communication is possible with and without noise.

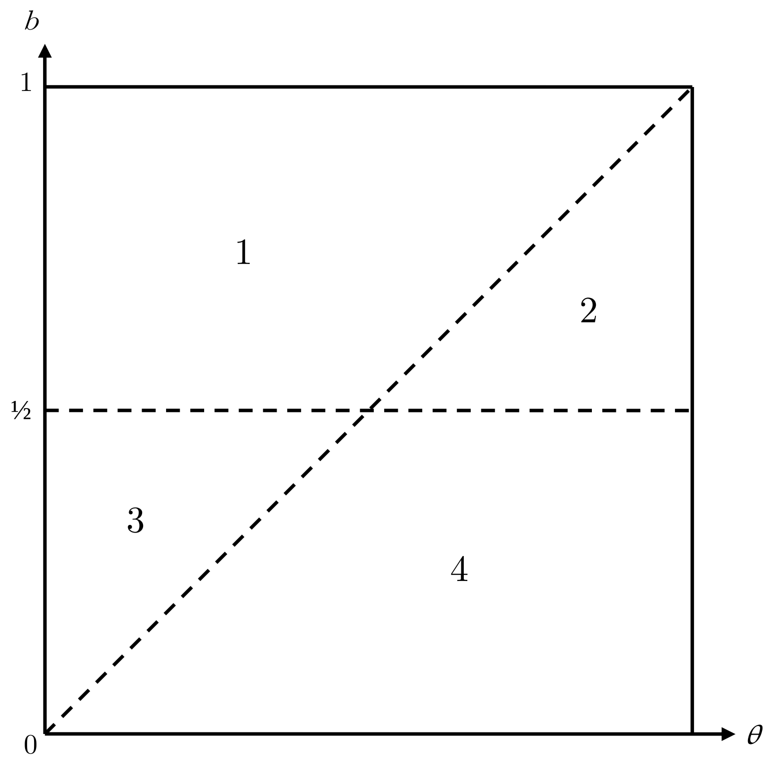

Figure 3: Existence of communicative equilibria with and without noise.

1. $b > \frac{1}{2}$, $b > \theta$: communication is not possible with or without noise;
2. $b > \frac{1}{2}$, $b < \theta$: communication is possible only with noise;
3. $b < \frac{1}{2}$, $b > \theta$: communication is possible without noise, and may be possible with noise;
4. $b < \frac{1}{2}$, $b < \theta$: communication is possible with and without noise.

Note that in region 2 noise generates a potential Pareto improvement: without it, no meaningful communication would be possible. This confirms our observation in earlier work [2] that noise can be efficiency enhancing.

2.3 Finding communicative equilibria and the excess demand for vagueness

As mentioned in the previous section, we are unable to find communicative equilibria for general parameter values using analytical methods. By showing that equilibrium can be characterized by a single first-order condition, however, we can compute equilibria numerically. First recall that, in equilibrium, the receiver chooses an action that is equal to his expectation of the sender's type, given his interpretation q. Let α denote this expectation as a function of the message m_0 and m_1 that the receiver expects the sender to use, i.e.

$$\alpha\left(q, m_0, m_1, \theta, \sigma\right) \equiv \frac{\theta \cdot \phi_{m_1,\sigma^2}\left(q\right)}{\left(1-\theta\right) \cdot \phi_{m_0,\sigma^2}\left(q\right) + \theta \cdot \phi_{m_1,\sigma^2}\left(q\right)}.$$

Next, define a function V as follows:

$$V\left(b, m_0, m'\right) \equiv \int_{-\infty}^{\infty} -\left(b - \alpha\left(q, m_0, 1, \theta, \sigma^2\right)\right)^2 \cdot \phi_{m',\sigma^2}\left(q\right) dq.$$

So V gives us the expected payoff of the type-0 sender from sending message m' when the receiver expects her to use message m_0 (assuming, as in the previous section, that the type-1 sender sends message $m_1 = 1$). We show in the appendix (Lemma 4) that the first-order condition $V_3\left(b, m_0, m'\right) = 0$ is sufficient for m' to be the unique global maximizer of $V\left(b, m_0, \cdot\right)$. If m' coincides with $m_0 \neq 1$, we have a communicative equilibrium. By plotting $V_3\left(b, m, m\right)$ between $m = 0$ and $m = 1$, then, we can find all such equilibria. Given the non-negativity constraint, two kinds of solution are possible:

1. $m_0^* = 0$: communicative equilibrium with maximal differentiation ($V_3\left(b, 0, 0\right) \leq 0$);

2. $m_0^* \in (0, 1)$: communicative equilibrium with intentional vagueness ($V_3\left(b, m_0^*, m_0^*\right) = 0$).

We find it useful to think of $z\left(b, m\right) \equiv V_3\left(b, m, m\right)$ as measuring the *excess demand for vagueness*: this function tells us how much the sender of type 0 would benefit from sending a slightly higher, and hence more vague, message than the receiver is expecting.

Figure 4 shows the excess demand for vagueness for various parameter values.

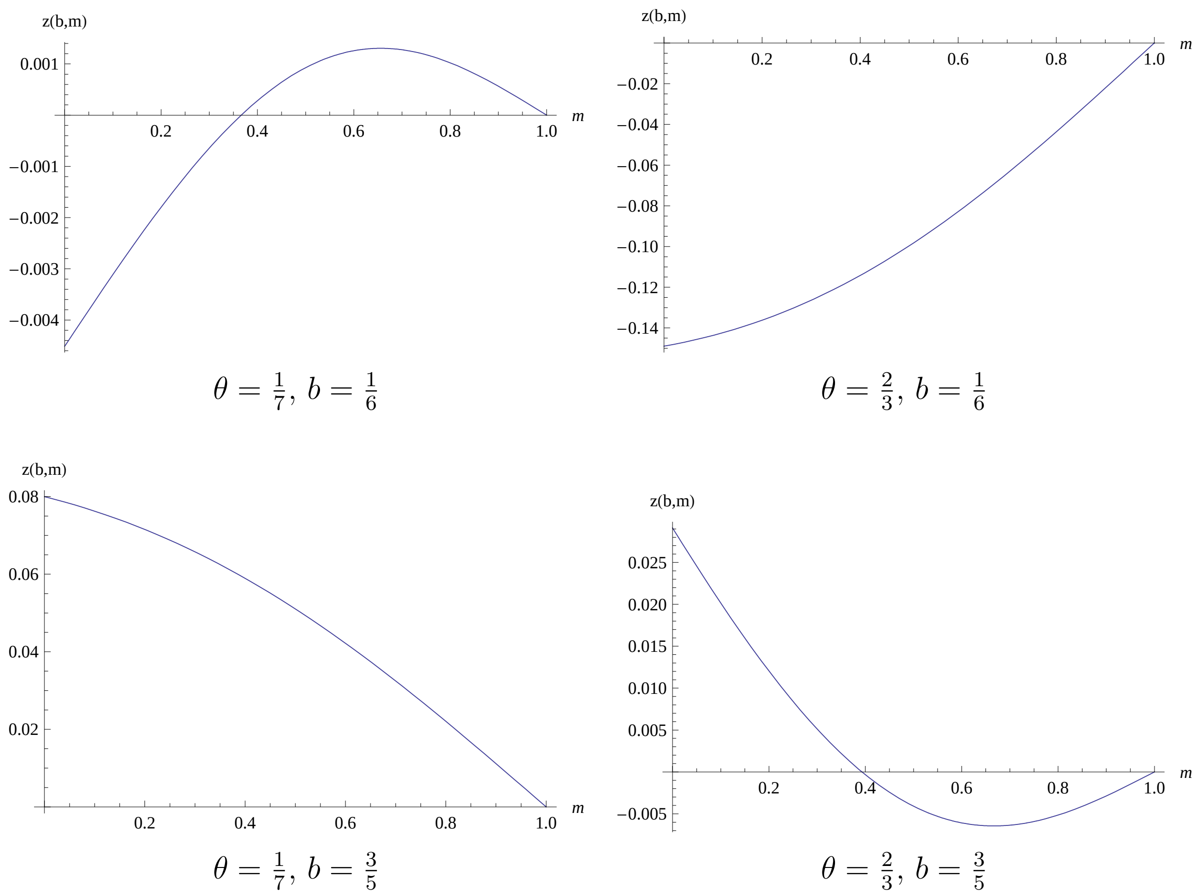

Figure 4: Excess demand for vagueness for different parameter values ($\sigma = 1$)

Notice that the excess demand function is always zero at $m = 1$: in this case the expectation function α is flat, so the sender is obtains no benefit (or loss) from changing her message. When $b < \theta$, the function is positively sloped at $m = 1$; this explains Proposition 4: the function must either still be negative when $m = 0$, in which case we have a communicative equilibrium with maximal differentiation (case (ii)) or it must be positive. If the latter, it follows from continuity and the Intermediate Value Theorem that we have a communicative equilibrium with intentional vagueness (case (iv)). When $b > \theta$, the function is negatively sloped at $m' = 1$, and there is no guarantee of a communicative equilibrium. Indeed, if $\theta \geq \frac{1}{2}$ or $b \geq \frac{1}{2}$, Propositions 5 and 6 tell us that there is no communicative equilibrium (case (iii)). If $\theta < \frac{1}{2}$ and $b < \frac{1}{2}$, on the other hand, one or more communicative equilibria may exist (case (i)).

2.4 Equilibrium Selection and Comparative statics in b: An Application of the Correspondence Principle

In this section we introduce a dynamic on the space of sender strategies that have the form $(m, 1)$, where the type-0 sender sends message m and the type-1 sender sends message 1.

Doing so serves the dual purposes of helping us obtain sharp comparative statics predictions for variations in the sender's bias and selecting equilibria in this class of strategies.

Recall the excess demand for vagueness function $z(b,m) \equiv V_3(b,m,m)$ introduced in the previous section. Analogous with the excess demand function in Walrasian general equilibrium theory, this function measures the degree to which the sender wishes to increase vagueness above the current level. Assume for simplicity that the receiver always best responds to the sender's strategy and that the type-0 sender adjusts her message m in the direction of her excess demand for vagueness. In continuous time, this suggests a dynamic of the form

$$\dot{m} = z(b,m),$$

that is the time derivative of m is equal to the excess demand for vagueness.[3] Call this the *vagueness dynamic*. A type-0 message $m^* \in (0,1)$ is a stationary point of the vagueness dynamic if $z(b,m^*) = 0$. If we call equilibria with $m^* \in (0,1)$ interior equilibria, then for messages in the range $m \in (0,1)$ there is a one-to-one correspondence between interior equilibria of the communication game and stationary points of the vagueness dynamic.

Of particular interest are stationary points m^* with the property $\frac{\partial z(b,m^*)}{\partial m} < 0$. Such stationary points are called *hyperbolically stable* (Hirsch and Smale [10], p. 187). Hyperbolic stability implies asymptotic stability (i.e. after any sufficiently small displacement the dynamic will take us back to m^*), local uniqueness, and structural stability (Hirsch and Smale [10], p. 305) (i.e. for sufficiently small perturbations of the function $z(b,\cdot)$ the resulting dynamical system retains a unique hyperbolic equilibrium in a neighborhood of the original equilibrium). Our principal reason for being interested in hyperbolic stability is that it has strong implications for comparative statics in our model. This is a manifestation of the *correspondence principle* that was formulated by Samuelson [22] and [23]: Dynamic properties of equilibrium frequently have implications for the comparative statics of equilibrium.[4]

Consider the comparative statics in b. Let m^* be an interior hyperbolically stable equilibrium. Then, given that z is continuously differentiable at (b,m^*), the Implicit Function Theorem tells us that there exists an open interval I that contains b and a local solution $m^*(\cdot): I \to \mathbb{R}$ that satisfies

$$z(b', m^*(b')) \equiv 0 \ \forall b' \in I$$

and $m^*(b) = m^*$. Differentiating the identity and evaluating at $b' = b$, we find that

$$\frac{dm^*(b)}{db} = -\frac{\partial z(b,m^*)}{\partial b} \Big/ \frac{\partial z(b,m^*)}{\partial m}.$$

[3] Our results would remain unchanged if we considered dynamics in the more general class $\dot{m} = \xi(z(b,m))$, where $\xi: \mathbb{R} \to \mathbb{R}$ is any continuously differentiable function with $\xi'(z) > 0$ for all $z \in \mathbb{R}$ and $\xi(0) = 0$.

[4] For a concise discussion of the correspondence principle, its history and related literature, see Echenique [6].

The denominator of the expression on the right–hand side is negative because m^* is hyperbolically stable; we show in appendix (Lemma 5) that $\frac{\partial z(b,m^*)}{\partial b} > 0$.

To summarize, call any equilibrium in which the sender uses the strategy $(m^*, 1)$ where m^* is a hyperbolically stable stationary point of the vagueness dynamic a *hyperbolically stable equilibrium with low message* m^*. Then we have the following important comparative statics result:

Proposition 7 *Near any hyperbolically stable equilibrium with low message* m^**, the low type's equilibrium message increases as a function of the sender's bias, i.e.*

$$\frac{dm^*(b)}{db} > 0.$$

This result has a natural interpretation. At any equilibrium that is dynamically stable increasing the sender's bias results in increased intentional vagueness: More bias implies more obfuscation.

A second reason for introducing the vagueness dynamic is that is suggests a way to select among equilibria on the basis of their stability properties. The following result establishes that (Lyapunov) stable equilibria always exist and shows that the vagueness dynamic rejects pooling if the bias is small relative to θ.

Proposition 8 *There exists at least one stable equilibrium. Pooling at* $m^* = 1$ *is asymptotically stable if* $b > \theta$ *and unstable if* $b < \theta$*.*

3 Monotone equilibria with an arbitrary finite number of types

In this section we generalize our model to any finite number of types and characterizing the set of monotone equilibria. Our reason for being interested in monotone equilibria is that they have an attractive stability property: If there is a communicative monotone equilibrium it forms a singleton (and therefore minimal) *curb* set (Basu and Weibull [1]) whereas no pooling equilibrium can be a member of a minimal *curb* set. We characterize monotone equilibria by showing that pooling can only occur at the top or the bottom of of the type space. The remaining *interior types* all send distinct messages. Our main result of this section is that all interior types are intentionally vague in the sense that they distort their message relative to the receiver's preferred message given the receiver's equilibrium strategy. Finally, using the characterization of monotone equilibria, we calculate equilibria for a few special cases in order to make some additional observations: (1) monotonicity can be infectious in that ordering the messages of some types may restrict the order of messages used by the remaining types, (2) dynamic stability retains some selective power, (3) noise can enable full separation with large biases, and (4) (in contrast to (3)) noise can prevent full separation even when sender and receiver have common interests.

We consider the same model as in the previous section, except that the type space is an arbitrary finite set $T \subset [0,1]$. There is a common prior distribution ν on T so that for any set $\Theta \subset T$, $\nu(\Theta)$ denotes the probability that the sender's type belongs to the set Θ. With n types, a pure strategy for the sender is a vector $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, where m_t denotes the message sent by type t. Say that the strategy $\mathbf{m}$ is *monotone* if $t > s \Rightarrow m_t \geq m_s$; it is *communicative* if there exists a pair of types t' and t'' with $m_{t'} \neq m_{t''}$.

Before characterizing monotone communicative equilibria, it is worth noting that when such an equilibrium exists we have a simple refinement argument that rules out pooling. We show in the appendix (Lemma 5) that in a monotone communicative equilibrium the receiver's action rule is strictly increasing. Therefore by Lemma 4 (appendix) the sender has a unique best reply, and consequently any monotone communicative equilibrium is strict. A strict equilibrium trivially satisfies the property that it is minimal among sets of strategies that include all their best replies; i.e. it is a minimal *curb* set as defined by Basu and Weibull [1]. Interestingly, whenever a monotone communicative equilibrium e_c exists, no pooling equilibrium, where the receiver's action is invariant to the interpretation, belongs to a minimal *curb* set. This can be seen by way of contradiction: If a pooling equilibrium e_p did belong to an minimal *curb* set C, then C would have to include every sender strategy and corresponding best reply of the receiver. Hence C would have to include e_c, which is a *curb* set in its own right. Therefore C could not be minimal. Our next result characterizes the monotone communicative equilibria of our model.

Proposition 9 *In a monotone communicative pure–strategy equilibrium, if distinct types s and t send a common message m, then either $m = 0$ or $m = 1$.*

In the case with two types we identified intentional vagueness with the low type sending a message that is less precise than he could be given the receiver's equilibrium action rule. With more than two types a natural generalization of this idea is to ask how close the sender's chosen message is to the one the receiver would want him to choose taking the receiver's equilibrium action rule as fixed. For types who do not send an extreme message of either 0 or 1 in equilibrium there is a simple answer:

Proposition 10 *Fix a monotone communicative equilibrium with equilibrium message $m_t \in (0,1)$ for some type t. Then, taking the equilibrium action rule as given, there is a message $m < m_t$ that the receiver would prefer the sender to send.*

Proof. The receiver's utility coincides with the utility of a sender whose type is less than t. Given the strict monotonicity of the receiver's action rule from Lemma 6, single crossing and SMLRP, this type would want to send a message less than m_t. ■

With more than two types we cannot expect the same strong comparative statics result to hold that relates the degree of intentional vagueness to the level of conflict in the two-type case, where we have at a maximum one interior type. With multiple interior types, the equilibrium response of one of these types to an increase in b indirectly affects the incentives that govern another interior type's equilibrium response to the same change. A given interior type's incentive to raise his message in response to an increase in b for a fixed action function of the receiver may be reversed under the influence of changes in the receiver's action function that result from the adjustments of other types. In addition, the message that the receiver would like a given type to send varies with the equilibrium messages of other types. Therefore, even if all interior types were to raise there equilibrium messages in response to and increase in b it need not be the case that that they are all becoming more intentionally vague according to our definition.

The forces that gave us the comparative statics result with two types, however, are still at work. In the n-type case let $\mathbf{a}(q, \mathbf{m})$ denote the receiver's best reply to interpretation q when he expects the sender to use the strategy $\mathbf{m}$. Then we can define the payoff of type t from sending message m_t' when the receiver expects the sender to use strategy $\mathbf{m}$ as

$$V(b, \mathbf{m}, m_t'; t) \equiv \int_{-\infty}^{\infty} U^s \left(\mathbf{a}\left(q, \mathbf{m}\right), t, b\right) \phi_{m_t', \sigma^2}\left(q\right) dq.$$

and adapt our definition of the excess demand for vagueness for type t as follows

$$z(b, \mathbf{m}; t) \equiv V_3(b, \mathbf{m}, m_t; t).$$

Then a straightforward generalization of Lemma 5 in the appendix (using Lemma 6) establishes

Proposition 11 *For any monotone communicative sender strategy* $\mathbf{m}$ *and type* t*,*

$$z_1(b, \mathbf{m}; t) > 0;$$

i.e., fixing the receiver's best reply to a monotone communicative sender strategy each sender type's excess demand for vagueness increases with the level of conflict.

Using our result that pooling can only occur at the top or the bottom of T, one can calculate equilibria with more than two types from the sender's first-order conditions. For example, with nine equally spaced types (type 0,…, type 8), a moderate variance ($\sigma = .3$) a small bias ($b = .03$) and a uniform prior there exists an equilibrium in which the three lowest types send a common message, 0, the four highest types send a common message, 1, and the two interior types 3 and 4 send distinct messages. Conditional, on the non-interior types behaving as specified, the first panel in the following figure plots the excess demand for vagueness of the two interior types as functions of their messages (m_3 for type 3 and m_4 for type 4), in red for type 3, and in green for type 4; for comparison purposes we also plot, in blue, the horizontal surface that corresponds to an excess demand for vagueness of zero. The second panel plots the excess demand for vagueness of type 2, the highest of the three low types who are meant to pool on message 0. The third panel plots the excess demand for vagueness of type 5, the lowest of the four high types who are meant to pool on message 1.

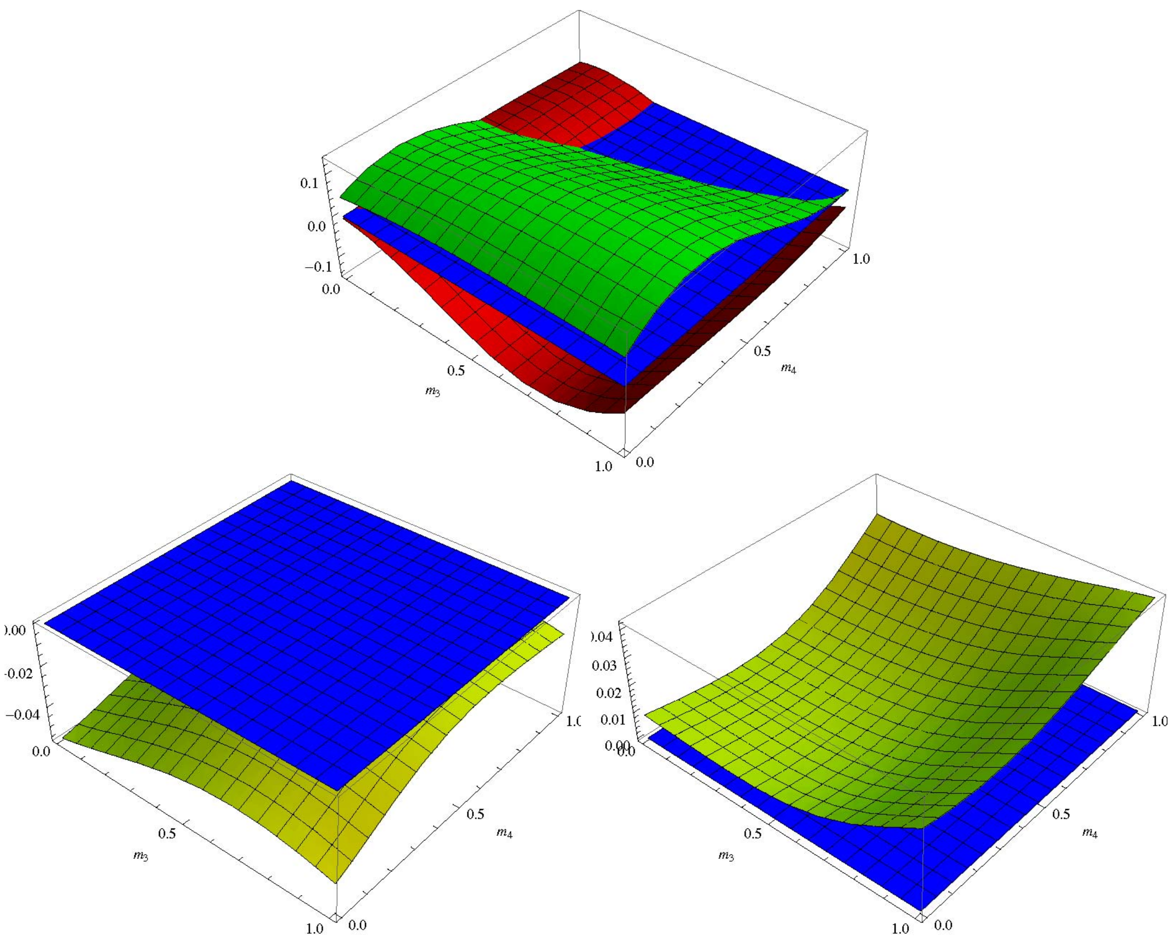

Figure 5: Equilibria with nine types and positive bias

We have an equilibrium with types 3 and 4 sending messages m_3^* and m_4^* provided (1) the two excess demands in the first panel simultaneously equal zero at $(m_3, m_4) = (m_3^*, m_4^*)$ (i.e. they intersect the blue surface at that point), (2) the excess demand for vagueness of type 2 is non-positive at $(m_3, m_4) = (m_3^*, m_4^*)$ (i.e. it is underneath the blue surface), and (3) the excess demand for vagueness of type 5 is non-negative at $(m_3, m_4) = (m_3^*, m_4^*)$ (i.e. it is above the blue surface). This uses the fact that with a monotone action rule for the receiver MLRP and single crossing imply that types 0 and 1 prefer to send a message that is no higher than type 2's message and types 6, 7 and 8 prefer to send a message that is no lower than the message sent by type 5.

It is noteworthy that conditional on the extreme types behaving as specified, the equilibrium with (m_3^*, m_4^*) is unique; for example, there is no equilibrium in which we can simply switch the ordering of messages of the two interior types, or an equilibrium in which these types pool with each other or with any of the remaining types. In this sense monotonicity for the types 0, 1, 2 and 5, 6, 7, and 8 is infectious for types 3 and 4. Note also that the equilibrium we have identified is asymptotically stable under a dynamic that generalizes the one discussed in the last section: Types 0, 1 and 2 have strictly negative excess demands for vagueness in a neighborhood of the equilibrium and therefore their messages would converge to the boundary where $m_0^* = m_1^* = m_2^* = 0$; types 5, 6, 7 and 8 have strictly positive excess demands for vagueness in a neighborhood of the equilibrium and therefore their messages would converge to the boundary where $m_5^* = m_6^* = m_7^* = m_8^* = 1$; the graphs of the excess demands for vagueness functions of types 3 and 4 intersect the zero surface and each other transversally and therefore in a neighborhood of the equilibrium their messages converge to m_3^* and m_4^*.

The effect of noise on sender separation is ambiguous and depends on the prior distribution, which we will demonstrate with two examples. In the first example, with four equally spaced types (type 0, ..., type 3 at 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1), if there is no noise, then a bias of .2 is not consistent with full separation (e.g. conditional on full separation, type 0 would want to mimic type 1), regardless of the prior distribution of types. If we introduce a moderate level of noise ($\sigma = .2$), then for a type distribution where type $i + 1$ is ten times as likely as type i, there exists a separating equilibrium in which type 0 sends message 0, type 3 sends message 1 and the messages sent by the remaining two types are given by the simultaneous intersection of the graphs of the excess demand functions for vagueness of type 1 (red) and type 2 (green) with the zero surface (blue) in Figure 6.

In our second example, with four types, 0, 1, 2 and 3, the effect of noise on separation is reversed. Suppose that sender and receiver have common interests, i.e. $b = 0$. Without noise there is a large number of separating equilibria, in which it does not matter which type sends which message as long as these messages are distinct. As the variance σ^2 increases, however, it becomes impossible to support full separation in a monotone equilibrium, as shown in Figure 7.

In both panels of the figure we plot the excess demand for vagueness of the types 1 and 2 as functions of their messages (given that type 0 sends 0 and type 3 sends 1). In the left

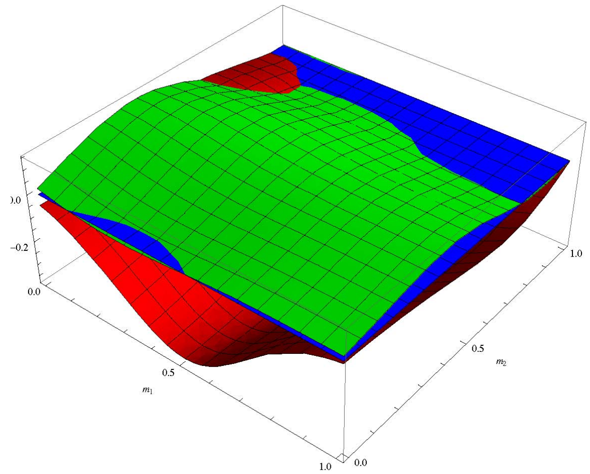

Figure 6: Noise makes separation possible

panel of the figure the variance is low, $\sigma = .1$, and there are three separating equilibria, one of which is unstable under a natural generalization of our vagueness dynamic. In the right panel, with $\sigma = 1$, type 2's excess demand is everywhere positive and type 1's is everywhere negative; i.e. type 1 will want to pool with type 0 and type 2 will want to pool with type 3.

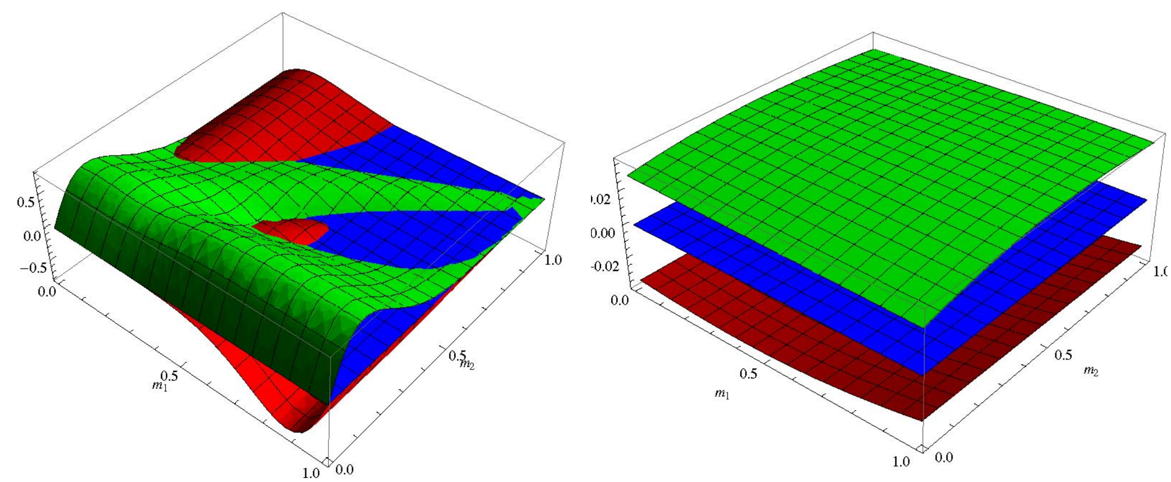

Figure 7: Equilibria with four types and common interests

This is reminiscent of an observation made by Nowak, Krakauer and Dress [15]. They allow for "the possibility of misunderstanding signals" and model a noise process in which the probability that one signal is understood as another depends on their similarity, modelled as distance in some metric space. They find that at the "evolutionary optimum" only a small number of signals is used to communicate a few valuable concepts. Jäger [12] reports on simulations that use a similar model in which normal noise is added to messages in a two–dimensional space in an exploratory study of the evolution of vowel systems. The vowel systems that result from simulating this model closely resemble those reported in a survey of vowel systems for natural languages by Schwartz, Boe, Vallé, and Abry [24]. In our model, the restricted number of signals that are in use and their configuration are a direct consequence of focussing on monotone communicative equilibria.

4 Related literature

In this section we review some closely related publications on vagueness and noisy communication.

A term is vague if it has borderline cases.[5] In a borderline case it is not clear whether the concept applies or not. The problems this causes are frequently illustrated by the sorites paradox, or the paradox of the heap: It is clear that a single grain of sand is not a 'heap', that adding or removing a single grain of sand cannot make a difference in whether a given amount of sand constitutes a 'heap,' and yet with enough sand we clearly have a heap. The word heap has borderline case where we cannot decide whether a given amount of sand is a heap or not. Therefore it is vague. Furthermore, there is higher–order vagueness: It is not clear where we enter the region of borderline cases. The term 'borderline cases' is itself vague. 'Vague' is vague.

The fact that there are case where we appear to be unable to decide whether a given statement, as 'This is a heap', is true or false poses a challenge to conventional logic and may affect the ability and manner in which we communicate. In the philosophy literature there are three principal approaches toward understanding and coping with vagueness, fuzzy logic (Zadeh [28]), supervaluationism (Fine [7]) and the epistemic view (Williamson [27]). Fuzzy logic replaces the two truth values 'true' or 'false' of standard logic by a continuum, so that in a borderline case it may for example assign a truth value of .6. Supervaluationism posits that in borderline cases vague predicates may be neither true nor false; there are truth–value gaps. Finally, the epistemic view holds that in borderline there is a fact of the matter, that is a vague predicate is either true or false, but we do not know which it is. To appreciate why there is a debate, it may help to note that in fuzzy logic tautologies involving vague predicates need not be true and that while supervaluationism rescue tautologies at the same it invalidates some familiar inference rule, e.g. it is no longer the case that showing that a statement is not true demonstrates its falsity.

While these three approaches are concerned with the use of language in individual reasoning, Rohit Parikh [17] addresses vagueness in communication and proposes what he calls a utility–based approach. According to him vague predicates can be useful in communication even if we cannot agree on their truth values in borderline cases: "There is no point in trying

[5]For a concise summary of vagueness in philosophy see `http://plato.stanford.edu/entries/vagueness/`.

to hide the differences in meaning of words between different people. But as long as these meanings are close enough, communication may be useful."

Rohit Parikh reports an experiment in which subjects were asked how many squares on a partial Munsell color chart are red and how many are blue. Of thirteen participants no two report the same numbers. In another similar experiment he shows that allowing for fuzzy truth values does not improve agreement. He argues that nevertheless "...if two people with slightly different extensions for the same words communicate, then this can be helpful to them even though we cannot say exactly what was conveyed. This is why I can tell you how to make tea, without telling you how to tell when the water is boiling, and also without being sure that your notion of boiling water is the same as mine."

Reiter and Sripada [20] provide additional evidence and discuss the practical difficulties this gives rise to when designing systems for natural language generation, e.g. the translation of meteorological data into weather forecasts, summaries of gas turbine sensors and summaries of sensor readings in neonatal intensive care units. For example, experts differed in their use of the term "oscillation" when describing patterns in gas turbine sensor data and in an informal experiment participants widely differed in their interpretation of the phrase "knowing Java" from "cannot program in Java, but knows that Java is a popular programming language" through "can use a tool such as JBuilder to write a very simple Java program ..." to "can create complex Java programs and classes ...."

Reiter and Sripada refer to the phenomenon that "people may not interpret words as expected" as "a type of 'semantic' noise." The phrase "semantic noise" is also employed by Warren Weaver in his effort to extend the reach of Shannon's model of noisy communication to include not only noise at the engineering level but also "the perturbations or distortions of meaning which are not intended by the source" (Shannon and Weaver [25], p.26). Weaver goes on to suggest that the sender take the noise into account so that "the sum of message meaning plus semantic noise is equal to the desired total message meaning at the destination."

Prashant Parikh [16] distinguishes between speaker meaning and addressee interpretation in a game–theoretic model of communication. An utterance may remain ambiguous until it is placed in context. Sometimes this gives the speaker a choice between making utterance that can be understood without contextual information and an utterance that requires that information. Vagueness may intervene in either case because speaker meaning and addressee meaning may only overlap rather than being identical. One could add that in an incomplete information setting context need not be common knowledge, which would be a source of noise.

De Jaegher [4] makes the point that vagueness in language can be efficiency enhancing. He argues that to account for vagueness in language it is not enough to look at Nash equilibria of a game with error–free communication. After all, the definition of Nash equilibrium requires that players know each others' strategies and therefore which messages communicate which concepts. His answer is to look for correlated equilibria, or equivalently Nash equilibria of an extended game with a correlation device. This technique plays a similar role to subjecting communication to noise in our model. In both cases language is conceived as a mechanism that takes messages as inputs and generates random outputs.

Lipman [13] asks "Why is language vague?" and rejects a number of explanations suggested by first-order intuition. As part of his argument he points out that there cannot be an advantage to mixing in a common-interest sender-receiver game. In our model, in contrast, agents do not have to resort to mixing to create noise, and conflict of interest is an important motivator. The exogenous component of vagueness in our setting is given by a language that acts as a noisy channel. We do show that vagueness in this sense can be compounded by strategic concerns.

Pinker, Novak and Lee [19] stress that both conflict and cooperation play a significant role in human communication. They draw on this interplay of objectives to explain why Grice's [9] principles of efficient communication fail to explain indirect speech. Part of the theory they advance relies on the logic of plausible deniability, which in turn depends on the interpretability of messages. They sketch a game-theoretic model in which the speaker chooses the degree of directness of his message and conclude that greater conflict results in less direct speech.

In a recent paper on leadership and obfuscation, Dewan and Myatt [5] model "clarity of communication" by essentially the same means as we model the exogenous component of vagueness in the present paper. A leader's message can induce different interpretations by party activists. Interpretations are draws from a normal distribution that is centered on the leader's message, with clarity of communication measured by the distribution's precision. In Dewan and Myatt's baseline model information transmission is not strategic and therefore there is no incentive for the leader to manipulate vagueness. In an extension of their model, listeners' attention may be limited, giving leaders an incentive to obfuscate in order to avoid attention drifting to others leaders. Obfuscation is modeled by allowing choice of variance, an option that is not available in our model.

In an earlier paper, Blume, Board and Kawamura [2] (henceforth BBK), we studied noisy communication with a different noise technology. In BBK messages either go through as sent or are drawn from an error distribution independent of the sent message. As in the present paper, there is exogenous vagueness expressed through the noise mechanism. Furthermore, noise can be beneficial. An attractive feature of the model in the present paper that is absent in both CS and BBK is that here the sender chooses distributions of interpretations and thereby exercises control over the probability that the receiver will end up with a concentrated posterior belief. A receiver with an interpretation that induces close to a uniform posterior belief faces a situation not unlike someone who must base a decision on a judgement in a borderline case.

5 Conclusion

We have shown that strategic agents are likely to add intentional vagueness to an exogenously vague language. In the two-type case the degree of intentional vagueness in equilibrium increases with the level of conflict between agents. In addition we confirm results from earlier work that exogenous vagueness can enhance efficiency by mitigating conflict. With an arbitrary finite number of types, interior types will be intentionally vague by distorting their messages upward in equilibrium relative to the messages the receiver would prefer them to send, given the receiver's equilibrium strategy. Also, as the level of conflict increases, so does the excess demand for vagueness of all sender types.

A Proofs

We start with some preliminaries. As in section 2.3, we define an expectation function α, which gives the expected value of the sender's type if she sends message m_0 when $t = 0$ and message m_1 when $t = 1$:

$$\alpha\left(q, m_0, m_1, \theta, \sigma\right) \equiv \frac{\theta \cdot \phi_{m_1,\sigma^2}\left(q\right)}{\left(1-\theta\right) \cdot \phi_{m_0,\sigma^2}\left(q\right) + \theta \cdot \phi_{m_1,\sigma^2}\left(q\right)}.$$

This function has 180° rotational symmetry; the following lemma describes this symmetry in the special case where $m_1 = 1$.

Lemma 1 *Suppose $m_0 \neq 1$. Then the action function α satisfies the following symmetry property:*

$$\alpha\left(q^* + x, m_0, 1, \theta, \sigma\right) - \frac{1}{2} = \frac{1}{2} - \alpha\left(q^* - x, m_0, 1, \theta, \sigma\right) \qquad \forall x \in \mathbb{R}.$$

where

$$q^* = \sigma^2 \frac{\ln\left(\frac{1-\theta}{\theta}\right)}{1-m_0} + \frac{1+m_0}{2}.$$

Proof. Notice that the condition

$$\alpha\left(q^* + x, m_0, 1, \theta, \sigma\right) - \frac{1}{2} = \frac{1}{2} - \alpha\left(q^* - x, m_0, \theta, \sigma\right) \qquad \forall x \in \mathbb{R}$$

is equivalent to

$$\frac{\theta \cdot \phi_{1,\sigma^2}\left(q\right)\left(q^*+x\right)}{\theta \cdot \phi_{1,\sigma^2}\left(q^*+x\right) + \left(1-\theta\right) \cdot \phi_{m_0,\sigma^2}\left(q^*+x\right)} = 1 - \frac{\theta \cdot \phi_{1,\sigma^2}\left(q^*-x\right)}{\theta \cdot \phi_{1,\sigma^2}(q^*-x) + (1-\theta) \cdot \phi_{m_0,\sigma^2}(q^*-x)} \qquad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \quad \frac{1}{1 + \frac{1-\theta}{\theta}\frac{\phi_{m_0,\sigma^2}(q^*+x)}{\phi_{1,\sigma^2}(q^*+x)}} = \frac{1}{1 + \frac{\theta}{1-\theta}\frac{\phi_{1,\sigma^2}(q^*-x)}{\phi_{m_0,\sigma^2}(q^*-x)}} \qquad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \quad \frac{1-\theta}{\theta}\frac{\phi_{m_0,\sigma^2}\left(q^*+x\right)}{\phi_{1,\sigma^2}\left(q^*+x\right)} = \frac{\theta}{1-\theta}\frac{\phi_{1,\sigma^2}\left(q^*-x\right)}{\phi_{m_0,\sigma^2}\left(q^*-x\right)} \qquad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \quad \frac{1-\theta}{\theta}\frac{e^{-\frac{\left(q^*+x-m_0\right)^2}{2\sigma^2}}}{e^{-\frac{\left(q^*+x-1\right)^2}{2\sigma^2}}} = \frac{\theta}{1-\theta}\frac{e^{-\frac{\left(q^*-x-1\right)^2}{2\sigma^2}}}{e^{-\frac{\left(q^*-x-m_0\right)^2}{2\sigma^2}}} \qquad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \quad \ln\left(\frac{1-\theta}{\theta}\right) + \frac{-\left(q^*+x-m_0\right)^2 + \left(q^*+x-1\right)^2}{2\sigma^2} = \ln\left(\frac{\theta}{1-\theta}\right) + \frac{-\left(q^*-x-1\right)^2 + \left(q^*-x-m_0\right)^2}{2\sigma^2} \qquad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \quad \ln\left(\frac{1-\theta}{\theta}\right) + \frac{2q^* m_0 - m_0^2 - 2q^* + 1}{2\sigma^2} = \ln\left(\frac{\theta}{1-\theta}\right) + \frac{2q^* - 1 - 2q^* m_0 + m_0^2}{2\sigma^2}$$

$$\Leftrightarrow \quad q^* = \sigma^2 \frac{\ln\left(\frac{1-\theta}{\theta}\right)}{1-m_0} + \frac{1+m_0}{2}.$$

■

We now present some results that describe several properties of communicative equilibria. Recall that we assume without loss of generality that $m_0 \leq m_1$; further, in a communicative equilibrium, $m_0 \neq m_1$, so $m_0 < m_1$. Lemma 2 states that, in any such equilibrium, the receiver's chosen action is a strictly monotone function of q.

Lemma 2 *In a communicative equilibrium, the receiver's action a is a strictly increasing function of the interpretation q.*

Proof. In any equilibrium, the receiver's action function satisfies

$$\mathbf{a}(q) = \alpha(q, m_0, m_1, \theta, \sigma) = \frac{\theta}{(1-\theta)\frac{\phi_{m_0,\sigma^2}(q)}{\phi_{m_1,\sigma^2}(q)} + \theta}.$$

For the normal distribution, the liklihood ratio $\frac{\phi_{m_0,\sigma^2}(q)}{\phi_{m_1,\sigma^2}(q)}$ is strictly decreasing in q for $m_0 < m_1$. ■

Since the sender, for given t, wants a higher action than the receiver, it follows that the type-1 will always choose an extremal message. Formally,

Lemma 3 *In a communicative equilibrium $m_1 = 1$.*

Proof. By Lemma 2 the receiver's action function $\mathbf{a}(q)$ is a strictly increasing function of q. Furthermore, $\mathbf{a}(q) < 1$ for all $q \in \mathbb{R}$. Therefore $-(1 + b - a(q))^2$, the type-1 sender's payoff from the interpretation q, is also a strictly increasing function of q. This and the fact that ϕ_{1,σ^2} strictly first-order stochastically dominates ϕ_{m_0,σ^2} for any $m_0 < 1$ implies that

$$\int_{-\infty}^{\infty} -(1 + b - a(q))^2 \phi_{1,\sigma^2}(q)\, dq > \int_{-\infty}^{\infty} -(1 + b - a(q))^2 \phi_{m_0,\sigma^2}(q)\, dq, \qquad \text{for all } m_0 < 1.$$

■

Henceforth, then, we assume that $m_1 = 1$. The optimization problem for the type-0 sender is much trickier to solve, since in a candidate equilibrium with $m_0 < m_1$, different messages not only shift, but also change the shape of the induced distribution of actions. Some care is required to ensure that sending the specified m_0 is globally optimal. To this end we repeatedly make use of an important technical observation. As long as the action function of the receiver satisfies equation (2) (see the definition of equilibrium in section 2.1), the expected payoff of a type-t sender is a convolution of two quasi-concave functions. Furthermore, the density of the normal distribution is log-concave. Ibragimov [11] shows that under these conditions the convolution itself will be quasi-concave. The following lemma adapts his result to the present environment.

Lemma 4 *If the receiver's action function* $\mathbf{a}$ *is strictly increasing in the interpretation* q*, then for any* t*, the sender's expected payoff from sending message* m

$$V^S(m,t,\mathbf{a}) \equiv \int_{-\infty}^{\infty} U^S(\mathbf{a}(q),t,b)\,\phi_{m,\sigma^2}(q)\,dq$$

is a strictly quasi-concave function of m*, and any* m^* *with* $\frac{dV^S}{dm*}(m^*,t)=0$ *is the unique global maximizer for type* t*.*

Proof. To simplify notation, we suppress reference to t, b and $\mathbf{a}$ and let $U(q) \equiv U^S(\mathbf{a}(q),t,b)$, so that

$$V^S(m) \equiv \int_{-\infty}^{\infty} U(q)\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(q-m)^2}{2\sigma^2}}dq.$$

Note that given our monotonicity assumption on $\mathbf{a}$ and by virtue of the fact that for any t there is a unique a_t that solves $\max_a U^S(a,t,b)$, U is either (i) strictly increasing, (ii) strictly decreasing, or (iii) there exists a value q_0 such that U is strictly increasing for $q<q_0$ and strictly decreasing for $q>q_0$. In cases (i) and (ii), the result follows because the normal distribution satisfies the strict-monotone-liklihood-ratio property and therefore strict first-order stochastic dominance. Otherwise, U has a unique maximizer q_0. For this case, consider

$$\begin{aligned}
\frac{dV^S}{dm} &= \int_{-\infty}^{\infty} U(q)\frac{d}{dm}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(q-m)^2}{2\sigma^2}}\right)dq \\
&= -\int_{-\infty}^{\infty} U(q)\frac{d}{dq}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(q-m)^2}{2\sigma^2}}\right)dq \\
&= \left[U(q)\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(q-m)^2}{2\sigma^2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(q-m)^2}{2\sigma^2}}\frac{dU}{dq}dq \\
&= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(q-m)^2}{2\sigma^2}}\frac{dU}{dq}dq,
\end{aligned}$$

using the fact $|U|$ is bounded. Define $\lambda \equiv q-q_0$. Note that we have $\frac{dU}{dq}(q_0+\lambda)<0$ and $\frac{dU}{dq}(q_0-\lambda)>0$ for all $\lambda>0$. Now suppose that $\frac{dV^S}{dm}(m^*)=0$ for some m^*. $\frac{dV^S}{dm}(m^*)$ can be re-written as

$$\frac{dV^S}{dm}(m^*) = \frac{1}{\sigma\sqrt{2\pi}}\left\{\int_0^{\infty} e^{\frac{-(m^*-(q_0+\lambda))^2}{2\sigma^2}}\frac{dU}{dq}(q_0+\lambda)\,d\lambda + \int_0^{\infty} e^{\frac{-(m^*-(q_0-\lambda))^2}{2\sigma^2}}\frac{dU}{dq}(q_0-\lambda)\,d\lambda\right\}.$$

Also, we have

$$\frac{dV^S}{dm}(m^*+\delta) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-\left(2\delta m^* - 2\delta q_0 + \delta^2\right)}{2\sigma^2}} \times$$
$$\left(\int_0^\infty e^{\frac{-(m^*-(q_0+\lambda))^2}{2\sigma^2}} e^{\frac{2\delta\lambda}{2\sigma^2}} \frac{dU}{dq}(q_0+\lambda)\, d\lambda + \int_0^\infty e^{\frac{-(m^*-(q_0-\lambda))^2}{2\sigma^2}} e^{\frac{-2\delta\lambda}{2\sigma^2}} \frac{dU}{dq}(q_0-\lambda)\, d\lambda\right).$$

Note that for $\delta > 0$ ($\delta < 0$) we are inflating (deflating) the negative terms and deflating (inflating) the positive terms in the integrand. Therefore $\frac{dV^S}{dm}(m^*+\delta) < 0$ for $\delta > 0$ and $\frac{dV^S}{dm}(m^*+\delta) > 0$ for $\delta < 0$; i.e. V^S is strictly quasi-concave and m^* is a global maximum. ■

We can now start to prove our main results. The proofs make use of the function V defined in section 2.3; V gives the expected utility of the type-0 sender with bias b if she sends message m when the reveiver expects her to send message m_0 and the type-1 sender to send message 1:

$$V(b, m_0, m') \equiv \int_\infty^\infty -(b - \alpha(q, m_0, 1, \theta, \sigma)^2 \cdot \phi_{m',\sigma}(q)\, dq.$$

Proof of Proposition 1. Let $b = \frac{1}{2}$ and suppose we have a communicative equilibrium where the sender chooses $\mathbf{m} = (m_0, 1)$ with $m_0 < 1$. From Lemma 1, we know that $\alpha(q, m_0, 1, \theta, \sigma)$ has 180° rotational symmetry about the point $\left(q^*, \frac{1}{2}\right)$, where $q^* = \sigma^2 \frac{\ln\left(\frac{1-\theta}{\theta}\right)}{1-m_0} + \frac{1+m_0}{2}$. It follows that $V\left(\frac{1}{2}, m_0, q^*-k\right) = V\left(\frac{1}{2}, m_0, q^*+k\right)$. Given Lemma 4, q^* is the unique maximizer of $V\left(\frac{1}{2}, m_0, m\right)$. If $\theta \leq \frac{1}{2}$, $q^* > m_0$, so the type-0 sender will deviate and send $m = \min\{q^*, 1\}$ instead of m_0; in this case, then, a communicative equilibrium does not exist. Suppose instead that $\theta > \frac{1}{2}$, Solving $m_0 = q^*$, we obtain

$$m_0^* = 1 - \sigma\sqrt{2\log\left(\frac{\theta}{1-\theta}\right)}.$$

Notice that this expression must be less than 1. If m_0^* lies between 0 and 1, we have a unique communicative equilibrium with the sender choosing $\mathbf{m} = (m_0^*, 1)$; if $m_0^* \leq 0$, we have a unique communicative equilibrium with the sender choosing $\mathbf{m} = (0, 1)$. ■

Proof of Proposition 2. In a communicative equilibrium, the sender chooses $\mathbf{m} = (m_0, 1)$ for some $m_0 \in [0, 1)$, and the receiver's action function is given by $\mathbf{a}(q) = \alpha(q, m_0, 1, \theta, \sigma)$. Since $\alpha(q, m_0, 1, \theta, \sigma)$ is strictly increasing in q and bounded below by 0, for $b = 0$, $-(b - \alpha(q, m_0, 1, \theta, \sigma))^2$ is a strictly decreasing function of q. This, and the fact that Φ_{m',σ^2} first-order stochastically dominates Φ_{m,σ^2} for any $m' > m$ implies that $V_3(0, m, m) < 0$ for all $m \in [0, 1)$. This means that whatever message the receiver expects the type-0 sender to send, she wants to send a lower message, and hence the unique communicative equilibrium (with $b = 0$) is with $\mathbf{m} = (0, 1)$. Given continuity of $V_3(b, m, m)$ in b, then, there is a non-empty interval $[0, \underline{b}]$ for which there is a unique communicative equilibrium which exhibits maximum differentiation. ■

Proof of Proposition 3. Fix some $m \in [0, 1)$. From the proof of Proposition 2 we know that $V_3(0, m, m) < 0$. By similar reasoning and the fact that $\alpha(q, m, 1, \theta, \sigma)$ is bounded above by 1, we have $V_3(1, m, m) > 0$. Continuity and the intermediate value theorem then imply that there exists a $b(m) \in (0, 1)$ such that

$$V_3(b(m), m, m) = 0.$$

From Lemma 4, then, it follows that $\mathbf{m} = (m, 1)$ is an equilibrium stratgey for the sender when $b = b(m)$. ■

Proof of Proposition 4. Differentiating V (the expected utility of the type-0 sender) with respect to the message she actually sends, and evaluating at the message m_0 that the receiver expects her to send, we obtain

$$V_3(b, m_0, m_0) = \int_{-\infty}^{\infty} (2b - \alpha)\, \alpha \phi_{m_0, \sigma^2}(q) \frac{q - m_0}{\sigma^2} dq.$$

The derivative of this expression with respect to m_0 evaluated at $m_0 = 1$ is equal to

$$\left.\frac{dV_3(b, m_0, m_0)}{dm_0}\right|_{m_0=1} = \frac{2(b - \theta)(-1 + \theta)\,\theta\sqrt{1/\sigma^2}}{\sigma}$$

This derivative is positive exactly when $b < \theta$. Using the fact that $V_3(b, 1, 1) = 0$, this implies that when $b < \theta$, there exists $m_0 < 1$ for which $V_3(b, m_0, m_0) < 0$. Since $V_3(b, m_0, m_0)$ is continuous in m_0, there are two possibilities. Either $V_3(b, m_0, m_0) < 0\ \forall m_0 \in (0, 1)$, in which case there is a communicative equilibrium with maximal differentiation, where the sender's strategy is $\mathbf{m} = (0, 1)$. Or there exists an $m_0 \in (0, 1)$ for which $V_3(b, m_0, m_0) = 0$, in which case Lemma 4 implies that we have a communicative equilibrium with intentional vagueness, where the sender's strategy is $\mathbf{m} = (m_0, 1)$. ■

Proof of Proposition 5. Suppose that $\theta \geq \frac{1}{2}$ and $b \geq \theta$, and the type-0 sender sends message $m_0 < 1$. We claim that she obtains at least as high an expected utility from sending message 1; combining this result with Lemma 4, we derive a contradiction. To compare $V(b, m_0, m_0)$ with $V(\theta, m_0, 1)$, first, note that $\theta - \alpha(m_0 + k, m_0, 1, \theta, \sigma) \geq$

$\alpha(1+k,m_0,1,\theta,\sigma)-\theta$ for all $k\in\left[-\frac{1-m_0}{2},\frac{1-m_0}{2}\right]$, so

$$\int_{\frac{3m_0-1}{2}}^{\frac{m_0+1}{2}}-\left(\theta-\alpha\left(q,m_0,1,\theta,\sigma\right)\right)^2\phi_{m_0,\sigma^2}\left(q\right)dq\leq\int_{\frac{m_0+1}{2}}^{\frac{3-m_0}{2}}-\left(\theta-\alpha\left(q,m_0,1,\theta,\sigma\right)\right)^2\phi_{1,\sigma^2}\left(q\right)dq.$$

Now consider any k with $k>\frac{1-m_0}{2}$. We show that

$$\begin{aligned}&-\left(\theta-\alpha\left(m_0+k,m_0,1,\theta,\sigma\right)\right)^2-\left(\theta-\alpha\left(m_0-k,m_0,1,\theta,\sigma\right)\right)^2\\ \leq\ &-\left(\theta-\alpha\left(1+k,m_0,1,\theta,\sigma\right)\right)^2-\left(\theta-\alpha\left(1-k,m_0,1,\theta,\sigma\right)\right)^2\end{aligned}$$

Since $\theta-\alpha(m_0-k,m_0,1,\theta,\sigma)\leq\theta-\alpha(1-k,m_0,1,\theta,\sigma)\leq 0$ and $\theta-\alpha(m_0-k,m_0,1,\theta,\sigma)\leq\alpha(1+k,m_0,1,\theta,\sigma)-\theta\leq 0$, it suffices to show that

$$\begin{aligned}&\alpha\left(m_0+k,m_0,1,\theta,\sigma\right)-\alpha\left(m_0-k,m_0,1,\theta,\sigma\right)\\ \geq\ &\alpha\left(1+k,m_0,1,\theta,\sigma\right)-\alpha\left(1-k,m_0,1,\theta,\sigma\right).\end{aligned}$$

Simplifying the action function, we obtain

$$\alpha\left(q,m_0,1,\theta,\sigma\right)=\frac{1}{1+e^{\frac{(1-m_0)(1+m_0-2q)}{2\sigma^2}}\left(\frac{1}{\theta}-1\right)}.$$

Hence

$$\begin{aligned}&\alpha\left(m_0+k,m_0,1,\theta,\sigma\right)-\alpha\left(m_0-k,m_0,1,\theta,\sigma\right)\\
=\ &\frac{1}{1+e^{\frac{(1-m_0)(1+m_0-2(m_0+k))}{2\sigma^2}}\left(\frac{1}{\theta}-1\right)}-\frac{1}{1+e^{\frac{(1-m_0)(1+m_0-2(m_0-k))}{2\sigma^2}}\left(\frac{1}{\theta}-1\right)}\\
=\ &\frac{\theta(1-\theta)\left(e^{\frac{(1-m_0)(1+m_0-2(m_0-k))}{2\sigma^2}}-e^{\frac{(1-m_0)(1+m_0-2(m_0+k))}{2\sigma^2}}\right)}{\left(\theta+e^{\frac{(1-m_0)(1+m_0-2(m_0+k))}{2\sigma^2}}(1-\theta)\right)\left(\theta+e^{\frac{(1-m_0)(1+m_0-2(m_0-k))}{2\sigma^2}}(1-\theta)\right)}\\
=\ &\frac{\theta(1-\theta)\left(1-\frac{e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}}{e^{\frac{(1-m_0)(1-m_0+2k)}{2\sigma^2}}}\right)}{\left(\theta+e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}(1-\theta)\right)\left(\frac{\theta}{e^{\frac{(1-m_0)(1-m_0+2k)}{2\sigma^2}}}+(1-\theta)\right)}\\
=\ &\frac{\theta(1-\theta)\left(1-e^{\frac{-2k(1-m_0)}{\sigma^2}}\right)}{\left(\theta+e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}(1-\theta)\right)\left(\theta e^{\frac{-(1-m_0)(1-m_0+2k)}{2\sigma^2}}+(1-\theta)\right)}\end{aligned}\qquad(3)$$

and

$$
\begin{aligned}
& \alpha\left(1+k, m_0, 1, \theta, \sigma\right)-\alpha\left(1-k, m_0, 1, \theta, \sigma\right) \\
= & \frac{1}{1+e^{\frac{(1-m_0)(1+m_0-2(1+k))}{2\sigma^2}}\left(\frac{1}{\theta}-1\right)}-\frac{1}{1+e^{\frac{(1-m_0)(1+m_0-2(1-k))}{2\sigma^2}}\left(\frac{1}{\theta}-1\right)} \\
= & \frac{\theta(1-\theta)\left(e^{\frac{(1-m_0)(1+m_0-2(1-k))}{2\sigma^2}}-e^{\frac{(1-m_0)(1+m_0-2(1+k))}{2\sigma^2}}\right)}{\left(\theta+e^{\frac{(1-m_0)(1+m_0-2(1+k))}{2\sigma^2}}(1-\theta)\right)\left(\theta+e^{\frac{(1-m_0)(1+m_0-2(1-k))}{2\sigma^2}}(1-\theta)\right)} \\
= & \frac{\theta(1-\theta)\left(1-\frac{e^{\frac{(1-m_0)(m_0-1-2k)}{2\sigma^2}}}{e^{\frac{(1-m_0)(m_0-1+2k)}{2\sigma^2}}}\right)}{\left(\theta+e^{\frac{(1-m_0)(m_0-1-2k)}{2\sigma^2}}(1-\theta)\right)\left(\frac{\theta}{e^{\frac{(1-m_0)(m_0-1+2k)}{2\sigma^2}}}+(1-\theta)\right)} \\
= & \frac{\theta(1-\theta)\left(1-e^{\frac{-2k(1-m_0)}{2\sigma^2}}\right)}{\left(\theta+e^{\frac{(1-m_0)(m_0-1-2k)}{2\sigma^2}}(1-\theta)\right)\left(\theta e^{\frac{-(1-m_0)(m_0-1+2k)}{2\sigma^2}}+(1-\theta)\right)}
\end{aligned}
\tag{4}
$$

Notice that numerators of (3) and (4) are the same, and positive; it follows that

$$
\alpha\left(m_0+k, m_0, 1, \theta, \sigma\right)-\alpha\left(m_0-k, m_0, 1, \theta, \sigma\right) \geq \alpha\left(1+k, m_0, 1, \theta, \sigma\right)-\alpha\left(1-k, m_0, 1, \theta, \sigma\right)
$$

if and only if

$$
\begin{aligned}
& \left(\theta+e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}(1-\theta)\right)\left(\theta e^{\frac{-(1-m_0)(1-m_0+2k)}{2\sigma^2}}+(1-\theta)\right) \\
\leq & \left(\theta+e^{\frac{(1-m_0)(m_0-1-2k)}{2\sigma^2}}(1-\theta)\right)\left(\theta e^{\frac{-(1-m_0)(m_0-1+2k)}{2\sigma^2}}+(1-\theta)\right) \\
& \theta^2 e^{\frac{-(1-m_0)(1-m_0+2k)}{2\sigma^2}}+e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}\left(1-2\theta+\theta^2\right) \\
\leq & \theta^2 e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}+e^{\frac{-(1-m_0)(1-m_0+2k)}{2\sigma^2}}\left(1-2\theta+\theta^2\right) \\
& \left(e^{\frac{(1-m_0)(1-m_0-2k)}{2\sigma^2}}-e^{\frac{(1-m_0)(m_0-1-2k)}{2\sigma^2}}\right)(1-2\theta) \leq 0 \\
& (1-2\theta) \leq 0 \\
& \theta \geq \frac{1}{2} \qquad \checkmark
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \int_{-\infty}^{\frac{3m_0-1}{2}}-\left(\theta-\alpha\left(q, m_0, \theta, \sigma\right)\right)^2 \phi_{m_0, \sigma}(q)\, dq+\int_{\frac{m_0+1}{2}}^{\infty}-\left(\theta-\alpha\left(q, m_0, \theta, \sigma\right)\right)^2 \phi_{m_0, \sigma}(q)\, dq \\
\leq & \int_{-\infty}^{\frac{m_0+1}{2}}-\left(\theta-\alpha\left(q, m_0, \theta, \sigma\right)\right)^2 \phi_{1, \sigma}(q)\, dq \int_{\frac{3-m_0}{2}}^{\infty}-\left(\theta-\alpha\left(q, m_0, \theta, \sigma\right)\right)^2 \phi_{1, \sigma}(q)\, dq
\end{aligned}
$$

Combining this with our earlier result, we obtain $V(\theta, m_0, 1) \geq V(\theta, m_0, m_0)$, as required.
■

Lemma 5 $z_1(b,m) = V_{31}(b,m,m) > 0 \; \forall m \in [0,1)$.

Proof. Notice that $\phi_{m,\sigma^2}(m+x)\frac{x}{\sigma^2} = -\phi_{m,\sigma^2}(m-x)\frac{(-x)}{\sigma^2}$. Furthermore, for all $m \in [0,1)$, $\frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m,\sigma^2}(q)}$ is a strictly increasing function of q and therefore gives greater weight to $\phi_{m_0,\sigma^2}(m+x)\frac{x}{\sigma^2}$ than to $\phi_{m,\sigma^2}(m-x)\frac{(-x)}{\sigma^2}$ for all $x > 0$. Therefore

$$V_{31}(b,m,m) = \int_{-\infty}^{\infty} 2\frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m,\sigma^2}(q)}\phi_{m,\sigma^2}(q)\frac{q-m}{\sigma^2} > 0.$$

■

Proof of Proposition 6. For $m_0 < 1$, define

$$q^*(m_0) \equiv \sigma^2\frac{\log\left(\frac{1-\theta}{\theta}\right)}{1-m_0} + \frac{1+m_0}{2}$$

(so $\left(q^*(m_0), \frac{1}{2}\right)$ is the point of symmetry of the expectation function $\alpha(q, m_0, 1, \theta, \sigma)$ — see Lemma 1 above). If $\theta < \frac{1}{2}$, then $q^*(m) > m$, in which case we claim that the existence of a communicative equilibrium requires that $b < \frac{1}{2}$. To see why, consider

$$\begin{aligned}
V_3(b,m_0,m_0) &= \int_{-\infty}^{\infty} -\left(b - \frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\right)^2 \phi_{m_0,\sigma^2}(q)\frac{q-m}{\sigma^2}dq \\
&= \int_{-\infty}^{\infty}\left(-b^2 + 2b\frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\right. \\
&\qquad \left. - \left(\frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\right)^2\right)\phi_{m_0,\sigma^2}(q)\frac{q-m}{\sigma^2}dq \\
&= \int_{-\infty}^{\infty}\left(2b - \frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\right) \\
&\qquad \frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\phi_{m_0,\sigma^2}(q)\frac{q-m}{\sigma^2}dq.
\end{aligned}$$

Setting $b = \frac{1}{2}$, we obtain

$$\begin{aligned}
V_3\left(\frac{1}{2}, m_0, m_0\right) &= \int_{-\infty}^{\infty}\left(1 - \frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\right) \\
&\qquad \frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\phi_{m_0,\sigma^2}(q)\frac{q-m}{\sigma^2}dq.
\end{aligned}$$

The quantity $\left(1-\frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}\right)\frac{\theta\phi_{1,\sigma^2}(q)}{\theta\phi_{1,\sigma^2}(q)+(1-\theta)\phi_{m_0,\sigma^2}(q)}$ is a function of q that is symmetric about $q^*(m_0)$, strictly increasing to the left of $q^*(m_0)$ and strictly decreasing to the right of $q^*(m)$. So as long as $q^*(m_0) > m$, it assigns greater weight to $\phi_{m_0,\sigma^2}(m+x)\frac{x}{\sigma^2}$ than to $\phi_{m_0,\sigma^2}(m-x)\frac{(-x)}{\sigma^2}$ for all $x > 0$. Hence, $V_3\left(\frac{1}{2}, m_0, m_0\right) > 0$.

Now consider raising b above $\frac{1}{2}$. From Lemma 5, $V_{31}(b, m, m) > 0$ $\forall m \in [0, 1)$ and in particular for $m = m_0$. Hence $V_3(b, m_0, m_0) > 0$ for all $b > \frac{1}{2}$, which is not consistent with equilibrium. ■

Proof of Proposition 8. Recall that $z(b, 1) = 0$ and

$$\frac{dz(b,1)}{dm} = \frac{2(b-\theta)(-1+\theta)\theta\sqrt{1/\sigma^2}}{\sigma}.$$

Therefore, if $b > \theta$, by continuity of z there exists an $\underline{m} \in (0, 1)$ such that for all $m' \in (\underline{m}, 1]$ we have $z(b, m') > 0$. Hence, for $b > \theta$ pooling at at message $m^* = 1$ is asymptotically stable. If $b < \theta$, then $\frac{dz(b,1)}{dm} > 0$, i.e. $m = 1$ is a hyperbolic source rather than a sink of the vagueness dynamic, and therefore unstable.

It remains to show that there is a stable equilibrium when $b \leq \theta$.

Consider the case $b < \theta$ first. Then there are two subcases: If $z(b, m) \leq 0$ for all $m \in [0, 1)$, then $m = 0$ is stable. Otherwise, there exists exists $m' \in [0, 1)$ with $z(b, m') > 0$. At the same time, given the case we are considering, there is $m'' \in (m', 1)$ with $z(b, m'') < 0$. Define $\overline{m} \equiv \inf\{m \mid z(m, b) < 0,\ m > m'\}$ and $\underline{m} \equiv \sup\{m \mid z(m, b) > 0,\ m < \overline{m}\}$. Note that $\underline{m} \leq \overline{m}$. If $\underline{m} < \overline{m}$, then $z(b, m) = 0$ for all m^0 in the open set $(\underline{m}, \overline{m})$, and therefore any such m^0 is stable. If $\underline{m} = \overline{m}$, then any open set $(\overline{m}, \overline{m} + \epsilon)$ contains a subinterval on which $z(b, m) < 0$ and any open set $(\overline{m} - \epsilon, \overline{m})$ contains a subinterval on which $z(b, m) > 0$, and therefore $\overline{m}$ is stable.

Finally, consider $b = \theta$. If $z(b, m)) \geq 0$ for all $m \in [0, 1]$, then $m^* = 1$ is stable. If $z(b, m'') < 0$ for some $m'' \in [0, 1)$, then either, then either $z(b, m) \leq 0$ for all $m \in [0, m'')$ or there exists $m' < m''$ with $z(b, m'') > 0$. In that case, the argument given for the case $b < \theta$ applies. ■

Lemma 6 *In any pure-strategy equilibrium the receiver's action rule is continuously differentiable. If in addition the sender's strategy is monotone and communicative, then the receiver's action rule is strictly increasing.*

Proof. If the sender uses a pure strategy, then with any equilibrium message m we can associate the set of types $\Theta(m)$ who use that message. Let M^* denote the set of equilibrium messages. Then the receiver's posterior belief about the sender's type given interpretation q

is

$$\mu(t \mid q) = \frac{\phi_{m_t,\sigma^2}(q))\nu(t)}{\sum_{m\in M^*}\phi_{m,\sigma^2}(q)\,\nu(\Theta(m))}.$$

In equilibrium, the receiver's action rule is given by

$$\begin{aligned}\mathbf{a}(q) &= \arg\max_a \sum_{t\in T} -(a-t)^2\,\mu(t \mid q) \\ &= \frac{\sum_{m\in M^*}\phi_{m,\sigma^2}(q)\,\nu(\Theta(m))\,E[t \mid t\in\Theta(m)]}{\sum_{m\in M^*}\phi_{m,\sigma^2}(q)\,\nu(\Theta(m))}.\end{aligned}$$

(Note that, as in the two-type case, the receiver's best response is to choose an action equal to the expectation of t.) Continuous differentiability of the receiver's action rule follows from continuous differentiability of ϕ_{m,σ^2} for any m and the fact that ϕ_{m,σ^2} is everywhere positive.

If the sender's strategy is monotone and communicative, more than one message is sent. Let there be $k > 1$ such messages. It will be convenient to reindex messages and to use m_i to denote the ith equilibrium message and Θ_i to denote the set of types who send that message. Then we can rewrite the receiver's action rule as

$$\mathbf{a}(q) = \frac{\sum_{i=1}^{k}\phi_{m_i,\sigma^2}(q)\,\nu(\Theta_i)\,E[t \mid t\in\Theta_i]}{\sum_{i=1}^{k}\phi_{m_i,\sigma^2}(q)\,\nu(\Theta_i)},$$

where $E[t \mid t\in\Theta_{i+1}] > E[t \mid t\in\Theta_i]$ for all $i = 1,\ldots,k-1$ (which can be satisfied because of monotonicity) and $\nu(\Theta_i) > 0$ for all $i = 1,\ldots,k$. To prove that $\mathbf{a}$ is a strictly increasing function of q, we proceed by induction. Define $\xi(q \mid m_i) \equiv \frac{\phi_{m_i,\sigma^2}(q)\nu(\Theta_i)}{\sum_{j=1}^{k}\phi_{m_j,\sigma^2}(q)\nu(\Theta_j)}$, so that

$$a(q) = \sum_{i=1}^{k}\xi(q \mid m_i)\,E[t \mid t\in\Theta_i].$$

Notice that $\sum_{i=1}^{k-1}\frac{\xi(q|m_i)}{\xi(q|m_k)} + 1 = \frac{1}{\xi(q|m_k)}$. SMLRP implies that each of the fractions on the left–hand side decrease as q increases. Hence $\xi(q \mid m_k)$ is (strictly) increasing in q. This establishes the claim for $k = 2$. We will now show that if it holds for k, then it holds for $k+1$. For $i = 1,\ldots,k$ define $\tilde{\xi}(q \mid m_i) \equiv \frac{\xi(q|m_i)}{\sum_{j=1}^{k}\xi(q|m_j)}$. Then

$$\begin{aligned}&\sum_{i=1}^{k+1}\xi(q \mid m_i)\,E[t \mid t\in\Theta_i] \\ &\qquad = (1-\xi(q \mid m_{k+1}))\left\{\sum_{i=1}^{k}\tilde{\xi}(q \mid m_i)\,E[t \mid t\in\Theta_i]\right\} + \xi(q \mid m_{k+1})\,E[t \mid t\in\Theta_{k+1}].\end{aligned}$$

The result follows because the expression in curly brackets, which is is strictly smaller than $E\left[t \mid t \in \Theta_{k+1}\right]$, by the induction hypothesis is strictly increasing in q and because $\xi\left(q \mid m_{k+1}\right)$ is strictly increasing in q. ■

Proof of Proposition 9. Given the receiver's equilibrium action rule $\mathbf{a}$, define type t's payoff from sending message m as

$$V^S\left(m, t, \mathbf{a}\right) \equiv \int_{-\infty}^{\infty} U^s\left(\mathbf{a}\left(q\right), t, b\right) \phi_{m,\sigma^2}\left(q\right) dq.$$

Then

$$\frac{\partial V^S\left(m, t, \mathbf{a}\right)}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial U^s(\mathbf{a}(q), t, b)}{\partial t} \phi_{m,\sigma^2}(q) dq.$$

The sender's payoff function satisfies the single–crossing condition

$$\frac{\partial^2 U^s(a, t, b)}{\partial t \partial a} > 0.$$

This and the fact that $\mathbf{a}$ is a strictly increasing function of q from Lemma 6 implies that $\frac{\partial U^s(\mathbf{a}(q),t,b)}{\partial t}$ is strictly increasing in q. Since ϕ_{m,σ^2} satisfies the strict monotone likelihood ratio property Φ_{m',σ^2} first–order stochastically dominates Φ_{m,σ^2} for any $m' > m$. Therefore

$$\frac{\partial^2 \tilde{V}(\mathbf{a}, t, m)}{\partial m \partial t} > 0.$$

Suppose that

$$\frac{\partial \tilde{V}(\mathbf{a}, s, m_s)}{\partial m} \geq 0$$

for a type $s < 1$. Then

$$\frac{\partial \tilde{V}(\mathbf{a}, \tau, m_s)}{\partial m} > 0$$

for any type $\tau > s$. Using Lemma 4, this implies that either $m_\tau > m_s$ or $m'_\tau = 1$ for all $\tau' \geq s$. Similarly, when

$$\frac{\partial \tilde{V}(\mathbf{a}, t, m_t)}{\partial m} \leq 0$$

for a type $t > 0$, we get that for any type $\tau < t$ either $m_\tau < m_t$ or $m'_\tau = 0$ for all $\tau' \leq t$. ■

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