

Modeling Periodicity in Point Processes

Nan Shao and Keh-Shin Lii

Department of Statistics
University of California, Riverside

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Outline

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Point Process

Point processes arise when events occur at random times.

- ▶ Initiation of phone calls at a service center.
- ▶ Earthquake in certain area.
- ▶ The arrival of ambulance at ER.
- ▶ Stock transactions.

Periodic Point Processes

- ▶ Stock transaction with daily cycle.
- ▶ Weekly effect in phone calls at customer center.
- ▶ Imoto et al. (1999) discussed the periodic pattern of seismicity observed in central Japan.

Definition of Point Process

- ▶ Let $\{t_i, i = 1, 2, \dots\}$ be a sequence of nonnegative random variables on a probability space (Ω, \mathcal{F}, P) with $0 \leq t_i \leq t_{i+1}$, then the sequence $\{t_i\}$ is called a **point process** on $[0, \infty]$.
- ▶ If there is no multiple occurrence, namely, $t_i < t_{i+1}$ for any i , the process is called a **simple point process**. We will restrict our attention to simple point process.
- ▶ Let the sequence $\{t_1, t_2, \dots\}$ be a point process. Let $N(s, t)$ be the number of points in time interval $(s, t]$, and denote $N(t) = N(0, t)$. Then $N(t)$ is called a **counting process**.

Intensity Function

The **intensity function** $\lambda(t)$ of a point process, also called the mean rate of occurrence in Cox and Isham (1980), is defined as

$$\begin{aligned}\lambda(t) &= \lim_{\delta \rightarrow 0^+} \frac{P\{N(t, t + \delta) > 0\}}{\delta} \\ &= \lim_{\delta \rightarrow 0^+} \frac{E\{N(t, t + \delta)\}}{\delta}.\end{aligned}$$

So $E\{dN(t)\} = \lambda(t)dt$. It describes the first-order moment property of the unconditional counting measure.

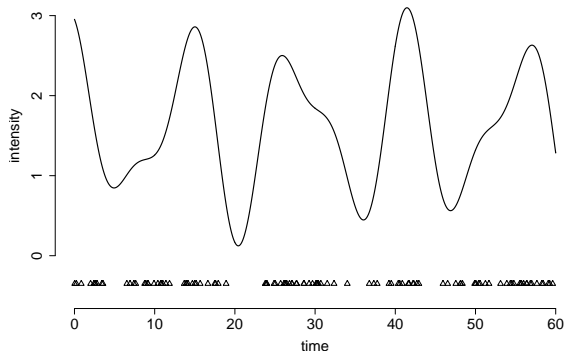


Figure: Solid line: the intensity function of a non-homogeneous Poisson process $\lambda(t) = \cos\left(\frac{\pi}{4\sqrt{3}}t\right) + 0.5 \cos\left(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}\right) + 1.6$. Triangle points: the occurrence time of events.

A Brief Review

- ▶ Lewis (1970, 1972) established the estimation and detection of a cyclic varying rate of a non-homogeneous Poisson process when the frequency is known *a priori*. The rate function took the following form

$$\lambda(t) = A \exp\{\rho \cos(\omega t + \phi)\}.$$

Vere-Jones (1982) proposed a method to estimate the frequency ω in above model and established its asymptotic property.

- ▶ Helmers et al. (2003) constructed a consistent kernel-type nonparametric estimate of the intensity function of a cyclic Poisson process where the intensity function is cyclic with only one period and the period is unknown.

Almost Periodic Point Processes

- ▶ General idea on ‘almost periodic’: any particular configuration that occurs once may recur not exactly, but within some accuracy.
- ▶ A sequence which appears to be ‘chaotic’ and ‘uninformative’ is a superposition of several periodic components and it is almost periodic.
- ▶ Almost periodic point process is much more general than the periodic one.

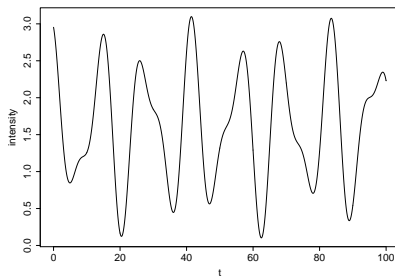


Figure: Almost periodic function $\lambda(t) = \cos\left(\frac{\pi}{4\sqrt{3}}t\right) + 0.5 \cos\left(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}\right) + 1.6$.

Our Model

The intensity function we consider is of the form

$$\lambda(t) = \sum_{i=1}^K A_k \cos(\omega_k t + \phi_k) + B, \quad (1)$$

where A_k, B, ω_k, ϕ_k are unknown parameters with $A_1 > A_2 > \dots > A_K > 0$, $\sum_{k=1}^K A_k < B$, $0 \leq \phi_k < 2\pi$ and $\omega_k > 0, k = 1, \dots, K$.

Periodogram

Periodogram in point processes is defined by Bartlett (1963) as follows

$$I_T(\omega) = d_T(\omega)\overline{d_T(\omega)},$$

where $(0, T)$ is the observation interval, and $d_T(\omega)$ is the **finite Fourier transform**, defined as

$$\begin{aligned}d_T(\omega) &= \frac{1}{\sqrt{2\pi T}} \int_0^T e^{-i\omega t} dN(t) \\ &= \frac{1}{\sqrt{2\pi T}} \sum_{j=1}^{N(t)} e^{-i\omega t_j}.\end{aligned}$$

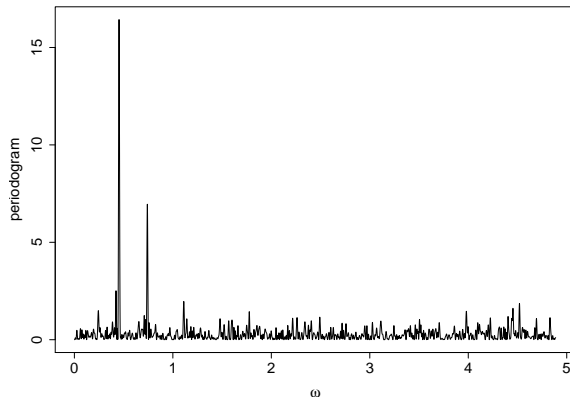


Figure: The periodogram of a non-homogeneous Poisson process with intensity function $\lambda(t) = \cos\left(\frac{\pi}{4\sqrt{3}}t\right) + 0.5 \cos\left(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}\right) + 1.6$. The process consists 778 observations with observation length $T = 500$.

Assumptions

- ▶ *Assumption 1* $N(t)$, $t \geq 0$, is a non-homogeneous Poisson process with a periodic or almost periodic intensity function (1), observed over the time interval $[0, T]$, and the number of periodic components K is given.
- ▶ Minimum separation:
Assumption 2 $T \min_{k \neq k'} (|\omega_k - \omega_{k'}|) \rightarrow \infty$, as $T \rightarrow \infty$.
- ▶ Searching range:
Assumption 3 $O(T^{\delta'-1}) \leq \omega_k \leq \Omega_T$, $k = 1, \dots, K$, where $0 < \delta' < 1$ and Ω_T is the upper bound, possibly determined by observations on the process in the interval $(0, T)$ with $E(\Omega_T) = O(T^{1-\delta})$, $\delta > 0$.

Estimation of the Frequencies

Let $\omega = (\omega_1, \dots, \omega_K)'$, we determine $\hat{\omega}_T$ as frequencies corresponding to the K largest local maxima of the periodogram in the range defined in Assumption 3 and the corresponding frequencies are well separated where their shortest distance cannot decrease sufficiently faster than $O(T^{-1})$ as $T \rightarrow \infty$.

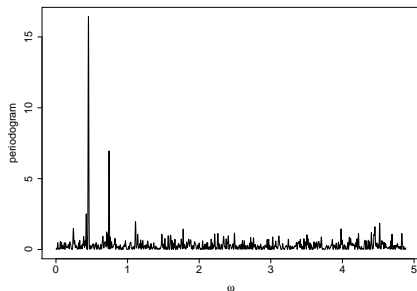
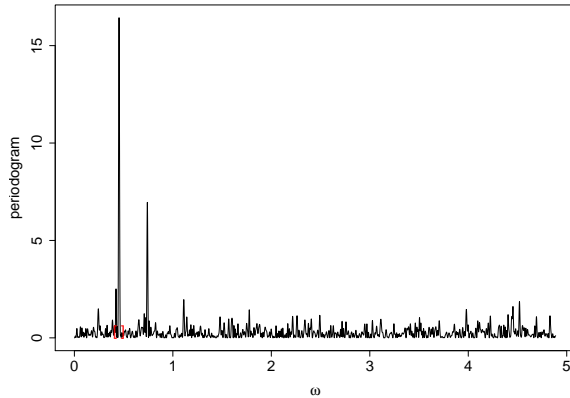
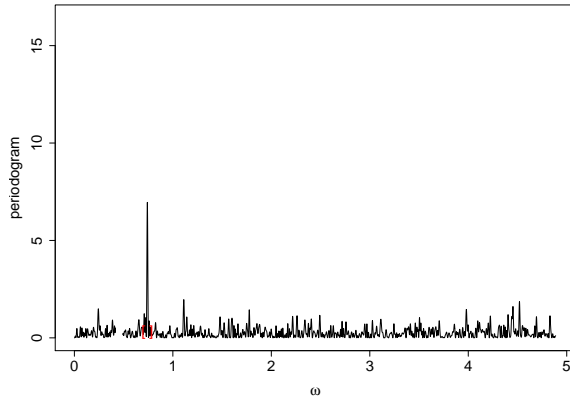


Figure: The periodogram of a non-homogeneous Poisson process with intensity function $\lambda(t) = \cos(\frac{\pi}{4\sqrt{3}}t) + 0.5 \cos(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}) + 1.6$. The process consists 778 observations with observation length $T = 500$.





Super Efficiency of $\hat{\omega}_T$

Proposition 1 Under Assumptions 1, 2 and 3, $\hat{\omega}_T$ is a consistent estimate of ω , and

$$(\hat{\omega}_{k,T} - \omega_k) = o(T^{-1}), \quad (a.s.), \quad k = 1, \dots, K.$$

Asymptotic Normality of $\hat{\omega}_T$

Theorem 1 Under Assumptions 1, 2 and 3, $T^{\frac{3}{2}}(\hat{\omega}_T - \omega)$ is asymptotically normally distributed as $T \rightarrow \infty$, with mean $\mathbf{0}$, and variance-covariance

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{3}{2}}(\hat{\omega}_{k,T} - \omega_k), T^{\frac{3}{2}}(\hat{\omega}_{k',T} - \omega_{k'})) = \frac{12}{A_k A_{k'}} \left[2B \delta_{k,k'} + \sum_{j=1}^K A_j \times \right. \\ \left. \left(-\cos(\phi_j - \phi_k - \phi_{k'}) \delta_{j,k+k'} + \cos(\phi_j - \phi_k + \phi_{k'}) \delta_{j,k-k'} + \cos(\phi_j + \phi_k - \phi_{k'}) \delta_{j,k'-k} \right) \right],$$

where $k, k' = 1, \dots, K$. In particular, the variance is

$$\lim_{T \rightarrow \infty} \text{var}(T^{\frac{3}{2}}(\hat{\omega}_{k,T} - \omega_k)) = \frac{12}{A_k^2} \left(2B - \sum_{j=1}^K A_j \cos(\phi_j - 2\phi_k) \delta_{j,k+k} \right).$$

Here

$$\delta_{k,k'} = I\{\omega_k = \omega_{k'}\} \quad \delta_{j,k+k'} = I\{\omega_j = \omega_k + \omega_{k'}\}, \\ \delta_{j,k-k'} = I\{\omega_j = \omega_k - \omega_{k'}\}, \quad \delta_{j,k'-k} = I\{\omega_j = \omega_{k'} - \omega_k\},$$

Estimation of the Amplitudes and Phases

Define $\hat{A}_{k,T}$ and $\hat{\phi}_{k,T}$ as the estimates of A_k and ϕ_k by

$$\hat{A}_{k,T}^2 = (8\pi/T)I_T(\hat{\omega}_{k,T}), \quad \text{so } \hat{A}_{k,T} = \sqrt{\hat{A}_{k,T}^2},$$

and

$$\tan \hat{\phi}_{k,T} = -T^{-1} \int_0^T \sin \hat{\omega}_{k,T} t dN(t) / T^{-1} \int_0^T \cos \hat{\omega}_{k,T} t dN(t),$$

$$\text{so } \hat{\phi}_{k,T} = \arctan \tan \hat{\phi}_{k,T},$$

where $k = 1, \dots, K$.

Asymptotic Normality of \hat{A}_k and $\hat{\phi}_k$

Theorem 2 Under Assumptions 1, 2 and 3, $T^{\frac{1}{2}}(\hat{A}_T - A)$ and $T^{\frac{1}{2}}(\hat{\phi}_T - \phi)$ are asymptotically normally distributed as $T \rightarrow \infty$, with mean $\mathbf{0}$, and variance-covariance

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{1}{2}}(\hat{A}_{k,T} - A_k), T^{\frac{1}{2}}(\hat{A}_{k',T} - A_{k'})) = 2B\delta_{k,k'} + \sum_{j=1}^K A_j \times$$

$$\left(\cos(\phi_j - \phi_k - \phi_{k'})\delta_{j,k+k'} + \cos(\phi_j - \phi_k + \phi_{k'})\delta_{j,k-k'} + \cos(\phi_j + \phi_k - \phi_{k'})\delta_{j,k'-k} \right),$$

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{1}{2}}(\hat{\phi}_{k,T} - \phi_k), T^{\frac{1}{2}}(\hat{\phi}_{k',T} - \phi_{k'})) = \frac{4}{A_k A_{k'}} \left[2B\delta_{k,k'} + \sum_{j=1}^K A_j \times \right.$$

$$\left. \left(-\cos(\phi_j - \phi_k - \phi_{k'})\delta_{j,k+k'} + \cos(\phi_j - \phi_k + \phi_{k'})\delta_{j,k-k'} + \cos(\phi_j + \phi_k - \phi_{k'})\delta_{j,k'-k} \right) \right].$$

where $k, k' = 1, \dots, K$.

In particular, the variances are

$$\lim_{T \rightarrow \infty} \text{var}(T^{\frac{1}{2}}(\hat{A}_{k,T} - A_k)) = 2B + \sum_{j=1}^K A_j \cos(\phi_j - 2\phi_k) \delta_{j,k+k},$$

$$\lim_{T \rightarrow \infty} \text{var}(T^{\frac{1}{2}}(\hat{\phi}_{k,T} - \phi_k)) = \frac{4}{A_k^2} \left(2B - \sum_{j=1}^K A_j \cos(\phi_j - 2\phi_k) \delta_{j,k+k} \right).$$

Estimation of the Baseline Constant

Define \hat{B}_T as

$$\hat{B}_T = \max\left(\sum_{k=1}^K \hat{A}_{k,T}, N(T)/T\right).$$

Let $\varepsilon = B - \sum_{k=1}^K A_k > 0$. We show that $N(T)/T - \sum_{k=1}^K \hat{A}_{k,T} \rightarrow \varepsilon > 0$ in probability as $T \rightarrow \infty$. So for large T , \hat{B} has the same asymptotic property as $N(T)/T$.

Theorem 3 Under Assumptions 1 and 2, $T^{\frac{1}{2}}(\hat{B}_T - B)$ is asymptotically normally distributed as $T \rightarrow \infty$, with mean 0, and variance B .

Covariance of the Parameter Estimates

Theorem 4 Under Assumptions 1, 2 and 3, $T^{\frac{3}{2}}(\hat{\omega}_T - \omega)$, $T^{\frac{1}{2}}(\hat{A}_T - A)$, $T^{\frac{1}{2}}(\hat{\phi}_T - \phi)$, and $T^{\frac{1}{2}}(\hat{B}_T - B)$ are jointly normally distributed as $T \rightarrow \infty$, with mean $\mathbf{0}$, and variance-covariance

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{3}{2}}(\hat{\omega}_{k,T} - \omega_k), T^{\frac{1}{2}}(\hat{\phi}_{k',T} - \phi_{k'})) = \frac{6}{A_k A_{k'}} \sum_{j=1}^K A_j \times$$

$$\left(\cos(\phi_j - \phi_k - \phi_{k'}) \delta_{j,k+k'} - \cos(\phi_j - \phi_k + \phi_{k'}) \delta_{j,k-k'} - \cos(\phi_j + \phi_k - \phi_{k'}) \delta_{j,k'-k} \right),$$

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{1}{2}}(\hat{A}_{k,T} - A_k), T^{\frac{1}{2}}(\hat{\phi}_{k',T} - \phi_{k'})) = \frac{1}{A_{k'}} \sum_{j=1}^K A_j \times$$

$$\left(\sin(\phi_j - \phi_k - \phi_{k'}) \delta_{j,k+k'} - \sin(\phi_j - \phi_k + \phi_{k'}) \delta_{j,k-k'} + \sin(\phi_j + \phi_k - \phi_{k'}) \delta_{j,k'-k} \right),$$

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{1}{2}}(\hat{A}_{k,T} - A_k), T^{\frac{1}{2}}(\hat{B}_T - B)) = B,$$

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{3}{2}}(\hat{\omega}_{k,T} - \omega_k), T^{\frac{1}{2}}(\hat{A}_{k',T} - A_{k'})) = 0,$$

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{3}{2}}(\hat{\omega}_{k,T} - \omega_k), T^{\frac{1}{2}}(\hat{B}_T - B)) = 0,$$

$$\lim_{T \rightarrow \infty} \text{cov}(T^{\frac{1}{2}}(\hat{\phi}_{k,T} - \phi_k), T^{\frac{1}{2}}(\hat{B}_T - B)) = 0,$$

where $k, k' = 1, \dots, K$.

Prediction

The one-step prediction \hat{T}_{n+1} is defined to minimize the mean squared error, and is given by

$$\begin{aligned}\hat{T}_{n+1} &= E(T_{n+1} | T_n = t_n, \dots, T_1 = t_1) \\ &= t_n + \int_{t_n}^{\infty} e^{-\Lambda(s) + \Lambda(t_n)} ds, \quad n \geq 1,\end{aligned}$$

where $\Lambda(t) = \int_0^t \lambda(s) ds$. And its mean squared error ν_n is given by

$$\begin{aligned}\nu_n &= E(T_{n+1} - \hat{T}_{n+1})^2 \\ &= E_{T_n} \left\{ 2 \int_{T_n}^{\infty} (s - T_n) e^{-\Lambda(s) + \Lambda(T_n)} ds - \left[\int_{T_n}^{\infty} e^{-\Lambda(s) + \Lambda(T_n)} ds \right]^2 \right\}, \quad n \geq 1.\end{aligned}$$

Note: $\Lambda(T_n) \sim \text{Gamma}(\alpha = n, \beta = 1)$.

The calculation of the MSE ν_n is carried out by Monte-Carlo integration with the following two steps.

1. Generate a large sample of $\Psi_n = \Lambda(T_n) \sim \text{Gamma}(\alpha = n, \beta = 1)$. Solve for T_n , namely, t_n is the unique root of $\Lambda(t_n) - \psi_n = 0$.
2. Calculate $2 \int_{t_n}^{\infty} (s - t_n) e^{-\Lambda(s) + \Lambda(t_n)} ds - [\int_{t_n}^{\infty} e^{-\Lambda(s) + \Lambda(t_n)} ds]^2$, and average over t_n . We obtain ν_n .

Simulation and Estimation

- ▶ Simulate 100 independent non-homogeneous Poisson process with intensity function $\lambda(t) = \cos(\frac{\pi}{4\sqrt{3}}t) + 0.5 \cos(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}) + 1.6$ by thinning.
- ▶ Cut off each process at $T = 500$. The sample size in each replicate ranges from 717 to 866.
- ▶ Table: sample means and standard errors of the parameter estimates.

	ω_1	ω_2	A_1	A_2	ϕ_1	ϕ_2	B
True mean	0.45345	0.74048	1	0.5	0	0.78540	1.6
Sample mean	0.45374	0.74116	1.01223	0.50671	-0.07806	0.59168	1.60616
Asymptotic sd	0.00055	0.00111	0.08000	0.08000	0.16000	0.32000	0.05657
Sample sd	0.00052	0.00112	0.07507	0.07838	0.16499	0.33445	0.05613

The sample means are close to the true means, and the sample standard deviations are close to the asymptotic standard deviations. So are the sample covariances.

More Simulations

- ▶ We consider four different periodic or almost periodic intensity functions of non-homogeneous Poisson processes. They are

$$\text{Case 1: } \lambda(t) = 1.6 + \cos\left(\frac{\pi}{4\sqrt{3}}t\right) + 0.5 \cos\left(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}\right),$$

$$\text{Case 2: } \lambda(t) = \sqrt{3.1 + 3 \cos\left(\frac{\pi}{3\sqrt{2}}t\right)},$$

$$\text{Case 3: } \lambda(t) = 0.1 + 0.5 \text{Mod}[t, 2\pi],$$

$$\text{Case 4: } \lambda(t) = 1.3 \exp\left\{\cos\left(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}\right)\right\}.$$

- ▶ We also conduct the ‘out-of-sample’ one-step-ahead prediction using the estimated intensity function, and compare the MSE with the MSE under homogeneous Poisson process model by taking their ratio, namely

$$\frac{1}{100} \sum_{i=1}^{100} (t_{n+1}^i - \hat{t}_{n+1}^i)^2 / \frac{1}{100} \sum_{i=1}^{100} (t_{n+1}^i - \tilde{t}_{n+1}^i)^2.$$

Case 1: $\lambda(t) = 1.6 + \cos\left(\frac{\pi}{4\sqrt{3}}t\right) + 0.5 \cos\left(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4}\right)$.

The number of data points used for estimation in 100 replicates ranges from 717 to 866. The observation length $T = 500$.

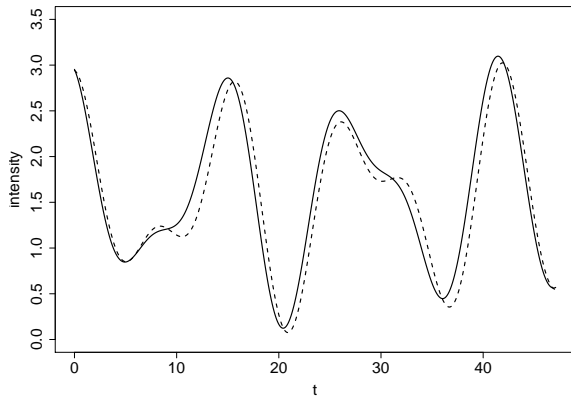


Figure: Solid line: true intensity function. Dashed line: estimated intensity function from one replicate.

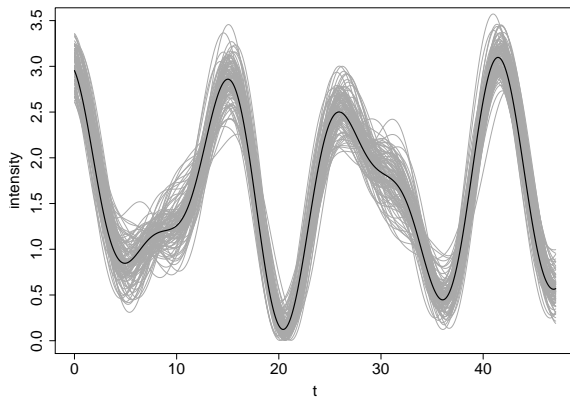


Figure: Dark solid line: true intensity function. Light solid lines: estimated intensity function from 100 replicates.

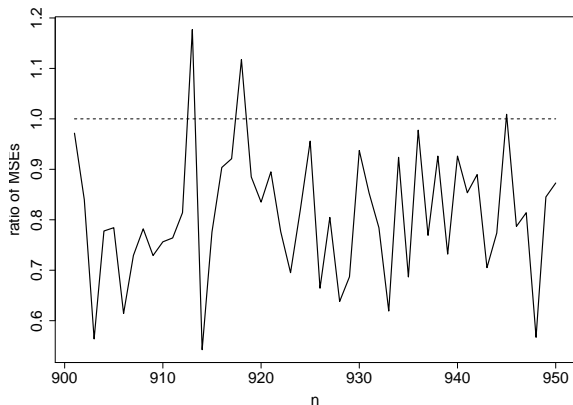


Figure: The ‘out-of-sample’ one-step-ahead prediction is carried out for the 901th to 950th data points. Solid line: ratio of MSE under our model and MSE under the homogeneous Poisson process model. The averaged reduction in MSE is 19.1%.

Case 2: $\lambda(t) = \sqrt{3.1 + 3 \cos(\frac{\pi}{3\sqrt{2}}t)}$.

The number of data points used for estimation in 100 replicates ranges from 749 to 872. The observation length $T = 500$.

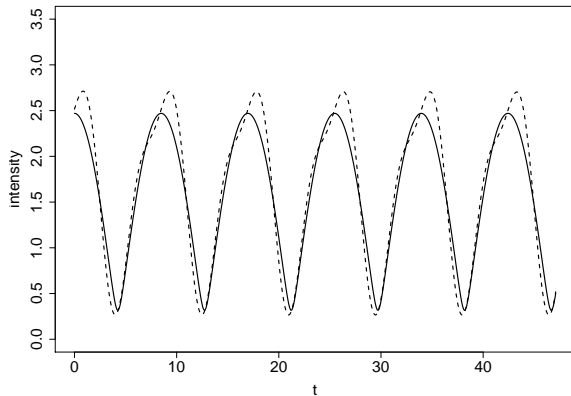


Figure: Solid line: true intensity function. Dashed line: estimated intensity function from one replicate.

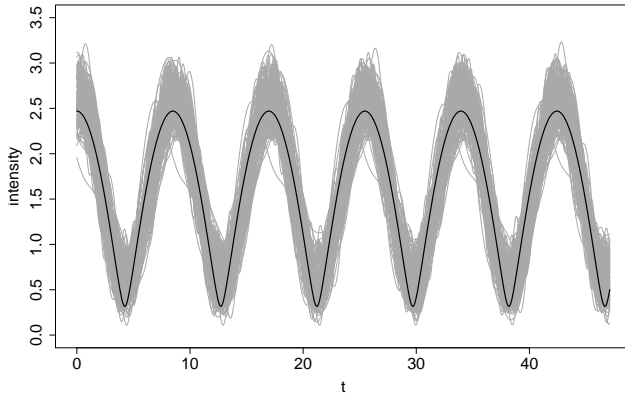


Figure: Dark solid line: true intensity function. Light solid lines: estimated intensity function from 100 replicates.

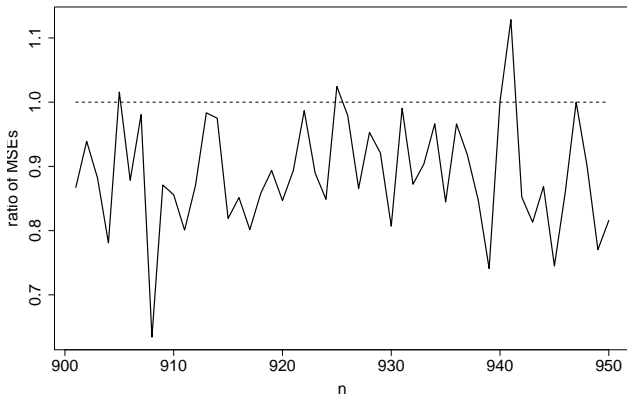


Figure: The ‘out-of-sample’ one-step-ahead prediction is carried out for the 901th to 950th data points. Solid line: ratio of MSE under our model and MSE under the homogeneous Poisson process model. The averaged reduction in MSE is 11.2%.

Case 3: $\lambda(t) = 0.1 + 0.5\text{Mod}[t, 2\pi]$.

The number of data points used for estimation in 100 replicates ranges from 733 to 906. The observation length $T = 500$.

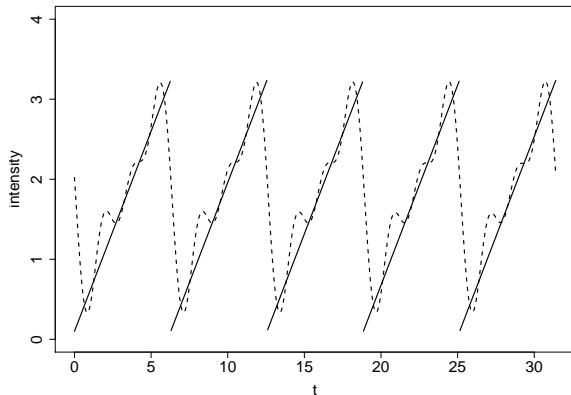


Figure: Solid line: true intensity function. Dashed line: estimated intensity function from one replicate.

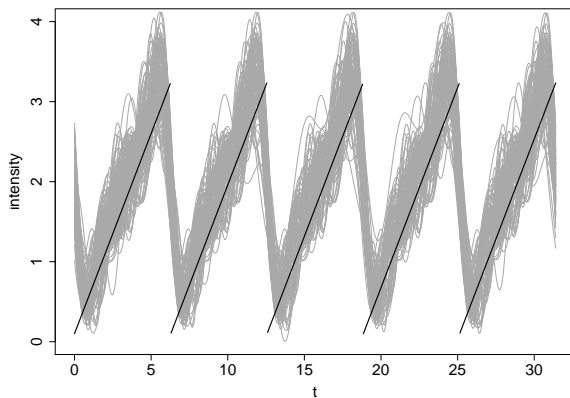


Figure: Dark solid line: true intensity function. Light solid lines: estimated intensity function from 100 replicates.

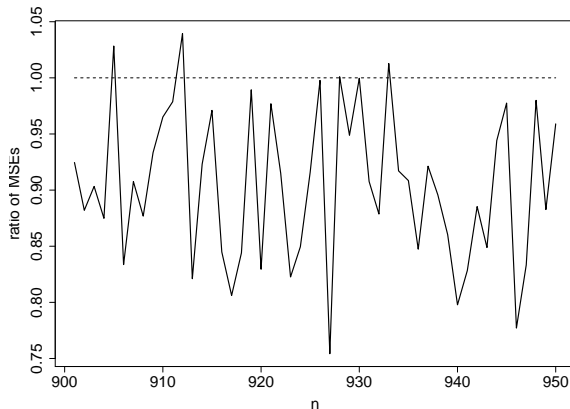


Figure: The ‘out-of-sample’ one-step-ahead prediction is carried out for the 901th to 950th data points. Solid line: ratio of MSE under our model and MSE under the homogeneous Poisson process model. The averaged reduction in MSE is 9.6%.

Case 4: $\lambda(t) = 1.3 \exp\{\cos(\frac{\pi}{3\sqrt{2}}t + \frac{\pi}{4})\}$.

The number of data points used for estimation in 100 replicates ranges from 733 to 906. The observation length $T = 500$.

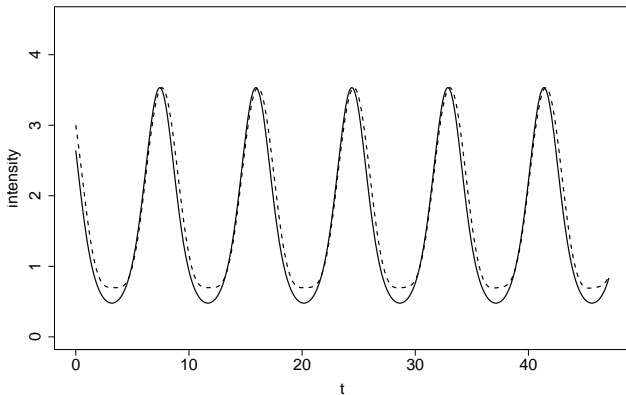


Figure: Solid line: true intensity function. Dashed line: estimated intensity function from one replicate.

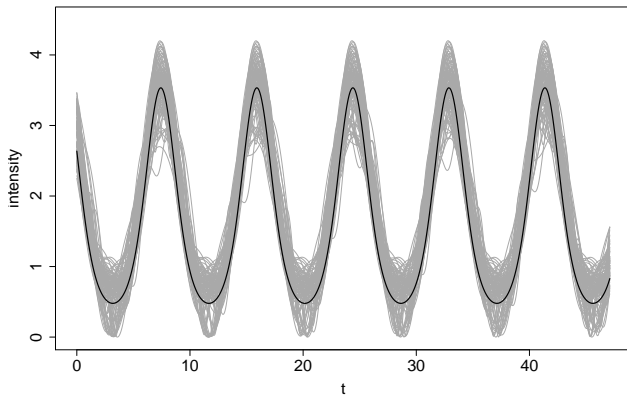


Figure: Dark solid line: true intensity function. Light solid lines: estimated intensity function from 100 replicates.

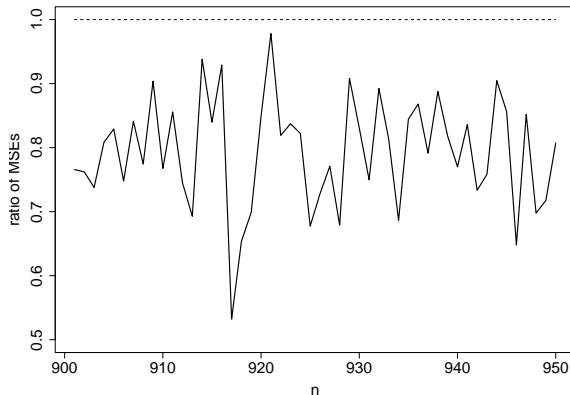


Figure: The ‘out-of-sample’ one-step-ahead prediction is carried out for the 901th to 950th data points. Solid line: ratio of MSE under our model and MSE under the homogeneous Poisson process model. The averaged reduction in MSE is 20.7%.

Computational Issues

- ▶ The frequency estimates $\hat{\omega}_T$ have to be located accurately.
- ▶ If the frequency estimates are not within $o(T^{-1})$ of the corresponding true frequencies, the amplitude and phase estimates are not consistent.
- ▶ The usual optimization algorithms do not work in here.

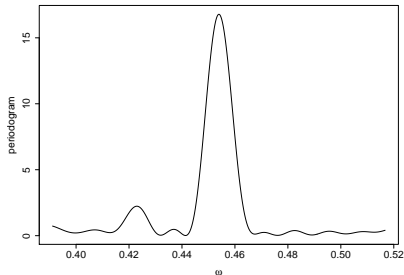


Figure: Periodogram at $\omega \in [\omega_1 - 10\pi/T, \omega_1 + 10\pi/T]$. It has many local maxima and local minima.

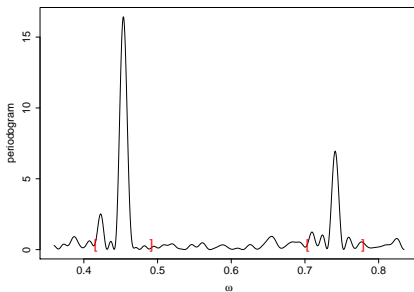
- ▶ Initially search the periodogram on a grid mesh with the mesh length $o(T^{-1})$, such as $2\pi T^{-3/2}$, and determine the initial values corresponding to the K largest ordinates subject to the minimum separation condition, and then do a more refined search in the neighborhood of the initial values.
- ▶ Note that the fast Fourier transform cannot be used in calculating the periodogram of a point process because the points $\{t_j : t_j \leq T\}$ are not equally spaced.

Two problems:

- ▶ How to choose the minimum separation for a particular set of data?
- ▶ How to determine the neighborhood of the initial values in the more refined search?

Problem 1

- ▶ The choice of minimum separation is $O(T^{-1+\beta})$ with $\beta > 0$, we can use $O(T^{-\frac{1}{2}})$, but it may be too large.
- ▶ The periodogram may go down to the noise level when it is outside $\hat{\omega}_{k,T} \pm 6\pi/T$.



- ▶ In practice, we determine $\hat{\omega}_{1,T}$ by maximizing the periodogram, and may determine $\hat{\omega}_{2,T}$ by maximizing the periodogram outside $\hat{\omega}_{1,T} \pm 6\pi/T$, and so on.
- ▶ The minimum separation can also be determined by prior knowledge of how far apart the frequencies are. The suggestion of $6\pi/T$ here may be the smallest choice since we assume the true frequencies to be well separated with minimum distance greater than $O(T^{-1})$.

Problem 2

- ▶ The initial search takes on a grid mesh with mesh length $o(T^{-1})$.
- ▶ The initial value in the search for $\hat{\omega}_{k,T}$ must fall in $[\omega_k - 2\pi/T, \omega_k + 2\pi/T]$.
- ▶ The refined search may take place in a narrower neighborhood than $[\omega_k - 2\pi/T, \omega_k + 2\pi/T]$; the suggested length of the narrower neighborhood is $T^{-5/4} \log T$.

Open Problems

- ▶ Determine the number of periodic terms K in the model. Possible solution: model selection criterion.
- ▶ Relax Assumption 1 that the process is non-homogeneous Poisson process.
- ▶ Construct and study the process with periodic or almost periodic correlation.
- ▶ ...

Questions?
Suggestions?
Datasets?

Thank You!