Dynamic Identification of DSGE Models

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Abstract

A DSGE model is identifiable when perturbing the parameters characterizing the forward looking optimizing model induces a distinguishable solution to the model. This paper studies identification of the parameters of a DSGE model using all the second moment properties of the data. We show that the solution of the model does not have the usual properties of a reduced form even when the shocks are uncorrelated and the system is stochastically non-singular. Hence, classical identification results do not apply. We use the restrictions that two linear dynamic models with identical spectrum must satisfy to obtain the rank and order conditions for identification. Both conditions depend on the solution parameters alone, and can be checked before any observations are considered. We also provide results for conditional identification under a priori restrictions, and partial identification of a subset of the parameter of interest. The results are established in a general set up that allows for fewer shocks than endogenous variables. Two examples are considered to illustrate the results.

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models have now reached the level of sophistication to permit analysis of important macroeconomic issues. Whereas the parameters in these models used to be calibrated, numerical advances in the last two decades have made it possible to estimate models with as many as a hundred parameters. Researchers are, however, aware that not all the parameters of interest can be consistently estimated because different structural parameters can lead to indistinguishable outcomes.[1] In spite of the recognition of this identification issue, a procedure has yet to be developed that tells us in a systematic manner how many parameters are identifiable, and if so which ones. The contribution of this paper is to propose rank and order conditions for local identifiability of log-linearized solutions to DSGE models, exploiting features that data generated by solutions with equivalent spectra must satisfy.

Since the solutions of DSGE models are merely systems of linear simultaneous equations with VARMA representations, it is useful to explain why new identification results are needed. First and foremost, economic theory often considers fewer shocks than observed endogenous variables, a condition known as 'stochastic singularity'. As a consequence, VARMA representations of DSGE models involve matrices that are generally not square. This violates the usual invertibility assumption used in the classical work by Hannan (1971), Zellner and Palm (1974), Hatanaka (1975), and Wallis (1977), for example. To circumvent the problem of stochastic singularity in DSGE models, it is not uncommon to complete the system by specifying enough shocks so as to match the number of endogenous variables. However, as argued in Chari, Kehoe, and McGrattan (2009), many shocks in moderate to large DSGE models are not structural. Adding measurement errors might enable estimation of singular systems but masks the issue of whether the model is fundamentally identified when an infinite amount of accurately measured data are observed. Furthermore, there is no guarantee that the parameters identified from a model augmented with measurement errors necessarily correspond to the parameters of interest. Alternatively, one can always drop enough variables so that the estimated model always has as many shocks as variables being analyzed. But this is valid if the abandoned variables are ancillary for the parameters of interest, and the assumption is not innocuous given the general equilibrium nature of the model. The identification results will generally not be invariant to the variables being dropped. Furthermore, the approach is inefficient as not all information available is used.

Even in models that are non-singular, identification of VARMA models requires the so-called left-co prime condition to rule out common factors in the autoregressive and moving average poly-

[1] The identification problem is discussed in Beyer and Farmer (2007), Cochrane (2007), and Canova and Sala (2009), among others.

nomials. The identified model is the canonical form obtained from solving the 'order' of the system, also known as the Mcmillian degree.[2] These methods are not easy to implement unless the model is small. Furthermore, we are not interested in identifying the VARMA parameters per se, but the ones that determine the VARMA parameters. More importantly, canonical forms are a useful (and perhaps the only) starting point if we have no other information about the model. When theory imposes enough restrictions on a linear dynamical system, we can identify the parameters of interest without appealing to the canonical form, an approach referred to as structural identification, see Hannan and Diestler (1988). In the DSGE context, we typically know the number of endogenous state variables and the number of exogenous shocks. Thus, unlike a typical VARMA model, we know the order of the model which simplifies the identification problem.

Classical work on the identification of linear simultaneous equations pioneered by Koopmans (1950) exploits the structure of the model, but several features of DSGE models make those results inapplicable. First, DSGE models are dynamic. Whereas in a static case, the parameters of two observationally equivalent structures must be related by a constant matrix, this definition of equivalence is inadequate because two models with different dynamic structures can have identical impulse response functions. Therefore, exclusion restrictions that permit identification of static models may not ensure identification of dynamic models. Rubio-Ramírez, Waggoner, and Zha (2007) faced the same issue when considering conditions for identification of linear structural VARs. Results for identification of linear dynamic simultaneous systems in Deistler (1976) are also inapplicable because they assume the presence of observed exogenous variables. DSGE models have no exogenous variables other than the shocks which are latent and thus cannot be used for identification.

Second, in traditional work on identification, latent shocks are assumed to be independent and identically distributed (iid) across time. The iid property implies that the distribution of the observables is time invariant. Identification then proceeds as though the econometrician observes the true distribution of the model variables. With time series data, the iid assumption on the shocks is overly restrictive because the distribution of the observed variables may well be changing over time. It is more appropriate to use the weaker white noise assumption and base the identification analysis solely on the autocovariances of the observables and not their entire distribution.

Finally, there is another important feature of DSGE models that invalidates the traditional approach. In many economic models covered by the classical identification analysis, the parameters of the reduced form are known to be identifiable from the distribution of the observables. In such cases, the structural parameters are (locally) identified if and only if the mapping from the structural to the reduced form parameters is (locally) one to one. The identification problem then

[2]These methods are discussed in Solo (1986), Reinsel (2003), and Lutkepohl (2005).

becomes a question of uniqueness of solutions to systems of equations. This leads to the well known rank conditions for (local) identification (see, e.g., Fisher, 1966; Rothenberg, 1971; Hausman and Taylor, 1983). What makes such analysis inapplicable here is that the parameters appearing in the solutions of DSGE models are generally *not identifiable* even if the shocks in the model are serially uncorrelated and the system is stochastically non-singular. As such, we do not have a reduced form model as conventionally defined.

Parameters that are not identifiable cannot be consistently estimated. In spite of the importance of this problem, the literature on identification of DSGE models is relatively small. Beyer and Farmer (2007) show that a determinate solution can be observationally equivalent to one that is not. While indeterminate solutions cannot be ruled out, most solutions are determinate, and identification of determinate solutions is not even completely understood. Canova and Sala (2009) suggest to examine the properties of the impulse responses evaluated at different parameter values, while Consolo, Favero, and Paccagnini (2009) compare the properties of the VAR implied by the DSGE model with those of a factor augmented VAR, also implied by the DSGE model. Both approaches, while useful, have not been shown to be necessary or sufficient for identification.

Depending on the assumptions used, different necessary and sufficient conditions for identification can be obtained. Iskrev (2009) is perhaps the only other analysis that developed formal conditions for identification of DSGE models. He suggests to identify the parameters of interest from the first T autocovariances of the observables. When T is fixed and finite, as would be the case in practice, Iskrev's (2009) conditions are only sufficient but not necessary. Our structural identification approach uses all (infinite) autocovariances as we exploit restrictions that two solutions with equivalent spectra must obey. This allows us to arrive at a finite set of conditions that are necessary and sufficient.

Almost every empirical DSGE exercise estimates a subset of the parameters and fixes many others.[3] At issue is how many restrictions are truly necessary. The challenge here is to identify the parameter of interest from the solution whose parameters may themselves be non-identifiable. We provide a rigorous understanding of why the solution equations have this feature and develop easy to compute rank and order conditions that can be used in a general setting. These results are new and have implications for frequentist analysis. The results are also useful in a Bayesian context. As Poirier (1998) pointed out, Bayesian analysis is not immune to identification problems as local non identification leads to pathological behavior of the posteriors when flat priors are used. This has serious consequences for the posterior computation in practice. The local non

[3]For example, Del Negro, Schorfheide, Smets, and Wouters (2007) fix 7 of the 47 model parameters, while Smets and Wouters (2007) fix 7 of the 39 parameters. Even with the simple stochastic growth model, Ruge-Murcia (2007) only estimates three of the six parameters and fixes the remaining three parameters.

identification problems lead to reducible Markov chains in which the region of locally unidentified parameter values becomes an absorbing state. This violates the convergence conditions typically imposed for Gibbs samplers. Knowing which parameters are locally unidentifiable would help decide which parameters need to be *fixed*. Appealing to informative priors (see, e.g., Chao and Phillips, 1998; Kleibergen and van Dijk, 1998) is of no help when the data have no information about the parameters, and this should be accepted as a fact.

The plan of the paper is as follows. In Section 2 we present our setup and study the properties of the observables. In Section 3, we provide an operational definition of observational equivalence and present rank and order conditions for identification. Section 4 studies conditional identification under a priori restrictions and partial identification of a subset of parameters of interest. Section 5 uses two examples to illustrate our results. Section 6 studies identification under weaker assumptions. Section 7 concludes. It should be mentioned that solutions based on higher order approximations necessitate a different setup and are beyond the scope of the present analysis.

2 Setup

This section introduces the solution to a generic DSGE model and studies the properties of the observables that are relevant for identification.

2.1 Model

Consider a generic nonlinear discrete time DSGE model in which the parameter of interest, denoted θ, belongs to a set $\Theta \subseteq \mathbb{R}^{n_\theta}$. In the analysis to follow, θ can represent the parameters of the optimizing model or those of the log-linearized model. We simply refer to θ is the structural parameter. Parameters known to be only identifiable from the steady state are excluded from θ; we will return to this issue subsequently. Let K_t be a vector of observed endogenous (state) variables whose values are known at time t, W_t be a vector of observed endogenous (jump) variables, Z_t be a vector of latent exogenous variables, and ϵ_t be a vector of unobserved shocks that satisfy $E_t\epsilon_{t+1} = 0$. These vectors (as well as their leads and lags) are of dimensions n_K, n_W, n_Z and n_Z, respectively. We shall work with DSGE model solutions in which the variables have a recursive equilibrium law of motion given by:

$$\begin{aligned} K_{t+1} &= P(\theta)K_t + Q(\theta)Z_t \\ W_t &= R(\theta)K_t + S(\theta)Z_t \\ Z_{t+1} &= \Psi(\theta)Z_t + \epsilon_{t+1}. \end{aligned} \tag{1}$$

The matrices $P(\theta)$, $Q(\theta)$, $R(\theta)$, $S(\theta)$ and $\Psi(\theta)$ are of dimensions $(n_K \times n_K)$, $(n_K \times n_Z)$, $(n_W \times n_K)$, $(n_W \times n_Z)$, and $(n_Z \times n_Z)$, respectively. Their expressions are assumed to be available either analytically—in the case of simple DSGE models that one can solve by hand—or else numerically as obtained via algorithms of Anderson and Moore (1985), Uhlig (1999), Sims (2002), Klein (2000), and King and Watson (2002), among others. When the solution to the log-linearized version of the DSGE model exists and is unique in the sense of Sims (2002), all the algorithms provide similar solutions (see, e.g., Anderson, 2008), and their outputs can always be written in the form of (1). Indeterminate solutions can be handled but would require K_t to be augmented with enough lags of the appropriate predetermined variables in the model.

We shall refer to the equations in (1) as the *solution* equations. These equations have the flavor of a reduced form model in which the dependent variables K_{t+1} and W_t are expressed in terms of the predetermined variables K_t and latent shocks Z_t. However, unlike in traditional reduced form models, the parameters of the solution are generally unidentified for two reasons. First, the contemporaneous effects of the shocks on the endogenous variables depend not only on the magnitude of the shocks, but also on the matrices $Q(\theta)$ and $S(\theta)$. Identification of the effects of Z_t is a potential problem because Z_t is latent. Second, the usual exogeneity conditions for identification of $P(\theta)$ and $R(\theta)$ generally fail to hold. This is because the latent shocks are allowed to be dynamic so Z_t is correlated with K_t whenever $\Psi(\theta) \neq 0$.

In DSGE models, variables defined by identities are often of interest. One important feature of the solution (1) is that it allows for identities to be included in W_t. Consider a set of new endogenous jump variables $\overline{W}_t$ defined through $\overline{n}_W$ identities: $\overline{W}_t = \overline{A}(\theta)K_{t+1} + \overline{B}(\theta)W_t$. Then letting $\widetilde{W}_t \equiv (W_t', \overline{W}_t')'$, the new solution equations can be written as:

$$\begin{aligned} K_{t+1} &= P(\theta)K_t + Q(\theta)Z_t \\ \widetilde{W}_t &= \widetilde{R}(\theta)K_t + \widetilde{S}(\theta)Z_t \end{aligned} \tag{2}$$

with a new $\widetilde{n}_W \times n_K$ matrix $\widetilde{R}(\theta)$ and a new $\widetilde{n}_W \times n_Z$ matrix $\widetilde{S}(\theta)$ ($\widetilde{n}_W \equiv n_W + \overline{n}_W$) defined as:

$$\widetilde{R}(\theta) \equiv \begin{pmatrix} R(\theta) \\ \overline{A}(\theta)P(\theta) + \overline{B}(\theta)R(\theta) \end{pmatrix} \quad \text{and} \quad \widetilde{S}(\theta) \equiv \begin{pmatrix} S(\theta) \\ \overline{A}(\theta)Q(\theta) + \overline{B}(\theta)S(\theta) \end{pmatrix}.$$

Our setup is thus quite general.

Hereafter, we collect all the observables into a vector $Y_t \equiv (K_t', W_t')'$, whose dimension we denote by $n_Y \equiv n_K + n_W$. We now state the main model assumptions:

Assumption 1 *For every $\theta \in \Theta$, $\{\epsilon_t\} \sim WN(0, \Sigma(\theta))$. That is, for every $(t, s) \geqslant 1$, $E(\epsilon_t) = 0$ and $E(\epsilon_t \epsilon_s') = \delta_{t-s}\Sigma(\theta)$ with $\Sigma(\theta)$ positive definite.*

Assumption 2 *For every* $\theta \in \Theta$ *and any* $z \in \mathbb{C}$, $\det(Id - \Psi(\theta)z)\det(Id - P(\theta)z) = 0$ *implies* $|z| > 1$. *Moreover, for every* $(\theta, \widetilde{\theta}) \in \Theta^2$, $P(\theta)$ *and* $\Psi(\widetilde{\theta})$ *have no eigenvalues in common.*

Assumption 3 $n_Z \leqslant n_Y$.

Assumption 4 *For every* $\theta \in \Theta$, *we have: (i)* $\begin{pmatrix} Q(\theta) & P(\theta)Q(\theta) & \dots & P^{n_Z+n_K-2}(\theta)Q(\theta) \end{pmatrix}$ *full rank; (ii)* $\begin{pmatrix} Q(\theta)' & S(\theta)' \end{pmatrix}'$ *full rank.*

Assumption 5 *Let* $\Lambda(\theta) \equiv (P(\theta), Q(\theta), R(\theta), S(\theta), \Psi(\theta), \Sigma(\theta))$ *denote the solution parameter. Let* Θ *be an open subset of* $\mathbb{R}^{n_\theta}$ *and* $n_\Lambda \equiv n_Y(n_Z + n_K) + 2n_Z^2$. *Then the mapping* $\Lambda : \Theta \to \mathbb{R}^{n_\Lambda}$ *which to each* $\theta \in \Theta$ *assigns* $\Lambda(\theta)$ *is continuously differentiable on* Θ.

To accommodate time series data, Assumption 1 only assumes white noise errors, which is weaker than iid.[4] The key feature of Assumption 1 is that the covariance matrix $\Sigma(\theta)$ is constant across time. Note that $\Sigma(\theta)$ is not required to be of the form σ^2Id (with $\sigma > 0$). We allow different components of the shock vector ϵ_t to be correlated as in Curdia and Reis (2009). Since $\Sigma(\theta)$ is non-singular, the shocks ϵ_t are required to be linearly independent.

The first requirement of Assumption 2 holds when all the eigenvalues of the matrices $\Psi(\theta)$ and $P(\theta)$ lie inside the unit circle, i.e. when $\Psi(\theta)$ and $P(\theta)$ are stable. The second restriction rules out situations where the internal propagating mechanism of one structure (as given by $P(\theta)$) coincides with the dynamics of the exogenous process in another structure (as given by $\Psi(\widetilde{\theta})$). Assumption 3 states that there cannot be more shocks than the total number of endogenous variables. Most DSGE models have this property.

When $n_Z \leqslant n_Y$, Assumptions 4(i) and 4(ii) ensure that the $n_K \times (n_Z(n_Z + n_K - 1))$ matrix $\begin{pmatrix} Q(\theta) & P(\theta)Q(\theta) & \dots & P^{n_Z+n_K-2}(\theta)Q(\theta) \end{pmatrix}$ and the $n_Y \times n_Z$ matrix $\begin{pmatrix} Q(\theta)' & S(\theta)' \end{pmatrix}'$ are of rank n_K and n_Z, respectively. When those rank conditions fail, then the dynamics of Z_t are no longer fully transmitted to Y_t; hence, not all information about the shocks can be recovered from the observables. Assumption 4(ii) allows for the identities to be included in the solution. If the original system of equations in (1) satisfies Assumptions 3 and 4(ii), then the new system in (2) is such that $n_Z \leqslant \widetilde{n}_Y \equiv n_K + \widetilde{n}_W$, and $\begin{pmatrix} Q(\theta)' & \widetilde{S}(\theta)' \end{pmatrix}'$ also has rank n_Z. Thus, the rank conditions continue to hold even when identities in variables are present. However, they rule out cases when the solution is unique but an expanded state vector is assumed.[5]

[4] Rational expectations models typically assume that $E_t\epsilon_{t+1} = 0$. Assumption 1 can be restated in terms of martingale difference errors since it does not prevent the observables to have higher moments that change in time, nor does it prevent the unobservables ϵ_t to be conditionally heteroskedastic. Imposing further restrictions on ϵ_t such as being homoskedastic or iid would imply further properties on the observables that could be used for identification.

[5] For example, the solution of an indeterminate system might involve additional lags of the predetermined variables, resulting in an expanded state vector relative to that of the determinate solution. Allowing for indeterminacy when the model is determinate means that some variables in the state vector would be redundant, violating the 'minimum state' assumption that will be used subsequently.

Finally, Assumption 5 puts smoothness requirements on the mapping from the parameter vector of interest θ to system matrices $P(\theta), Q(\theta), R(\theta), S(\theta), \Psi(\theta)$, and the covariance matrix $\Sigma(\theta)$ in Assumption 1. We collect all the components of those matrices in the solution parameter

$$\Lambda(\theta) \equiv (P(\theta), Q(\theta), R(\theta), S(\theta), \Psi(\theta), \Sigma(\theta))$$

which is of dimension $n_\Lambda \equiv n_Y(n_Z + n_K) + 2n_Z^2$.

Given the DSGE model solution (1), the objective of the exercise is to study the identifiability of (functions of) θ. For this, we need to be precise about the properties of the observables, $\{Y_t\}$.

2.2 The Observables $\{Y_t\}$

Under Assumptions 1 and 2, the process for $\{Y_t\}$ has the following VMA(∞) representation:

$$Y_t = \begin{pmatrix} K_t \\ W_t \end{pmatrix} = \sum_{j=0}^{\infty} h(j;\theta)\epsilon_{t-j}. \tag{3}$$

The $n_Y \times n_Z$ matrices $h(j;\theta)$ are the Markov parameters of the sequence $\{Y_t\}$, obtained from the transfer function (also called impulse response function):

$$\begin{aligned} H(z;\theta) &= \begin{pmatrix} H_K(z;\theta) \\ H_W(z;\theta) \end{pmatrix} \equiv \sum_{j=0}^{\infty} h(j;\theta) z^j \\ &= \begin{pmatrix} z[\mathrm{Id} - P(\theta)z]^{-1}Q(\theta)[\mathrm{Id} - \Psi(\theta)z]^{-1} \\ \left\{R(\theta)z[\mathrm{Id} - P(\theta)z]^{-1}Q(\theta) + S(\theta)\right\}[\mathrm{Id} - \Psi(\theta)z]^{-1} \end{pmatrix}. \end{aligned} \tag{4}$$

Under our assumptions, the sequence of endogenous variables $\{Y_t\}$ has two important properties which we now present in the form of two Lemmas.

Lemma 1 *(Covariance Stationarity) Let Assumptions 1 and 2 hold. Then for every $\theta \in \Theta$, $\{Y_t\}$ is weakly stationary with $E(Y_t) = 0$ and $E(Y_t Y_s') \equiv \Gamma(s-t;\theta) = \sum_{j=0}^{\infty} h(j;\theta)\Sigma(\theta)h'(j+s-t;\theta)$, for all $(t,s) \geqslant 1$.*

Lemma 1 is an immediate consequence of (3) and (4). When the process is weakly stationary, the autocovariances (which the econometrician eventually observes) completely summarize the properties of $\{Y_t\}$. For any $z \in \mathbb{C}$, the autocovariance generating function is defined as:

$$\Omega(z;\theta) \equiv \sum_{j=-\infty}^{+\infty} \Gamma(j;\theta) z^{-j}.$$

With a slight abuse of terminology, we shall hereafter refer to $\Omega(z;\theta)$ as the spectral density (or spectrum) of the observables $\{Y_t\}$.[6] Note that for every $\theta \in \Theta$, the autocovariance matrices $\Gamma(j;\theta)$ satisfy $\Gamma(j;\theta) = \Gamma(-j;\theta)'$. Thus, $\Omega(z;\theta)$ is a square $n_Y \times n_Y$ matrix of rank $n_Z (\leq n_Y)$ that satisfies $\Omega(z^{-1};\theta)' = \Omega(z;\theta)$, for every $z \in \mathbb{C}$. In addition, we have (see, e.g., Deistler, 2006):

$$\Omega(z;\theta) = H(z;\theta)\Sigma(\theta)H(z^{-1};\theta)' \tag{5}$$

with the transfer function for $H(z;\theta)$ as defined in (4).

From the VMA(∞) representation in (3) it follows that the process $\{Y_t\}$ is causal. Under our assumptions $\{Y_t\}$ is also invertible meaning that $\{\epsilon_t\}$ is spanned by the past and present values of $\{Y_t\}$, or that $\{\epsilon_t\}$ is 'fundamental' for $\{Y_t\}$.

Lemma 2 *(Invertibility) Let Assumptions 1 through 4 hold. Then for every $\theta \in \Theta$, $\{Y_t\}$ is invertible and we can write $\epsilon_t = \sum_{j=0}^{\infty} g(j;\theta)Y_{t-j}$ for all $t \geqslant 1$.*

A proof of Lemma 2 is in Appendix A. When $H(z;\theta)$ is a square matrix, conditions for invertibility are usually stated in terms of the eigenvalues of $H(z;\theta)$. As in our setup $H(z;\theta)$ is a matrix of dimension $n_Y \times n_Z$ which is not necessarily square, we need the rank $H(z;\theta)$ to be n_Z in order to obtain a VAR(∞) representation of the model (see, e.g., Hannan and Diestler, 1988) and thus ensure fundamentalness. Economic theory usually cannot rule out non-fundamentalness, a situation that arises when agents have more information than the econometrician.[7] We need invertibility of $\{Y_t\}$ to show that the spectral density matrix $\Omega(z;\theta)$ has constant rank n_Z almost everywhere in $\mathbb{C}$. Causal and invertible systems have a *minimum phase* (or 'miniphase') property that will allow us to go from the spectral density matrix to the transfer function that generated it.

3 Identification Analysis

In this section, we derive necessary and sufficient conditions under which two values θ_0 and θ_1 of the structural parameter θ are observationally equivalent. We then use this result to derive rank and order conditions for θ to be identifiable from the spectrum of the observables.

[6] Often, the term spectral density is used to denote the real function $\Omega(\exp(i\omega);\theta)$ which is defined for any $\omega \in \mathbb{R}$ and is always positive semi definite. The autocovariances $\Gamma(j;\theta)$ are then obtained from the spectral density $\Omega(\exp(i\omega);\theta)$ via the inversion formula: $\Gamma(j;\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Omega(\exp(i\omega))\exp(-i\omega j)d\omega$.

[7] Giannone and Reichlin (2006) argue that superior information is much less likely to happen when there are more variables than shocks in the system. Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007) suggest a way to check fundamentalness.

3.1 Observational Equivalence

In static analysis, the distribution of the observed variables Y_t is typically time invariant, so observational equivalence can be defined in terms of the distribution (or density) of Y_t. In a dynamic setting, this distribution may well vary across time. However, for processes that are known to be covariance stationary, observational equivalence can be defined in terms of the autocovariance structure of the observables, or equivalently, their spectral density.

The question of identification from the spectral density can be stated as follows: having observed the spectral density function $\Omega(z;\theta_0)$, under what conditions is it possible to uncover the value θ_0 that generated it? It is possible that two different values of θ lead to the same spectral density.

Definition 1 *The DSGE model solution is said to be locally identifiable from the spectral density of $\{Y_t\}$ at a point $\theta_0 \in \Theta$ if there exists an open neighborhood of θ_0 such that for every θ_1 in this neighborhood, θ_0 and θ_1 are observationally equivalent with $\Omega(z;\theta_0) = \Omega(z;\theta_1)$ for every $z \in \mathbb{C}$ only if $\theta_1 = \theta_0$.*

Definition 1 says that θ_0 is identifiable if, in its neighborhood, no other value θ_1 generates the same spectral density of the observables. At this point, we could have just analyzed the conditions under which the infinite system of equations $\Gamma(j;\theta_0) = \Gamma(j;\theta_1)$ for all $j = 0, \ldots, \infty$ holds. However, understanding the source of equivalent spectra allows us to characterize observational equivalence through a system of equations that is finite.

To proceed with this structural identification approach, first note from (4) and (5) that the spectral density $\Omega(z;\theta)$ depends on the structural parameter θ only through the solution parameter $\Lambda(\theta)$ and we have $\Omega(z;\theta) = \Omega(z;\Lambda(\theta))$. Each value θ_0 of θ implies the value $\Lambda_0 \equiv \Lambda(\theta_0)$ which completely characterizes the spectrum of $\{Y_t\}$. As already pointed out, however, the crucial assumption of Fisher (1966) and Rothenberg (1971)—that $\Lambda(\theta)$ itself is (globally or locally) identifiable—does not hold in DSGE models. To see this, let U be a full rank $n_Z \times n_Z$ matrix, and consider the system:

$$\begin{aligned} K_{t+1} &= P(\theta)K_t + \widetilde{Q}(\theta)\widetilde{Z}_t \\ W_t &= R(\theta)K_t + \widetilde{S}(\theta)\widetilde{Z}_t \\ \widetilde{Z}_t &= \widetilde{\Psi}(\theta)\widetilde{Z}_{t-1} + \tilde{\epsilon}_t \end{aligned} \tag{6}$$

with $\widetilde{Z}_t \equiv U^{-1}Z_t$, and $\widetilde{Q}(\theta) \equiv Q(\theta)U$, $\widetilde{S}(\theta) = S(\theta)U$, $\widetilde{\Psi}(\theta) = U^{-1}\Psi(\theta)U$, $\widetilde{\Sigma}(\theta) = U^{-1}\Sigma(\theta)U^{-1\prime}$. The transformed solution parameter $\widetilde{\Lambda}(\theta) = (\widetilde{P}(\theta), \widetilde{Q}(\theta), \widetilde{R}(\theta), \widetilde{S}(\theta), \widetilde{\Psi}(\theta), \widetilde{\Sigma}(\theta))$ generates the same spectrum as $\Lambda(\theta)$, since $\Omega(z;\Lambda(\theta)) = \Omega(z;\widetilde{\Lambda}(\theta))$ for every $z \in \mathbb{C}$. Yet, $\Lambda(\theta)$ is different from $\widetilde{\Lambda}(\theta)$. The solution parameter $\Lambda(\theta)$ is thus not identifiable from the spectral density of the observables.

This means that Rothenberg's (1971) rank condition alone will in general not be sufficient to guarantee the identifiability of θ.

Now, the transformation in (6) is one example of how observationally equivalent solution parameters can be obtained; however, we need to find *all* such transformations. Recall that $\Omega(z;\theta_1) = \Omega(z;\theta_0)$ when

$$H(z;\theta_0)\Sigma(\theta_0)H(z^{-1};\theta_0)' = H(z;\theta_1)\Sigma(\theta_1)H(z^{-1};\theta_1)'. \tag{7}$$

Thus equivalent spectra can arise because: (i) each $H(z;\theta)$ can potentially be obtained from a multitude of quintuples $(P(\theta), Q(\theta), R(\theta), S(\theta), \Psi(\theta))$ in (4), and (ii) there can be transfer functions $H(z;\theta)$ and shock covariance matrices $\Sigma(\theta)$ which jointly satisfy (7). We now derive the necessary and sufficient conditions for θ_0 and θ_1 to be observationally equivalent.

Proposition 1 *Let Assumptions 1 through 4 hold. Consider two structural parameters θ_0 and θ_1 with $(\theta_0, \theta_1) \in \Theta^2$. Then θ_0 and θ_1 are observationally equivalent if and only if:*

$$P(\theta_1) = P(\theta_0), \quad R(\theta_1) = R(\theta_0)$$

and there exists a full rank $n_Z \times n_Z$ matrix U such that:

$$Q(\theta_1) = Q(\theta_0)U, \quad S(\theta_1) = S(\theta_0)U, \quad \Psi(\theta_1) = U^{-1}\Psi(\theta_0)U, \quad \Sigma(\theta_1) = U^{-1}\Sigma(\theta_0)U^{-1\prime}.$$

Proposition 1 says the following: assume that it is possible for the econometrician to observe the spectral density $\Omega(z;\theta)$. Then, the matrices $P(\theta)$ and $R(\theta)$ are exactly identified from the spectrum. However, the matrices $Q(\theta)$, $S(\theta)$, $\Psi(\theta)$, and $\Sigma(\theta)$ can only be recovered up to an unknown full rank matrix U. Note that restricting the errors ϵ_t to have unit variance, i.e. $\Sigma(\theta) = \mathrm{Id}_{n_Z}$, is not sufficient for U to be identity as any orthogonal $n_Z \times n_Z$ matrix U satisfies $UU' = \mathrm{Id}_{n_Z}$.

The result that not all the solution parameters of DSGE models are identifiable applies whether or not the model is stochastically singular. It arises as a consequence of the fact that the contemporaneous effects of Z_t on the endogenous variables depend on $\Sigma(\theta)$, $Q(\theta)$ and $S(\theta)$, and Z_t is not observed to tie down $\Sigma(\theta)$. This result is important in its own right as non-identifiable parameters are not consistently estimable. Estimators that attempt to recover θ from estimates of the solution parameters, as well as inference and model diagnostics that rely on estimates of these solution parameters will not be valid irrespective of the sample size, and whether a frequentist or Bayesian approach is used.

The proof of Proposition 1 exploits two important features of the DSGE model solution: the minimality of its state space representation, and the 'miniphase' property of its spectrum. We now briefly discuss both properties and their implications. A formal analysis is given in the appendix.

The first feature concerns the equivalence of transfer functions, for which precise statements can be made for minimal systems. A state space representation is said to be minimal if the dimension of the latent state vector is no larger than any other system with the same output spectral density. To show minimality, we need to establish that the system is controllable and observable.[8] We then use the fact that two minimal realizations can have the same transfer function $H(z;\theta)$ if and only if they are linearly related through a similarity transformation, which is simply a linear change in the coordinates of the latent state variables.[9] Similarity transformations make precise the sense in which two structural parameter values θ_0 and θ_1 can generate identical transfer functions $H(z;\theta_0) = H(z;\theta_1)$. Our proof of minimality is new as some of our state variables are observed and we take account of the structure that the DSGE model imposes on the solution equations.

The second feature concerns the equivalence of spectral densities. The problem of finding the transfer function $H(z;\theta)$ from a given spectral density $\Omega(z;\theta)$ is known as 'spectral factorization' (see, e.g., Anderson, 1969). Its solution requires that the rank of the spectral density matrix $\Omega(z;\theta)$ be constant and known almost everywhere in $\mathbb{C}$. When $\{Y_t\}$ is causal and invertible, the constant rank property holds provided the shocks ϵ_t have a full rank covariance matrix $\Sigma(\theta)$. If the rank of $\Omega(z;\theta)$ can also be shown to equal the size of the shocks (n_Z), the spectral density is said to have a 'minimum phase' (or 'miniphase') property. The miniphase property allows us to make precise the conditions under which two couples $(H(z;\theta_0), \Sigma(\theta_0))$ and $(H(z;\theta_1), \Sigma(\theta_1))$ can yield the same spectral density.

In summary, we can go from the spectrum to the transfer function if the spectral density has minimum phase, and we know how the transfer functions of equivalent systems are related if the systems are minimal state. Combining these properties lead to the characterization of observational equivalence in Proposition 1.

3.2 Rank and Order Conditions

Having shown how two structural parameters with the same spectrum must be related, we now use those restrictions to derive conditions that are necessary and sufficient for local identification. Our approach mirrors that of Hannan (1971) and Deistler (1976) who used conditions for equivalent spectra, and Hannan and Diestler (1988) who used relations of similarity transformations to study identification of stochastically non-singular linear dynamic models. We provide results specifically for DSGE models.

[8] Controllability means that for any initial state, it is always possible to design an input sequence that puts the system to a desired final state. Observability means that we can always reconstruct the initial state from observing the evolution of the output, provided we also know the evolution of the input. Formally, these conditions hold when the controllability and observability matrices are full rank. See, e.g., Chapter 8 in Gourieroux and Monfort (1997).

[9] See, e.g., Theorem 3.10 of Antsaklis and Michel (1997).

Let $\lambda : \Theta \times \mathbb{R}^{n_Z^2} \to \mathbb{R}^{n_\Lambda}$ be a mapping which, to each structural parameter $\theta \in \Theta$ and $n_Z \times n_Z$ full rank matrix U, assigns a transformed solution parameter vector $\lambda(\theta, U)$ of dimension $n_\Lambda \equiv n_Y(n_Z + n_K) + 2n_Z^2$ defined by:[10]

$$\lambda(\theta, U) \equiv \begin{pmatrix} \text{vec}(P(\theta)) \\ \text{vec}(Q(\theta)U) \\ \text{vec}(R(\theta)) \\ \text{vec}(S(\theta)U) \\ \text{vec}(U^{-1}\Psi(\theta)U) \\ \text{vec}(U^{-1}\Sigma(\theta)U^{-1\prime} \end{pmatrix}. \tag{8}$$

Note that some of the components of λ are always equal to zero (or to some other constant). This is important for identification of θ as this puts restrictions on the equivalence matrix U in Proposition 1. Indeed, θ_0 is identifiable from the spectrum of $\{Y_t\}$ if and only if the system of n_Λ equations given by $\lambda(\theta_0, \text{Id}) = \lambda(\theta, U)$ has a unique solution $\theta = \theta_0$ and $U = \text{Id}$. We have the following result:

Lemma 3 *Let Assumptions 1 through 5 hold. Then the DSGE model solution is locally identifiable from the spectral density of $\{Y_t\}$ at a point $\theta_0 \in \Theta$ if and only if the mapping λ is locally injective at $\theta = \theta_0$ and $U = Id$.*

A proof of Lemma 3 is in Appendix C. The mapping in (8) defines a system of $n_\Lambda = n_Y(n_Z + n_K) + 2n_Z^2$ equations in $(n_\theta + n_Z^2)$ unknowns. We now look for necessary and sufficient conditions under which a (locally) unique solution to this system is $\theta = \theta_0$ and $U = \text{Id}$. It is a well known result (Fisher, 1966) that if the rank of the matrix of partial derivatives of $\lambda(\theta, U)$ with respect to the components of θ and U remains constant in a neighborhood of (θ_0, Id), then a necessary and sufficient condition for λ to be locally injective is that:

$$\text{rank}\begin{pmatrix} \frac{\partial \lambda(\theta_0, \text{Id})}{\partial \theta} & \frac{\partial \lambda(\theta_0, \text{Id})}{\partial \text{vec} U} \end{pmatrix} = n_\theta + n_Z^2. \tag{9}$$

See also Glover and Willems (1974) and Glover (1973) for a similar result. Necessary and sufficient conditions for identifiability of θ from the spectrum are given in the following Proposition:

[10] If $A \in \mathbb{R}^{m \times n}$ then $\text{vec}(A)$ is defined to be the nm-vector formed by stacking the columns of A on top of one another, i.e. $\text{vec}(A) \in \mathbb{R}^{nm}$. Note that $\text{vec}(A') = T_{m,n}\text{vec}(A)$ where $T_{m,n}$ is an $nm \times nm$ permutation matrix that satisfies $T_{n,m}T_{m,n} = \text{Id}_{mn}$, $T_{n,m} = T_{m,n}^{-1}$, and $T_{m,n} = T_{n,m}'$. Moreover, if $B \in \mathbb{R}^{p \times q}$ then $A \otimes B$ denotes the Kronecker product (or tensor product) of A and B, i.e. $A \otimes B \in \mathbb{R}^{mp \times nq}$.

Proposition 2 *Let Assumptions 1 through 5 hold. Let*

$$\Delta(\theta) \equiv \left(\frac{\partial \lambda(\theta, Id)}{\partial \theta} \quad \frac{\partial \lambda(\theta, Id)}{\partial vecU} \right) = \begin{pmatrix} \frac{\partial vecP(\theta)}{\partial \theta} & 0_{n_K^2 \times n_Z^2} \\ \frac{\partial vecQ(\theta)}{\partial \theta} & Id_{n_Z} \otimes Q(\theta) \\ \frac{\partial vecR(\theta)}{\partial \theta} & 0_{n_K n_W \times n_Z^2} \\ \frac{\partial vecS(\theta)}{\partial \theta} & Id_{n_Z} \otimes S(\theta) \\ \frac{\partial vec\Psi(\theta)}{\partial \theta} & Id_{n_Z} \otimes \Psi(\theta) - \Psi(\theta)' \otimes Id_{n_Z} \\ \frac{\partial vec\Sigma(\theta)}{\partial \theta} & -(Id_{n_Z^2} + T_{n_Z, n_Z})(\Sigma(\theta) \otimes Id_{n_Z}) \end{pmatrix}.$$

If the rank of $\Delta(\theta)$ remains constant in a neighborhood of θ_0, then a necessary and sufficient rank condition for the DSGE model solution to be locally identified from the spectrum of $\{Y_t\}$ at a point θ_0 in Θ is:

$$rank\, \Delta(\theta_0) = n_\theta + n_Z^2.$$

A necessary order condition is:

$$n_\theta \leqslant n_Y(n_Z + n_K) + \frac{n_Z(n_Z+1)}{2}.$$

What makes Proposition 2 important is that θ can be identified from the spectrum even though the solution parameter $\Lambda(\theta)$ is itself not identifiable. The Proposition, proved in Appendix C, contains two results. The first is a rank condition which is both necessary and sufficient for identification. This condition is entirely new and not yet seen in the literature. It extends the rank condition of Glover and Willems (1974)—who identify θ from the transfer function when the shocks ϵ_t are observed—to situations where the shocks are unobserved, which is always the case in DSGE models. The second result of Proposition 2 is an order restriction on n_θ without which θ cannot be identified from the spectrum. The intuition behind the order condition is simple: the number of linearly independent equations defined by the mapping $\lambda(\theta, U)$ in (8) needs to be at least as large as the number of unknowns $n_\theta + n_Z^2$. Since the last n_Z^2 equations correspond to the matrix $U^{-1}\Sigma(\theta)U^{-1\prime}$ which is always symmetric, they only contain $n_Z(n_Z+1)/2$ independent components. Hence the order restriction follows. It is important to note that our rank and order conditions do not depend on the method used to estimate θ; they hold whether one eventually pursues likelihood or moment based methods.

Whether or not identification is possible from the spectrum crucially depends on the dynamics of Z_t in (1). This is because in both the rank and order conditions, the dimension n_θ of θ depends on the assumptions placed on the system matrix Ψ and the covariance matrix Σ. In particular, when the process for Z_t consists of mutually uncorrelated univariate AR(1) processes, Ψ and Σ are diagonal matrices that contribute to $2n_Z$ unknowns in n_θ. If on the other hand, the shocks are VAR(1), the Ψ and Σ matrices consist of $n_Z^2 + n_Z(n_Z+1)/2$ unknowns (since Σ is always required

to be symmetric). The order condition is more likely to be satisfied when the shocks are assumed to be univariate processes as this reduces the dimensionality of θ.

The dimension of the Jacobian matrix $\Delta(\theta)$ in Proposition 2 depends only the dimension of the structural parameter (n_θ), the number of observables (n_Y), the number of endogenous state variables (n_K), and number of shocks (n_Z). Iskrev (2009) looks for conditions under which θ is identifiable from the first T autocovariances of the observables. Unless an infinite number of autocovariances can be evaluated, so that the Jacobian matrix is infinite dimensional, his conditions are sufficient but not necessary. The reason why we are able to work with a Jacobian matrix that is of finite dimension and yet use all the information contained in infinite autocovariances is that we exploit the minimal state and miniphase property of the solution to narrow down systems in the equivalence class.

As written, the results in Iskrev (2009) seem to be valid only for stochastically non-singular systems, for otherwise the $n_Y \times n_Y$ matrix $\Gamma(j;\theta)$ will be singular. Although the results can be modified to accommodate stochastic singularity, his approach would require the researcher to take a stand on which observables to analyze. Parameters not identified by one subset of observables may well be identified by others. Identities and stochastic singularity are explicitly taken into account in our setup, and we use the entire $\Gamma(j;\theta)$ matrix for every j to establish identification.

An immediate consequence of the rank condition in Proposition 2 is the following simple *necessary* condition for identification:

$$\text{rank}\left(\frac{\partial \text{vec}\Lambda(\theta_0)}{\partial \theta}\right) = n_\theta. \tag{10}$$

While necessary, the above condition by itself is not sufficient to guarantee identifiability. Not surprisingly, our rank condition in Proposition 2 is stronger than (10). This is because the solution parameter $\Lambda(\theta)$ is not identifiable so a stronger requirement is needed to identify θ. This result has a classical flavor, even though we work with assumptions that would not be valid in a classical setup. Indeed, Canova and Sala (2009) and Iskrev (2007, 2008) suggest to check the rank of $\frac{\partial \text{vec}\Lambda(\theta_0)}{\partial \theta}$, just as suggested in (10). However, their recommendation comes as a consequence of considering $\frac{\partial \log L}{\partial \theta} = \frac{\partial \log L}{\partial \text{vec}\Lambda}\frac{\partial \text{vec}\Lambda}{\partial \theta}$. But the solution parameter $\Lambda(\theta)$ is in fact not identifiable as Proposition 1 shows, and $\frac{\partial \log L}{\partial \text{vec}\Lambda}$ is not full rank. The rank condition on $\frac{\partial \text{vec}\Lambda(\theta_0)}{\partial \theta}$ alone is still necessary for identification; it is however not sufficient.

Finally, we comment on the constant rank requirement in Proposition 2. Requiring that the rank of $\Delta(\theta)$ remain constant in a neighborhood of θ_0 ensures that (θ_0, Id) is a regular point of the mapping λ in (8). This regularity property is used to show that our rank and order conditions are necessary for λ to be locally one-to-one. However, if we are only interested in conditions that are sufficient, then a simple application of the Implicit Function Theorem shows that our rank condition

is sufficient for identification, even if θ_0 fails to be a regular point. It is worth emphasizing that the regularity requirement is not satisfied almost everywhere in the parameter space except when the determinant of $\Delta(\theta)'\Delta(\theta)$ is an analytic function (see, e.g., Fisher, 1966).[11]

4 Partial and Conditional Identifiability

When the order condition in Proposition 2 fails, it is not possible to identify the entire structural parameter vector θ from the spectrum alone. In such situations, two approaches are possible: (i) *conditional* identification that imposes additional a priori restrictions on some of the components of θ in order to recover the identifiability of the entire structural parameter vector, and (ii) *partial identification* that examines if certain components of θ are identifiable from the spectrum.

4.1 Conditional Identification

We first consider the case in which the entire structural parameter vector θ is to be conditionally identified from the spectrum under a set of a priori restrictions on some of the components of θ. Let θ_0 satisfy a set of a priori restrictions:

$$\varphi(\theta_0) = 0 \tag{11}$$

where $\varphi : \Theta \to \mathbb{R}^{n_\varphi}$ is assumed to be continuous. Calibrated parameters fit into this category.[12]

Consider the mapping $\chi : \Theta \times \mathbb{R}^{n_Z^2} \to \mathbb{R}^{n_\varphi + n_\Lambda}$, which to each parameter θ and full rank $n_Z \times n_Z$ matrix U, assigns $\chi(\theta, U)$ defined by:

$$\chi(\theta, U) \equiv \begin{pmatrix} \varphi(\theta) \\ \lambda(\theta, U) \end{pmatrix}$$

where the constraints imposed by Proposition 1 enter through the mapping λ in (8). We now look for conditions under which $\chi(\theta, U)$ is locally injective at $\theta = \theta_0$ and $U = \text{Id}$. As before, this condition is necessary and sufficient for θ to be locally identifiable from the spectrum and the a priori restrictions (11). If the rank of the matrix of partial derivatives of $\chi(\theta, U)$ remains constant in a neighborhood of (θ_0, Id), then a necessary and sufficient condition for χ to be locally injective is that:

$$\text{rank} \begin{pmatrix} \frac{\partial \chi(\theta_0, \text{Id})}{\partial \theta} & \frac{\partial \chi(\theta_0, \text{Id})}{\partial \text{vec} U} \end{pmatrix} = n_\theta + n_Z^2. \tag{12}$$

This leads to the following necessary and sufficient conditions for θ to be conditionally identifiable from the spectrum of Y_t under the a priori restrictions (11).

[11] Our Example 1 illustrates this point.

[12] For a discussion, see Rios-Rull, Schorfheide, Fuentes-Albero, Kryshko, and Santaeulalia-Llopis (2009).

Proposition 3 *Let Assumptions 1 through 5 hold and assume that φ is continuously differentiable on Θ. Let*

$$\Delta_\varphi(\theta) \equiv \left(\frac{\partial \chi(\theta, Id)}{\partial \theta} \quad \frac{\partial \chi(\theta, Id)}{\partial vecU} \right) = \begin{pmatrix} \frac{\partial \varphi(\theta)}{\partial \theta} & 0_{n_\varphi \times n_Z^2} \\ \frac{\partial vecP(\theta)}{\partial \theta} & 0_{n_K^2 \times n_Z^2} \\ \frac{\partial vecQ(\theta)}{\partial \theta} & Id_{n_Z} \otimes Q(\theta) \\ \frac{\partial vecR(\theta)}{\partial \theta} & 0_{n_K n_W \times n_Z^2} \\ \frac{\partial vecS(\theta)}{\partial \theta} & Id_{n_Z} \otimes S(\theta) \\ \frac{\partial vec\Psi(\theta)}{\partial \theta} & Id_{n_Z} \otimes \Psi(\theta) - \Psi(\theta)' \otimes Id_{n_Z} \\ \frac{\partial vec\Sigma(\theta)}{\partial \theta} & -(Id_{n_Z^2} + T_{n_Z, n_Z})(\Sigma(\theta) \otimes Id_{n_Z}) \end{pmatrix}.$$

If the rank of $\Delta_\varphi(\theta)$ remains constant in a neighborhood of θ_0, then a necessary and sufficient rank condition for the DSGE model solution satisfying the a priori restrictions (11) *to be locally conditionally identified from the spectrum of $\{Y_t\}$ at θ_0 is:*

$$rank\, \Delta_\varphi(\theta_0) = n_\theta + n_Z^2.$$

A necessary order condition for conditional identification is:

$$n_\varphi \geqslant n_\theta - \left[n_Y(n_Z + n_K) + \frac{n_Z(n_Z+1)}{2} \right].$$

A proof is given in Appendix D. When the rank condition in Proposition 3 holds, then θ is locally conditionally identified at θ_0, where the conditioning information is given by the constraints $\varphi(\theta) = 0$. Proposition 3 shows that in this case that is likely true for many models, the full rank condition on $\Delta(\theta_0)$ is not the appropriate identification condition as we also need to take into account the additional a priori restrictions. Typically, those restrictions would involving fixing the values of some components of θ. Proposition 3 shows how many of these restrictions are necessary for identification, and is very useful in empirical work.

The result that identifiability of θ depends on whether or not $n_\theta \leqslant n_Y(n_Z+n_K)+n_Z(n_Z+1)/2$ has not been seen elsewhere in the literature. Aforementioned work by Canova and Sala (2009) and Iskrev (2007, 2008) does not include a priori restrictions on θ. It is worth emphasizing that our identification conditions do not depend on any population moments of the data. This is unlike the problem of weak identification (see, e.g., Guerron-Quintana, Inoue, and Kilian, 2009), which is a finite sample problem. Our rank condition also does not involve the spectrum or autocovariance matrix itself. This also distinguishes our approach from either likelihood or moment based identification methods, which typically require full rank Fisher information matrix (see, e.g., Iskrev, 2007), or a full rank Hessian of the GMM objective function. These matrices, unlike $\Delta(\theta)$ and $\Delta_\varphi(\theta)$, depend on the sample moments of the data. Using our approach, the researchers can establish whether or not their DSGE models are identified prior to collecting any data, without confounding the identification problem with the sampling error.

4.2 Partial Identification

When the entire structural parameter vector θ fails to be identifiable from the spectrum, it is still possible that some of its components remain to be so. To analyze such partial identification situations, partition the n_θ vector θ into two components θ_i and θ_{-i} of respective sizes $n_{\theta,i}$ and $n_{\theta,-i}$ (with $n_{\theta,i}+n_{\theta,-i}=n_\theta$). Without loss of generality, we order the components so that $\theta=({\theta_{-i}}',{\theta_i}')'$. We shall say that the DSGE model solution is partially identifiable in θ_i from the spectrum of $\{Y_t\}$ whenever the equations derived in Proposition 1 are sufficient to uniquely determine the sub-vector θ_i of the structural parameter vector θ. Nothing is said however about the remaining sub-vector θ_{-i}.

Formally, partial identification obtains if and only if any solution (θ,U) to the system of equations $\lambda(\theta_0,\mathrm{Id})=\lambda(\theta,U)$ with λ defined in (8) is such that $\theta_i=\theta_{0,i}$ and $U=\mathrm{Id}$. To look for necessary and sufficient conditions for this uniqueness property to hold, we assume that the ranks of the matrices of partial derivatives $\left(\frac{\partial\lambda}{\partial\theta}\quad\frac{\partial\lambda}{\partial \mathrm{vec}U}\right)$ and $\frac{\partial\lambda}{\partial\theta_{-i}}$ remain constant in a neighborhood of (θ_0,Id). Then as in Fisher (1966) and Glover (1973), a necessary and sufficient condition for $\theta_i=\theta_{0,i}$ and $U=\mathrm{Id}$ to be the only solution to $\lambda(\theta_0,\mathrm{Id})=\lambda(\theta,U)$ is that:

$$\mathrm{rank}\left(\frac{\partial\lambda(\theta_0,\mathrm{Id})}{\partial\theta}\quad\frac{\partial\lambda(\theta_0,\mathrm{Id})}{\partial \mathrm{vec}U}\right)=\mathrm{rank}\left(\frac{\partial\lambda(\theta_0,\mathrm{Id})}{\partial\theta_{-i}}\right)+\left(n_{\theta,i}+n_Z^2\right) \tag{13}$$

Note that the rank condition (13) can be viewed as a generalization of the rank condition (9) to cases where θ_{-i} is non-empty.

Consider again the matrix $\Delta(\theta)$ defined as in Proposition 2. We then have the following result.

Proposition 4 *Let Assumptions 1 through 5 hold. If the ranks of $\Delta(\theta)$ and $\frac{\partial\lambda(\theta,Id)}{\partial\theta_{-i}}$ remain constant in a neighborhood of θ_0, then a necessary and sufficient rank condition for the DSGE model solution to be locally partially identified in θ_i from the spectrum of $\{Y_t\}$ at a point θ_0 in Θ is:*

$$rank\,\Delta(\theta_0)=rank\left(\frac{\partial\lambda(\theta_0,Id)}{\partial\theta_{-i}}\right)+\left(n_{\theta,i}+n_Z^2\right).$$

A necessary order condition is:

$$rank\left(\frac{\partial\lambda(\theta_0,Id)}{\partial\theta_{-i}}\right)+n_{\theta,i}\leqslant n_Y(n_Z+n_K)+\frac{n_Z(n_Z+1)}{2}.$$

The proof of Proposition 4 is analogous to the proof of Proposition 2 and hence omitted. It is important to note that even though one might be interested in a subset of parameters, its identifiability will, in general, depend on the parameters that are not of interest.

5 Examples

This section consists of two examples: a simple growth model with fewer shocks than variables in the system, and a model with three shocks and four endogenous variables, of which one is defined through an identity.[13]

5.1 Example 1

Consider a one-sector stochastic growth model with inelastic labor supply, allowing for the possibility of costly adjustment of capital. The planner's problem is to choose consumption ($\mathcal{C}_t$) and capital ($\mathcal{K}_t$) to maximize her expected utility: $E_t \sum_{s=0}^{\infty} \beta^{t+s} \left(\frac{\mathcal{C}_{t+s}^{1-\nu}}{1-\nu}\right)$ subject to the given production technology and a feasibility constraint: $\mathcal{Q}_t = Z_t \mathcal{K}_t^{\alpha}(1-\Phi_t)$ and $\mathcal{Q}_t = \mathcal{C}_t + \mathcal{K}_{t+1} - (1-\delta)\mathcal{K}_t$. The exogenous technology process is specified as $\mathcal{Z}_t = \mathcal{Z}_0 \exp(z_t)$ where $z_t = \psi z_{t-1} + \epsilon_t$ and $\epsilon_t \sim WN(0, \sigma^2)$ with $\mathcal{Z}_0$ given. The adjustment cost function is: $\Phi_t(\mathcal{K}_{t+1}, \mathcal{K}_t) = \frac{\phi}{2}[\frac{\mathcal{K}_{t+1}}{\mathcal{K}_t} - 1]^2$.

The solution to the log-linearized model takes the form:

$$\begin{aligned} k_{t+1} &= P_{kk} k_t + Q_{kz} z_t \\ c_t &= R_{ck} k_t + S_{cz} z_t \end{aligned}$$

with $z_t = \psi z_{t-1} + \epsilon_t$ and $\epsilon_t \sim WN(0, \sigma^2)$. Here, $P(\theta) = P_{kk}$, $Q(\theta) = Q_{kz}$, $R(\theta) = R_{ck}$, $S(\theta) = S_{cz}$, $\Psi(\theta) = \psi$ and $\Sigma(\theta) = \sigma^2$. The variables in (1) are $Z_t \equiv z_t$, $K_t \equiv k_t$, $W_t \equiv c_t$, with $n_Z = 1$, $n_K = 1$, and $n_W = 1$. The parameter of interest is $\theta \equiv \left(\alpha, \beta, \delta, \phi, \nu, \psi, \sigma^2\right)$. The solution parameter is $\Lambda(\theta) = (P_{kk}, Q_{kz}, R_{ck}, S_{cz}, \psi, \sigma^2)$.

We consider three versions of the stochastic growth model, all with

$$\theta_0 = (\alpha_0, \beta_0, \delta_0, \phi_0, \nu_0, \psi_0, {\sigma_0}^2) = (0.36, 0.95, 0.025, \phi, \nu, 0.85, 0.04).$$

The three models differ in the restrictions imposed on ν and ϕ.

Model I: With $\nu = 1, \phi = 0$, we have a log utility and capital is costless to adjust. Thus the dimension of θ is $n_\theta = 5$. Since $n_Y(n_Z + n_K) + n_Z(n_Z + 1)/2 = 5$, the order condition of Proposition 2 holds. The matrix $\Delta(\theta)$ evaluated at $\theta_0 = (0.36, 0.95, 0.025, 0.85, 0.04)$ has rank $6 = n_\theta + n_Z^2$ which satisfies the rank condition in Proposition 2. Hence, the value θ_0 of the parameter θ in Model I is identifiable from the spectrum of $\{(k_t, c_t)'\}$.

[13] An appendix that explains how to take the numerical results from several solution algorithms to the form in (1) is available upon request. In addition, the appendix contains two other examples to help understand how the proposed rank and order conditions work.

Model II: With $\nu > 0$ unrestricted and $\phi = 0$, we have a power utility and capital adjustment remains costless. Now, $n_\theta = 6$ and the order condition of Proposition 2 fails to hold. Since $n_\theta - n_Y(n_Z + n_K) + n_Z(n_Z + 1)/2 = 1$, we need to introduce at least one restriction. Consider fixing a particular component of θ to its true value. For example, fixing β to 0.95, corresponds to letting $\varphi(\theta) = \beta - 0.95$ in (11). By Proposition 3, conditional identification requires that the rank of the matrix $\Delta_\varphi(\theta)$ be $n_\theta + n_Z^2 = 7$. Table 1 summarizes the rank results obtained by successively fixing one component of θ at a time. As seen from Table 1 fixing any component of θ to its true value other than β or ψ is sufficient for θ_0 to be conditionally identifiable from the spectrum of $\{(k_t, c_t)'\}$.

θ_i	α	β	δ	ν	ψ	σ^2
	7	6	7	7	6	7

Table 1: Rank of $\Delta_\varphi(\theta_0)$ in Model II

To check for partial identifiability of each of the components of θ, we let θ_i be one among 6 components of θ. For the model to be partially identifiable in θ_i, the rank of $\frac{\partial \lambda(\theta_0, \mathrm{Id})}{\partial \theta_{-i}}$ should be equal to $\mathrm{rank}\Delta(\theta_0) - (n_{\theta,i} + n_Z^2) = 4$. As can be seen from Table 2, none of the components of θ are partially identifiable in this model. Indeed, in general equilibrium models, most parameters cannot be identified in isolation.

θ_i	α	β	δ	ν	ψ	σ^2
	5	5	5	5	5	5

Table 2: Rank of $\frac{\partial \lambda(\theta_0, \mathrm{Id})}{\partial \theta_{-i}}$ in Model II

Model III: With both $\nu > 0$ and $\phi > 0$ unrestricted, we extend Model II to allow for costly adjustment of capital. Now, $n_\theta = 7$ and $n_\theta - n_Y(n_Z + n_K) + n_Z(n_Z + 1)/2 = 2$, so at least two restrictions are necessary to satisfy the order condition in Proposition 2. We consider all restrictions obtained by fixing the values of two distinct components θ_i and θ_j of θ. Table 3 reports the ranks of $\Delta_\varphi(\theta)$ evaluated at $\theta_0 = (0.36, 0.95, 0.025, 0.03, 2, 0.85, 0.04)$. Recall that for θ to be conditionally identified under $\varphi(\theta) = 0$ we now need $\mathrm{rank}\, \Delta_\varphi(\theta_0) = n_\theta + n_Z^2 = 8$.

As seen from Table 3 the two restrictions on α and β, for example, are sufficient to identify θ. The rank condition is, however, not trivially satisfied for all parameter restrictions. For example, restricting ψ and any other component of θ gives a reduced rank of $\Delta_\varphi(\theta)$.

θ_i \ θ_j	β	δ	ϕ	ν	ψ	σ
α	8	8	8	8	7	8
β	-	8	8	8	7	8
δ		-	8	8	7	8
ϕ			-	8	7	8
ν				-	7	8
ψ					-	7

Table 3: Rank of $\Delta_\varphi(\theta_0)$ in Model III

Note that when $\phi = 0$ as is assumed for Model II, the required rank is 7. When ϕ is a free parameter as in Model III, the required rank for identification is 8. This discontinuity in the rank condition around $\phi = 0$ implies that when ϕ is a free parameter, the rank of the derivative matrix is not constant in the local neighborhood of $\phi = 0$, holding other parameters fixed. Thus, it cannot be taken for granted that every θ in the parameter space is a regular point.

Finally, we check for partial identifiability of each of the components of θ. For Model III to be partially identifiable in θ_i, the rank of $\frac{\partial\lambda(\theta_0,\mathrm{Id})}{\partial\theta_{-i}}$ should be equal to $\mathrm{rank}\Delta(\theta_0) - (n_{\theta,i} + n_Z^2) = 4$. As can be seen from Table 4, none of the components of θ are partially identifiable in this model.

θ_i	α	β	δ	ϕ	ν	ψ	σ^2
	6	6	6	6	6	5	5

Table 4: Rank of $\frac{\partial\lambda(\theta_0,\mathrm{Id})}{\partial\theta_{-i}}$ in Model III

5.2 Example 2

An and Schorfheide (2007) consider a model whose log-linearized solution is given by:

$$\begin{aligned}
y_t &= E_t y_{t+1} + g_t - E_t g_{t+1} - \frac{1}{\tau}\big(r_t - E_t\pi_{t+1} - E_t z_{t+1}\big) \\
\pi_t &= \beta E_t\pi_{t+1} + \frac{\tau(1-\nu)}{\nu\overline{\pi}^2\phi}(y_t - g_t) \\
c_t &= y_t - g_t \\
r_t &= \rho_r r_{t-1} + (1-\rho_r)\psi_1\pi_t + (1-\rho_r)\psi_2(y_t - g_t) + e_{rt} \\
g_t &= \rho_g g_{t-1} + \epsilon_{gt} \\
z_t &= \rho_z z_{t-1} + \epsilon_{zt}
\end{aligned}$$

with $e_{rt} = \epsilon_{rt}$, $\epsilon_{rt} \sim WN(0, \sigma_r^2)$, $\epsilon_{gt} \sim WN(0, \sigma_g^2)$, and $\epsilon_{zt} \sim WN(0, \sigma_z^2)$ mutually uncorrelated. In

the above model $\overline{\pi}$ is steady state inflation rate. The parameter vector of interest is

$$\theta = (\tau, \beta, \nu, \phi, \overline{\pi}, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r^2, \sigma_g^2, \sigma_z^2).$$

The dimension of θ is $n_\theta = 13$.

This model has three exogenous shocks $\epsilon_t \equiv (\epsilon_{rt}, \epsilon_{gt}, \epsilon_{zt})'$ ($n_Z = 3$), one observed state variable $K_t \equiv r_{t-1}$ ($n_K = 1$), and three additional endogenous variables $W_t \equiv (y_t, \pi_t, c_t)'$ ($n_W = 3$). Letting $Z_t \equiv (e_{rt}, g_t, z_t)'$, the P, Q, R, S representation of An and Schorfheide's (2007) model is:

$$\begin{aligned}
K_{t+1} = \quad r_t \quad &= \quad P_{rr}(\theta)\, K_t + \begin{pmatrix} Q_{rr}(\theta) & 0 & Q_{rz}(\theta) \end{pmatrix} Z_t \\
W_t = \begin{pmatrix} y_t \\ \pi_t \\ c_t \end{pmatrix} &= \begin{pmatrix} R_{yr}(\theta) \\ R_{\pi r}(\theta) \\ R_{cr}(\theta) \end{pmatrix} K_t + \begin{pmatrix} S_{yr}(\theta) & 1 & S_{yz}(\theta) \\ S_{\pi r}(\theta) & 0 & S_{\pi z}(\theta) \\ S_{cr}(\theta & 0 & S_{cz}(\theta) \end{pmatrix} Z_t \\
Z_t = \begin{pmatrix} e_{rt} \\ g_t \\ z_t \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Psi_g & 0 \\ 0 & 0 & \Psi_z \end{pmatrix} \begin{pmatrix} e_{rt-1} \\ g_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{rt} \\ \epsilon_{gt} \\ \epsilon_{zt} \end{pmatrix}
\end{aligned}$$

in which $\Psi_r = \rho_r$, $\Psi_g = \rho_g$, and the P, Q, R, S coefficients are nonlinear functions of θ.

In this example we have $n_\Lambda = n_Y(n_Z + n_K) + 2n_Z^2 = 34$. Since $n_\theta = 13 < n_Y(n_Z + n_K) + n_Z(n_Z + 1)/2 = 22$ the order condition in Proposition 2 is satisfied. The order condition would have failed if the shocks were specified as an unrestricted VAR(1) with a non-diagonal covariance matrix. To check the rank condition, we choose θ_0 as in Table 3 of An and Schorfheide (2007), $\theta_0 = (2, 0.9975, 0.1, 53.6797, 1.008, 1.5, 0.125, 0.75, 0.95, 0.9, 4 \times 10^{-6}, 36 \times 10^{-6}, 9 \times 10^{-6})$. Then, in Proposition 2 we have that $\text{rank}\Delta(\theta_0) = 21 < n_\theta + n_Z^2 = 22$ which shows that the entire parameter vector θ_0 is not identifiable from the spectrum of $\{(r_{t-1}, y_t, \pi_t, c_t)'\}$.

To see if the parameters can be partially identified, we first let θ_i be each of the individual components of θ, and then check the rank condition in Proposition 4. For example, to partially identify β in this model, it is necessary and sufficient that the rank of the partial derivatives of $\lambda(\theta, \text{Id})$ with respect to the remaining components of θ be equal to $21 - (1 + 3^2) = 11$ when evaluated at θ_0. Table 5 reports the rank results. As can be seen from Table 5, the model is not partially identifiable in the parameters τ, β, ν, ϕ and $\overline{\pi}$.

θ_i	τ	β	ν	ϕ	$\overline{\pi}$	ψ_1	ψ_2	ρ_r	ρ_g	ρ_z	σ_r^2	σ_g^2	σ_z^2
	12	12	12	12	12	11	11	11	11	11	11	11	11

Table 5: Rank of $\frac{\partial \lambda(\theta_0, \text{Id})}{\partial \theta_{-i}}$

Now let $\kappa \equiv \frac{\tau(1-\nu)}{\nu \overline{\pi}^2 \phi}$, and consider a new parameterization $\theta = (\tau, \beta, \kappa, \psi_1, \psi_2, \rho_r, \rho_g, \rho_z, \sigma_r^2, \sigma_g^2, \sigma_z^2)$ of dimension $n_\theta = 11$. The new definition of θ reflects the fact that ν, $\overline{\pi}$, and ϕ are not

separately identified as noted in An and Schorfheide (2007). Consider then the value $\theta_0 = (2, 0.9975, 0.33, 1.5, 0.125, 0.75, 0.95, 0.9, 4 \times 10^{-6}, 36 \times 10^{-6}, 9 \times 10^{-6})$. We have that $\text{rank}\Delta(\theta_0) = 20 = n_\theta + n_Z^2$ which shows that the new parameter θ_0 is identifiable from the spectrum.

6 Identification Without Fundamentalness

In the preceding sections, we obtained rank and order conditions for identification of θ from the spectral density $\Omega(z;\theta)$ of the observables by assuming the system to be minimal and the spectrum to have a miniphase property. There are situations when these assumptions might fail to hold. For example, fundamentalness (invertibility) is required for the miniphase property to hold, but this feature is usually not an implication of economic theory. When there are more shocks than variables as in Smets and Wouters (2003), the rank of the spectrum no longer equals the number of shocks. In such a case, the minimum phase property also does not hold, even though the system might still be controllable and observable (hence minimal). On the other hand, the minimal state property is violated when the solution is determinate but the state vector has been expanded to accommodate indeterminacy.

While the transformations in Proposition 1 still yield observationally equivalent solution parameters, without minimality or minimum phase, we can no longer rule out the possibility that other transformations of the solution parameter also lead to observational equivalence. This implies that the rank conditions derived in Propositions 3 and 4 are no longer sufficient for identification. They are however still *necessary*.

Proposition 5 *Let Assumptions 1, 2 and 5 hold. Then the rank condition of Proposition 2 is necessary for the DSGE model solution to be locally identified from the spectrum of $\{Y_t\}$ at a point $\theta = \theta_0$. Similarly, the rank conditions derived in Propositions 3 and 4 are still necessary for conditional and partial identification, respectively.*

Deriving conditions that are *sufficient* for identification when the system is not minimal and does not have minimum phase is nontrivial. In such cases, we need to study directly the equivalence relation (7):

$$H(z;\theta_1)\Sigma(\theta_1)H(z^{-1};\theta_1)' = H(z;\theta_0)\Sigma(\theta_0)H(z^{-1};\theta_0)',$$

for all z in $\mathbb{C}$. A complete analysis requires one to look for the necessary and sufficient conditions under which the a priori restrictions along with the infinite system of equalities: $\Gamma(j;\theta_1) = \Gamma(j;\theta_0)$ obtained for every $j = 0, \ldots, \infty$ can only hold if $\theta_1 = \theta_0$. That for any finite T, $\Gamma(j;\theta_1) = \Gamma(j;\theta_0)$ for every $j = 0, \ldots, T$ implies $\theta_1 = \theta_0$, is of course sufficient for identification. Thus, the conditions of Iskrev (2009) remain sufficient for ascertaining whether θ can be identifiable from the first T

autocovariances of the observables. However, in order to obtain a condition that is also necessary, one needs to check that the infinite set of equations $\Gamma(j,\theta_0) = \Gamma(j,\theta_1)$ for all $j = 0,\ldots,\infty$ has a unique solution $\theta_1 = \theta_0$. Studying local injectivity properties of an infinite system of nonlinear equations is both theoretically challenging and computationally difficult, which is why formally showing that the conditions in Iskrev's (2009) are both necessary and sufficient seems hard.

On the other hand, following the structural identification approach, precise identification results can still be obtained without the minimum phase property since restrictions imposed on similarly transformed systems can be used. However, we need to strengthen the full rank condition in Assumption 4(ii) by replacing it with the following:

Assumption 4(ii)' *For every* $\theta \in \Theta$, $\begin{pmatrix} S(\theta)' & Q(\theta)' & \ldots & (\Psi(\theta)')^{n_Z+n_K-2}Q(\theta)' & (\Psi(\theta)')^{n_Z+n_K-1}S(\theta)' \end{pmatrix}'$ *is full rank.*

Note that Assumption 4(ii) implies the above holds; however, the converse is not true. When the above $n_Y(n_Z+n_K)\times n_Z$ matrix is of full rank n_Z then the solution equations (1) are observable. Taken together Assumptions 4(i) and 4(ii)' guarantee that the solution still has the minimal state property. However, since Assumption 4(ii)' does not require $\begin{pmatrix} Q(\theta) & S(\theta) \end{pmatrix}$ to be full rank, the minimum phase property may not hold. In such situations, we are still able to characterize all observationally equivalent transformations by studying the restrictions on the contemporaneous covariances of the observables.

Proposition 6 *Let Assumptions 1, 2, 3, 4(i), and 4(ii)' hold. Consider two structural parameters* θ_0 *and* θ_1 *with* $(\theta_0,\theta_1) \in \Theta^2$. *Then* θ_0 *and* θ_1 *are observationally equivalent if and only if:*

$$P(\theta_1) = P(\theta_0), \quad R(\theta_1) = R(\theta_0)$$

and there exists a full rank $n_Z \times n_Z$ *matrix* U *and a symmetric* $n_Z \times n_Z$ *matrix* $M = M'$ *such that:*

$$\begin{aligned} Q(\theta_1) &= Q(\theta_0)U, \quad S(\theta_1) = S(\theta_0)U, \quad \Psi(\theta_1) = U^{-1}\Psi(\theta_0)U, \\ M - \Psi(\theta_1)M\Psi(\theta_1)' &= \Sigma(\theta_1) - U^{-1}\Sigma(\theta_0)U^{-1\prime}, \quad Q(\theta_1)M\Psi(\theta_1)' = 0, \quad S(\theta_1)M\Psi(\theta_1)' = 0, \\ Q(\theta_1)MQ(\theta_1)' &= 0, \quad S(\theta_1)MQ(\theta_1)' = 0, \quad \textit{and} \quad S(\theta_1)MS(\theta_1)' = 0. \end{aligned}$$

A proof of Proposition 6 is in Appendix E. We now say that θ_0 is locally identified from the output spectrum of minimal systems if and only if the above equations have a unique solution $\theta_1 = \theta_0$, $U = \text{Id}$, and $M = 0$. The conditions rest on restrictions on the contemporaneous covariances of the observables in similar systems rather than on all the autocovariances. For this reason, we again have a finite number of conditions to check. Furthermore, the conditions do

not depend on the choice of moments or on the data. However, compared to Proposition 1 that additionally uses minimum phase, we now have an additional unknown $n_Z \times n_Z$ symmetric matrix M. Thus less can be identified from minimal systems without the minimum phase property.

7 Concluding Remarks

We study the identification problem by fully exploiting restrictions that need to exist between observationally equivalent parameters of the DSGE model (1). We provide an operational definition of spectral equivalence and show that some solution parameters of DSGE models are generally not identifiable. We then use these restrictions to develop necessary and sufficient conditions for identification.

We close with two practical notes. First, we only evaluate θ at selected values in the examples. Local identification can hold in some parts of the parameter space Θ but not in others. In practice, in order to ensure that every value of θ in Θ is locally identifiable, we would need to determine the rank of $\Delta(\theta)$ over a large grid of points in the parameter space along the lines of Iskrev (2009). This approach should however be taken with caution as it does not guarantee that θ is globally identified in Θ. Second, as noted in the introduction, we have assumed that parameters only identifiable in the steady state have been isolated, such as by analyzing the matrix of derivatives corresponding to the steady state equations. In principle, this can be done jointly with the identification of θ. This is not pursued here to focus on the solution equations.[14]

[14]Christiano, Motto, and Rostagno (2007) split the model parameters into two groups: a set of 26 parameters that control the steady state which they fixed at values taken from the literature, a set of 55 parameters that control the dynamics which they estimate.

Appendix A Proofs of Section 2

Proof of Lemma 2. Recall that for every $t \geqslant 1$ we have:

$$Y_t = H(L;\theta)\epsilon_t$$

where the transfer function $H(z;\theta)$ is an $n_Y \times n_Z$ matrix of polynomials defined in (4). To show invertibility we use the following result: If for every $\theta \in \Theta$, $H(z^{-1};\theta)$ is of rank n_Z in $|z| > 1$, then $H(z^{-1};\theta)$ has a left inverse, i.e. there exists a matrix $G(z^{-1};\theta)$ of dimension $n_Z \times n_Y$ such that $G(z^{-1};\theta)H(z^{-1};\theta) = \mathrm{Id}$. This implies that we can write $G(L;\theta)Y_t = \epsilon_t$ so the process is invertible. We now show that for every $\theta \in \Theta$, $\operatorname{rank} H(z^{-1};\theta) = n_Z$ for every $z \in \mathbb{C}$ such that $|z| > 1$. From (4) we have that:

$$\begin{aligned}
\operatorname{rank} H(z^{-1};\theta) &= \operatorname{rank} \begin{pmatrix} z^{-1}[\mathrm{Id} - P(\theta)z^{-1}]^{-1}Q(\theta)[\mathrm{Id} - \Psi(\theta)z^{-1}]^{-1} \\ \left\{R(\theta)z^{-1}[\mathrm{Id} - P(\theta)z^{-1}]^{-1}Q(\theta) + S(\theta)\right\}[\mathrm{Id} - \Psi(\theta)z^{-1}]^{-1} \end{pmatrix} \\
&= \operatorname{rank} \left[\begin{pmatrix} z[\mathrm{Id} - P(\theta)z^{-1}] & 0 \\ -R(\theta) & \mathrm{Id} \end{pmatrix}^{-1} \begin{pmatrix} Q(\theta) \\ S(\theta) \end{pmatrix} [\mathrm{Id} - \Psi(\theta)z^{-1}]^{-1} \right] \\
&= \operatorname{rank} \begin{pmatrix} Q(\theta) \\ S(\theta) \end{pmatrix} \\
&= n_Z \quad \text{for every } |z| > 1
\end{aligned}$$

where the second to last equality follows by Assumption 2, while the last equality follows by combining Assumptions 3 and 4(ii). □

Appendix B Proofs of Section 3.1

Hereafter, we define the state vector

$$S_t \equiv \begin{pmatrix} Z_t \\ K_t \end{pmatrix}$$

and we use the following state space representation of the DSGE model (1):

$$\begin{aligned}
S_{t+1} &= A(\theta)S_t + B(\theta)\epsilon_{t+1} \\
Y_t &= C(\theta)S_t
\end{aligned} \tag{B.1}$$

where $Y_t = \begin{pmatrix} K_t' & W_t' \end{pmatrix}'$ as before, while the matrices $A(\theta)$, $B(\theta)$, $C(\theta)$ are given by:

$$A(\theta) \equiv \begin{pmatrix} \Psi(\theta) & 0 \\ Q(\theta) & P(\theta) \end{pmatrix},\ B(\theta) \equiv \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix},\ C(\theta) \equiv \begin{pmatrix} 0 & \mathrm{Id} \\ S(\theta) & R(\theta) \end{pmatrix} \tag{B.2}$$

Recall that for every $t \geqslant 1$ we have $Y_t = H(L;\theta)\epsilon_t$. The transfer function $H(z;\theta)$ can be related to the matrices $(A(\theta), B(\theta), C(\theta))$ in (B.2) as:

$$H(z;\theta) = C(\theta)\left[\mathrm{Id} - zA(\theta)\right]^{-1} B(\theta) \tag{B.3}$$

In what follows, we call $(A(\theta), B(\theta), C(\theta))$ a *realization* of the transfer function $H(z;\theta)$. For simplicity, we drop θ from the notations below.

We begin with two useful Lemmas whose proof will be given after the proof to the main proposition.

Lemma B-1 *(Similarity Transform) Let Assumptions 1 through 4 hold. Let $(A(\theta_0), B(\theta_0), C(\theta_0))$ and $(A(\theta_1), B(\theta_1), C(\theta_1))$ be realizations of the transfer functions $H(z;\theta_0)$ and $H(z;\theta_1)$, respectively, with $(\theta_0, \theta_1) \in \Theta^2$. Then the two realizations are equivalent, i.e. $H(z;\theta_0) = H(z;\theta_1)$ for every $z \in \mathbb{C}$, if and only if there exists a nonsingular $(n_Z + n_K) \times (n_Z + n_K)$ matrix $\mathcal{T}$ such that $A(\theta_1) = \mathcal{T} A(\theta_0)\mathcal{T}^{-1}$, $B(\theta_1) = \mathcal{T} B(\theta_0)$, and $C(\theta_1) = C(\theta_0)\mathcal{T}^{-1}$.*

Lemma B-2 *(Minimum Phase) Let Assumptions 1 through 4 hold. Consider two couples $(H(z;\theta_0), \Sigma(\theta_0))$ and $(H(z;\theta_1), \Sigma(\theta_1))$ with spectral densities $\Omega(z;\theta_0)$ and $\Omega(z;\theta_1)$, respectively, where $(\theta_0, \theta_1) \in \Theta^2$. Then the two couples are equivalent, i.e. $\Omega(z;\theta_0) = \Omega(z;\theta_1)$ for every $z \in \mathbb{C}$, if and only if there exists a full rank $n_Z \times n_Z$ matrix U such that:*

$$\text{for every } z \in \mathbb{C} \quad H(z;\theta_1) = H(z;\theta_0)U, \quad \text{and} \quad U\Sigma(\theta_1)U' = \Sigma(\theta_0).$$

Proof of Proposition 1. The proof is obtained by combining the results of Lemmas B-1 and B-2 stated below and whose proofs are found at the end of this Appendix.

From Lemma B-2 we know that $\Lambda(\theta_0)$ and $\Lambda(\theta_1)$ are observationally equivalent if and only if, for every $z \in \mathbb{C}$ we have $H(z;\theta_1) = H(z;\theta_0)U$ with U nonsingular. Expressing this equality in terms of the (A, B, C) matrices in (B.3) and using the similarity transformation result of Lemma B-1 then give the following necessary and sufficient condition for observational equivalence of θ_0 and θ_1:

$$A(\theta_1) = \mathcal{T} A(\theta_0)\mathcal{T}^{-1}, \quad B(\theta_1) = \mathcal{T} B(\theta_0)U, \quad C(\theta_1) = C(\theta_0)\mathcal{T}^{-1}, \tag{B.4}$$

where U is a nonsingular $n_Z \times n_Z$ matrix, and $\mathcal{T}$ is a nonsingular $(n_Z + n_K) \times (n_Z + n_K)$ matrix. Let

$$\mathcal{T} \equiv \begin{pmatrix} T_4 & T_3 \\ T_2 & T_1 \end{pmatrix}$$

be a full rank $(n_Z + n_K) \times (n_Z + n_K)$ matrix, with sub-matrices (T_1, T_2, T_3, T_4) of dimensions $n_K \times n_K$, $n_K \times n_Z$, $n_Z \times n_K$, and $n_Z \times n_Z$, respectively. Then, writing the above equations for

$B(\theta_1) = \mathcal{T} B(\theta_0) U$, $A(\theta_1)\mathcal{T} = \mathcal{T} A(\theta_0)$, and $C(\theta_1)\mathcal{T} = C(\theta_0)$ gives the following:

$$0 = T_2\, U \tag{B.5}$$

$$\mathrm{Id} = T_4\, U \tag{B.6}$$

$$P(\theta_1)T_1 + Q(\theta_1)T_3 = T_1 P(\theta_0) \tag{B.7}$$

$$P(\theta_1)T_2 + Q(\theta_1)T_4 = T_1 Q(\theta_0) + T_2 \Psi(\theta_0) \tag{B.8}$$

$$\Psi(\theta_1)T_3 = T_3 P(\theta_0) \tag{B.9}$$

$$\Psi(\theta_1)T_4 = T_3 Q(\theta_0) + T_4 \Psi(\theta_0) \tag{B.10}$$

$$R(\theta_1)T_1 + S(\theta_1)T_3 = R(\theta_0) \tag{B.11}$$

$$R(\theta_1)T_2 + S(\theta_1)T_4 = S(\theta_0) \tag{B.12}$$

$$T_1 = \mathrm{Id} \tag{B.13}$$

$$T_2 = 0 \tag{B.14}$$

Now, we can rewrite Equation (B.9) as:

$$\left(\mathrm{Id}_{n_K} \otimes \Psi(\theta_1)\right) \mathrm{vec}(T_3) = \left(P(\theta_0)' \otimes \mathrm{Id}_{n_Z}\right) \mathrm{vec}(T_3)$$

where $\mathrm{vec}(T_3)$ denotes the vectorization of the matrix T_3 formed by stacking the columns of T_3 into a single column vector. It follows that:

$$\left[\left(\mathrm{Id}_{n_K} \otimes \Psi(\theta_1)\right) - \left(P(\theta_0) \otimes \mathrm{Id}_{n_Z}\right)'\right] \mathrm{vec}(T_3) = 0$$

which has a unique solution $T_3 = 0$ if and only if $\Psi(\theta_1)$ and $P(\theta_0)$ have no eigenvalues in common (Assumption 2). Then, combining the above with (B.13), (B.14), and (B.6) gives that the matrix $\mathcal{T}$ is necessarily of the form:

$$\mathcal{T} = \begin{pmatrix} U^{-1} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

Then, from (B.7) and (B.8) we get:

$$P(\theta_1) = P(\theta_0) \quad \text{and} \quad Q(\theta_1) = Q(\theta_0)\, U \tag{B.15}$$

Results for $R(\theta_1)$, $S(\theta_1)$, and $\Psi(\theta_1)$ are obtained from Equations (B.11), (B.12), and (B.10), respectively. Using the fact that in addition $\Sigma(\theta_1) = U^{-1}\Sigma(\theta_0)U^{-1'}$ from Lemma B-2 then gives the desired result. □

Proof of Lemma B-1. For simplicity, we drop θ from the notations below. We first give several definitions that have been adapted from the standard setup used in control theory (see, e.g., Antsaklis and Michel, 1997) to the dynamic equations in (B.1):

Definition 2 (Controllability) *Let $\mathcal{C}$ be the $(n_Z + n_K) \times (n_Z(n_Z + n_K))$ matrix defined as:*

$$\mathcal{C} \equiv \begin{pmatrix} B & AB & \dots & A^{n_Z+n_K-1}B \end{pmatrix}$$

A realization (A, B, C) is controllable if and only if rank$\mathcal{C} = n_Z + n_K$.

Definition 3 (Observability) *Let $\mathcal{O}$ be the $n_Y(n_Z + n_K) \times (n_Z + n_K)$ matrix defined as:*

$$\mathcal{O} \equiv \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n_Z+n_K-1} \end{pmatrix}$$

A realization (A, B, C) is observable if and only if rank$\mathcal{O} = n_Z + n_K$.

Note that the matrix $\mathcal{C}$ is called the controllability matrix, while $\mathcal{O}$ is referred to as the observability matrix.

Definition 4 (Minimality) *A realization (A, B, C) is minimal (irreducible, of least order) if and only if it is both controllable and observable.*

In the proof of Lemma B-1 we use the following Lemma whose proof is found at the end of this Appendix.

Lemma B-3 *Let Assumptions 1 through 4 hold. Then, for every $\theta \in \Theta$, the realization $(A(\theta), B(\theta), C(\theta))$ in* (B.2) *is minimal.*

According to Lemma B-3, in our DSGE model (1), under Assumptions 1 to 4, every realization $(A(\theta), B(\theta), C(\theta))$ of the transfer function $H(z; \theta)$ in (B.3) is minimal. We can then use the following result which immediately follows from Theorem 3.10 in Antsaklis and Michel (1997):

Theorem (Similarity Transform). *Let $(A(\theta_0), B(\theta_0), C(\theta_0))$ and $(A(\theta_1), B(\theta_1), C(\theta_1))$ be two realizations of $H(z; \theta_0)$ in* (B.3)*. If $(A(\theta_0), B(\theta_0), C(\theta_0))$ is minimal, then $(A(\theta_1), B(\theta_1), C(\theta_1))$ is also minimal if and only if there exists a nonsingular $(n_Z + n_K) \times (n_Z + n_K)$ matrix $\mathcal{T}$ such that $A(\theta_1) = \mathcal{T}A(\theta_0)\mathcal{T}^{-1}$, $B(\theta_1) = \mathcal{T}B(\theta_0)$, and $C(\theta_1) = C(\theta_0)\mathcal{T}^{-1}$.*

Now our model can be put in an (A, B, C) form like (B.1)-(B.2) and by Lemma B-3, all realizations are minimal. By the above theorem, they then must be related by a similarity transformation which gives the result as desired. □

Proof of Lemma B-2. In the spectral factorization problem, one is given a $n \times n$ matrix $\phi(z)$ of real rational functions of a complex variable z, with $\phi(z^{-1})' = \phi(z)$ and $\phi(\exp(i\omega))$ positive semi-definite for all real ω. One is required to find a $W(z)$ such that $W(z^{-1})'W(z) = \phi(z)$, with $W(z)$ real and rational, of entries analytic in $|z| \geqslant 1$, and possibly with $W(z)$ of constant rank in $|z| > 1$. A spectral factor $W_d(z)$ of dimension $r \times n$ is termed *minimum phase* if $W_d(z^{-1})'W_d(z) = \phi(z)$ and $W_d(z)$ has rank r everywhere in $|z| > 1$ where r is equal to the rank almost everywhere of $\phi(z)$, i.e. rank$\phi(z) = r$ for almost every $z \in \mathbb{C}$. As shown in Anderson (1969) or Anderson, Hitz, and Diem (1974), for example, minimum phase spectral factors are unique to within left multiplication by an arbitrary real constant orthogonal $r \times r$ matrix V, i.e. any other minimum phase factor $\widetilde{W}_d(z)$ is of the form:

$$\widetilde{W}_d(z) = VW_d(z) \quad \text{where} \quad V'V = \mathrm{Id}$$

Now, fix $\theta_0 \in \Theta$ and consider a solution parameter $\Lambda(\theta_0) = (P(\theta_0), Q(\theta_0), R(\theta_0), S(\theta_0), \Psi(\theta_0), \Sigma(\theta_0))$ that generates the spectral density $\Omega(z;\theta_0)$ of the observables $\{Y_t\}$. Under Assumption 1, the matrix $\Sigma(\theta_0)$ is real, symmetric and positive definite and $L(\theta_0)$ is the lower triangular matrix obtained by the Cholesky decomposition of $\Sigma(\theta_0)$, i.e. $\Sigma(\theta_0) = L(\theta_0)L(\theta_0)'$. Note that since $\Sigma(\theta_0)$ has real entries, $L(\theta_0)$ can be assumed to have real entries as well. In addition, $L(\theta_0)$ is positive definite. Now consider the following $n_Z \times n_Y$ matrix:

$$W_d(z) \equiv L(\theta_0)'H(z^{-1};\theta_0)'. \tag{B.16}$$

We have

$$\begin{aligned} W_d(z^{-1})'W_d(z) &= H(z;\theta_0)L(\theta_0)L(\theta_0)'H(z^{-1};\theta_0)' \\ &= H(z;\theta_0)\Sigma(\theta_0)H(z^{-1};\theta_0)' \\ &\equiv \Omega(z;\theta_0) \end{aligned}$$

for every $z \in \mathbb{C}$. Note that by (4), $\Omega(z;\theta_0)$ is a matrix of real rational functions of a complex variable z, with $\Omega(z^{-1};\theta_0)' = \Omega(z;\theta_0)$ and $\Omega(\exp(i\omega);\theta_0)$ positive semi-definite for all real ω.

We now show that $\operatorname{rank} W_d(z) = n_Z$ for every $z \in \mathbb{C}$ such that $|z| > 1$. From the proof of Lemma 2 we know that (4) $\operatorname{rank} H(z^{-1};\theta_0) = n_Z$ for every $|z| > 1$. Using (B.16) and the fact that $L(\theta_0)$ is positive definite, we have that $\operatorname{rank} W_d(z) = n_Z$ for every $|z| > 1$. It remains to be shown that $\operatorname{rank} \Omega(z;\theta_0) = n_Z$ for almost every $z \in \mathbb{C}$. For this, recall again from the proof of Lemma 2 that:

$$H(z^{-1};\theta_0) = \begin{pmatrix} z[\mathrm{Id} - P(\theta_0)z^{-1}] & 0 \\ -R(\theta_0) & \mathrm{Id} \end{pmatrix}^{-1} \begin{pmatrix} Q(\theta_0) \\ S(\theta_0) \end{pmatrix} [\mathrm{Id} - \Psi(\theta_0)z^{-1}]^{-1}$$

which is rank deficient only for those $z \in \mathbb{C}$ that are such that $\det[z\mathrm{Id} - P(\theta_0)]\det[z\mathrm{Id} - \Psi(\theta_0)] = 0$, i.e. only at the eigenvalues of the matrices $P(\theta_0)$ and $\Psi(\theta_0)$. Given that there is at most $n_Z + n_K$

different eigenvalues, $H(z^{-1};\theta_0)$ is of full rank almost everywhere in $\mathbb{C}$. Since by Assumption 1, $\Sigma(\theta_0)$ is full rank, it follows that $\Omega(z;\theta_0)$ is of rank n_Z almost everywhere in $\mathbb{C}$.

We can now apply the result in Anderson, Hitz, and Diem (1974) to show that if $\widetilde{W_d}(z) \equiv L(\theta_1)'H(z^{-1};\theta_1)'$ is another minimum phase factor, then necessarily we have:

$$L(\theta_1)'H(z^{-1};\theta_1)' = \widetilde{W_d}(z) = VW_d(z) = VL(\theta_0)'H(z^{-1};\theta_0)'$$

where V is an orthogonal $n_Z \times n_Z$ matrix, $V'V = \text{Id}$. Now let:

$$U \equiv L(\theta_0)V'L(\theta_1)^{-1}$$

be a full rank $n_Z \times n_Z$ matrix. It then follows that $H(z;\theta_1) = H(z;\theta_0)U$ for every $z \in \mathbb{C}$ and $UL(\theta_1)V = L(\theta_0)$, so $UL(\theta_1)VV'L(\theta_1)'U^{-1} = L(\theta_0)L(\theta_0)'$, i.e. $U\Sigma(\theta_1)U' = \Sigma(\theta_0)$. □

Proof of Lemma B-3. The proof of Lemma B-3 is done in two steps.

Step 1. We first show controllability. For this, note that for any $1 \leqslant k \leqslant n_Z + n_K - 1$, we have:

$$A^kB = \begin{pmatrix} \Psi^k \\ P^{k-1}Q + P^{k-2}Q\Psi + \ldots + PQ\Psi^{k-2} + Q\Psi^{k-1} \end{pmatrix}$$

so it holds that for every $1 \leqslant k \leqslant n_Z + n_K - 1$:

$$A^kB - A^{k-1}B\Psi = \begin{pmatrix} 0 \\ P^{k-1}Q \end{pmatrix} \tag{B.17}$$

Now consider the following $(n_Z(n_Z + n_K)) \times (n_Z(n_Z + n_K))$ matrix $\mathcal{J}$:

$$\mathcal{J} \equiv \begin{pmatrix} \text{Id} & -\Psi & & (0) \\ & \ddots & \ddots & \\ & & \ddots & -\Psi \\ (0) & & & \text{Id} \end{pmatrix}$$

Since $\mathcal{J}$ is nonsingular, we have that:

$$\text{rank}(\mathcal{C}\mathcal{J}) = \text{rank}(\mathcal{C})$$

Now using the property in (B.17), we have that:

$$\mathcal{C}\mathcal{J} = \begin{pmatrix} \text{Id} & 0 & 0 & \ldots & 0 \\ 0 & Q & PQ & \ldots & P^{n_Z+n_K-2}Q. \end{pmatrix}$$

Under Assumption 4(i) we have that $\text{rank}\begin{pmatrix} Q & PQ & \ldots & P^{n_Z+n_K-2}Q \end{pmatrix} = n_K$. Then, it holds that $\text{rank}(\mathcal{C}\mathcal{J}) = n_Z + n_K$, so (A, B, C) is controllable.

STEP 2. To show observability, first note that for any $1 \leqslant k \leqslant n_Z + n_K - 1$, we have:

$$CA^k = \begin{pmatrix} P^{k-1}Q + P^{k-2}Q\Psi + \ldots + PQ\Psi^{k-2} + Q\Psi^{k-1} & P^k \\ RP^{k-1}Q + RP^{k-2}Q\Psi + \ldots + RPQ\Psi^{k-2} + RQ\Psi^{k-1} + S\Psi^k & RP^k \end{pmatrix}$$

Now consider the $n_Y(n_Z + n_K) \times n_Y(n_Z + n_K)$ matrix $\mathcal{Q}$ defined as:

$$\mathcal{Q} \equiv \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 & & (0) \\ 0 & \mathrm{Id} & 0 & 0 & & \\ -P & 0 & \mathrm{Id} & 0 & & \\ 0 & 0 & -R & \mathrm{Id} & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & -P & 0 & \mathrm{Id} & 0 \\ (0) & & 0 & 0 & -R & \mathrm{Id} \end{pmatrix}$$

Note that $\det\mathcal{Q} = 1$ so Q is of full rank and we have:

$$\mathrm{rank}(\mathcal{Q}\mathcal{O}) = \mathrm{rank}\mathcal{O}$$

Now, straightforward calculations yield:

$$\mathcal{Q}\mathcal{O} = \begin{pmatrix} 0 & \mathrm{Id} \\ S & R \\ Q & 0 \\ S\Psi & 0 \\ \vdots & \vdots \\ Q\Psi^{n_Z+n_K-2} & 0 \\ S\Psi^{n_Z+n_K-1} & 0 \end{pmatrix}$$

Under Assumption 4(ii) it follows that

$$\mathrm{rank}\begin{pmatrix} 0 & \mathrm{Id} \\ S & R \\ Q & 0 \end{pmatrix} = n_Z + n_K$$

which shows that $\mathrm{rank}(\mathcal{O}) = \mathrm{rank}(\mathcal{Q}\mathcal{O}) = n_Z + n_K$, and (A, B, C) is observable. □

Appendix C Proofs of Section 3.2

Proof of Lemma 3 The proof is similar to that of Theorem 1 in Glover and Willems (1974). The necessity is straightforward. We now prove sufficiency by proving the contrapositive: suppose that θ_0 is not locally identifiable. Then there exists an infinite sequence of parameter vectors $\{\theta_1, \ldots, \theta_k, \ldots\}$ (of dimension n_θ) approaching θ_0 such that:

$$\Omega(z; \theta_k) = \Omega(z; \theta_0) \quad \text{for all } z \in \mathbb{C}.$$

From Proposition 1 we know that the above holds if and only if there exists an infinite sequence of full rank $n_Z \times n_Z$ matrices $\{T_1, \ldots, T_k, \ldots\}$ such that:

$$P(\theta_k) = P(\theta_0), \quad Q(\theta_k)T_k = Q(\theta_0), \quad R(\theta_k) = R(\theta_0), \quad S(\theta_k)T_k = S(\theta_0), \tag{C.1}$$

$$T_k^{-1}\Psi(\theta_k)T_k = \Psi(\theta_0) \quad \text{and} \quad T_k^{-1}\Sigma(\theta_k)T_k^{-1\prime} = \Sigma(\theta_0). \tag{C.2}$$

In other words, using the notation in (8) we have that:

$$\lambda(\theta_k, T_k) = \lambda(\theta_0, \mathrm{Id})$$

In order to show that the mapping λ is not locally injective, it suffices to show that the sequence $\{T_1, \ldots, T_k, \ldots\}$ approaches the identity matrix Id. For this, note that from (C.1) we have:

$$\begin{pmatrix} Q(\theta_k) \\ S(\theta_k) \end{pmatrix} T_k = \begin{pmatrix} Q(\theta_0) \\ S(\theta_0) \end{pmatrix}$$

Now, from Assumption 4(ii) we know that the $n_Y \times n_Z$ matrix $(Q(\theta_k)', S(\theta_k)')'$ is of rank n_Z; hence, it is left invertible and we have

$$T_k = \left[\begin{pmatrix} Q(\theta_k)' & S(\theta_k)' \end{pmatrix} \begin{pmatrix} Q(\theta_k) \\ S(\theta_k) \end{pmatrix} \right]^{-1} \begin{pmatrix} Q(\theta_k)' & S(\theta_k)' \end{pmatrix} \begin{pmatrix} Q(\theta_0) \\ S(\theta_0) \end{pmatrix}$$

By continuity of $Q(\theta)$ and $S(\theta)$ (Assumption 5), it then follows that the sequence $\{T_1, \ldots, T_k, \ldots\}$ approaches the identity matrix Id. Hence, λ is not injective in the neighborhood of $(\theta_0, \mathrm{Id}_{n_Z})$. □

Proof of Proposition 2 The proof of Proposition 2 is done in two steps.

Step 1. We start by establishing the necessary and sufficient (rank) condition. To compute the matrix of partial derivatives of λ, we use the product rule: if $F(X)$ and $G(X)$ are, respectively, $m \times p$- and $p \times q$-variate and differentiable functions with respect to X, then

$$\frac{\partial \mathrm{vec}(F(X)G(X))}{\partial \mathrm{vec}X} = (G(X)' \otimes \mathrm{Id}_m)\frac{\partial \mathrm{vec}(F(X))}{\partial \mathrm{vec}X} + (\mathrm{Id}_q \otimes F(U))\frac{\partial \mathrm{vec}(G(X))}{\partial \mathrm{vec}X}$$

For example, if X is an $n_Z \times n_Z$ matrix ($X \in \mathbb{R}^{n_Z \times n_Z}$), then applying the above rule to $F(X) = X^{-1}$ (respectively $F(X) = X'$) and $G(X) = X$ we get the following useful results:

$$\frac{\partial \mathrm{vec}(X^{-1})}{\partial \mathrm{vec}X} = -(X^{-1\prime} \otimes X^{-1}) \quad \text{and} \quad \frac{\partial \mathrm{vec}(X^{-1\prime})}{\partial \mathrm{vec}X} = -T_{n_Z,n_Z}(X^{-1\prime} \otimes X^{-1})$$

where T_{n_Z,n_Z} is the $n_Z^2 \times n_Z^2$ permutation matrix that transforms $\mathrm{vec}X$ into $\mathrm{vec}X'$, i.e. $T_{n_Z,n_Z}\mathrm{vec}X = \mathrm{vec}X'$. Note that T_{n_Z,n_Z} is an orthogonal matrix: $T_{n_Z,n_Z} = T_{n_Z,n_Z}^{-1}$ and $T_{n_Z,n_Z} = T_{n_Z,n_Z}'$. In addition, note that we have $\mathrm{rank}(\mathrm{Id}_{n_Z^2} + T_{n_Z,n_Z}) = \frac{n_Z(n_Z+1)}{2}$.

STEP 1A. We first compute the partial derivatives of $\lambda(\theta, U)$ with respect to the components of θ. Using the product rule, it immediately follows that:

$$\frac{\partial \lambda(\theta,U)}{\partial \theta} = \begin{pmatrix} \frac{\partial P(\theta)}{\partial \theta} \\ (U' \otimes \mathrm{Id}_{n_Z}) \frac{\partial Q(\theta)}{\partial \theta} \\ \frac{\partial R(\theta)}{\partial \theta} \\ (U' \otimes \mathrm{Id}_{n_Z}) \frac{\partial S\theta)}{\partial \theta} \\ (U' \otimes U^{-1}) \frac{\partial \Psi(\theta)}{\partial \theta} \\ (U^{-1} \otimes U^{-1}) \frac{\partial \Sigma(\theta)}{\partial \theta} \end{pmatrix} \tag{C.3}$$

STEP 1B. We now compute the derivatives of $\lambda(\theta, U)$ with respect to U. For $\mathrm{vec}(Q(\theta)U)$ and $\mathrm{vec}(S(\theta)U)$ applying the product rule immediately gives

$$\frac{\partial \mathrm{vec}(Q(\theta)U)}{\partial \mathrm{vec}U} = (\mathrm{Id}_{n_Z} \otimes Q(\theta)) \tag{C.4}$$

$$\frac{\partial \mathrm{vec}(S(\theta)U)}{\partial \mathrm{vec}U} = (\mathrm{Id}_{n_Z} \otimes S(\theta)) \tag{C.5}$$

For $\mathrm{vec}(U^{-1}\Psi(\theta)U)$, we let $F(U) = U^{-1}$ and $G(U) = \Psi(\theta)U$. Then,

$$\begin{aligned} \frac{\partial \mathrm{vec}(U^{-1}\Psi(\theta)U)}{\partial \mathrm{vec}U} &= -\big((\Psi(\theta)U)' \otimes \mathrm{Id}_{n_Z}\big)(U^{-1\prime} \otimes U^{-1}) + (\mathrm{Id}_{n_Z} \otimes U^{-1})(\mathrm{Id}_{n_Z} \otimes \Psi(\theta)) \\ &= -\big((U'\Psi(\theta)'U^{-1\prime}) \otimes U^{-1}\big) + \big(\mathrm{Id}_{n_Z} \otimes (U^{-1}\Psi(\theta))\big) \\ &= \big(U' \otimes U^{-1}\big)\big((\mathrm{Id}_{n_Z} \otimes \Psi(\theta)) - (\Psi(\theta)' \otimes \mathrm{Id}_{n_Z})\big)\big((U^{-1})' \otimes \mathrm{Id}_{n_Z}\big) \end{aligned} \tag{C.6}$$

where the second and third equality use the fact that for any conformable matrices A, B, C, D we have $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. Finally, letting $F(U) = U^{-1}$ and $G(U) = \Sigma(\theta)U^{-1\prime}$ we get:

$$\begin{aligned} \frac{\partial \mathrm{vec}(U^{-1}\Sigma(\theta)U^{-1\prime})}{\partial \mathrm{vec}U} &= ((U^{-1}\Sigma(\theta)) \otimes \mathrm{Id}_{n_Z}) \frac{\partial \mathrm{vec}(U^{-1})}{\partial \mathrm{vec}U} + (\mathrm{Id}_{n_Z} \otimes U^{-1}) \frac{\partial \mathrm{vec}(U^{-1\prime})}{\partial \mathrm{vec}U} \\ &= -(\mathrm{Id}_{n_Z^2} + T_{n_Z,n_Z})((U^{-1}\Sigma(\theta)U^{-1\prime}) \otimes U^{-1}) \end{aligned} \tag{C.7}$$

Combining (C.4) through (C.7) we then get:

$$\frac{\partial \lambda(\theta,U)}{\partial \mathrm{vec}U} = \begin{pmatrix} 0_{n_K^2 \times n_Z^2} \\ \mathrm{Id}_{n_Z} \otimes Q(\theta) \\ 0_{n_K n_W \times n_Z^2} \\ \mathrm{Id}_{n_Z} \otimes S(\theta) \\ \big(U' \otimes U^{-1}\big)\big((\mathrm{Id}_{n_Z} \otimes \Psi(\theta)) - (\Psi(\theta)' \otimes \mathrm{Id}_{n_Z})\big)\big((U^{-1})' \otimes \mathrm{Id}_{n_Z}\big) \\ -(\mathrm{Id}_{n_Z^2} + T_{n_Z,n_Z})((U^{-1}\Sigma(\theta)U^{-1\prime}) \otimes U^{-1}) \end{pmatrix} \tag{C.8}$$

Now let

$$\Delta(\theta) \equiv \begin{pmatrix} \frac{\partial \lambda(\theta,\mathrm{Id})}{\partial \theta} & \frac{\partial \lambda(\theta,\mathrm{Id})}{\partial \mathrm{vec} U} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathrm{vec} P(\theta)}{\partial \theta} & 0_{n_K^2 \times n_Z^2} \\ \frac{\partial \mathrm{vec} Q(\theta)}{\partial \theta} & \mathrm{Id}_{n_Z} \otimes Q(\theta) \\ \frac{\partial \mathrm{vec} R(\theta)}{\partial \theta} & 0_{n_K n_W \times n_Z^2} \\ \frac{\partial \mathrm{vec} S(\theta)}{\partial \theta} & \mathrm{Id}_{n_Z} \otimes S(\theta) \\ \frac{\partial \mathrm{vec} \Psi(\theta)}{\partial \theta} & \mathrm{Id}_{n_Z} \otimes \Psi(\theta) - \Psi(\theta)' \otimes \mathrm{Id}_{n_Z} \\ \frac{\partial \mathrm{vec} \Sigma(\theta)}{\partial \theta} & -(\mathrm{Id}_{n_Z^2} + T_{n_Z,n_Z})(\Sigma(\theta) \otimes \mathrm{Id}_{n_Z}) \end{pmatrix} \tag{C.9}$$

and note that using (C.8) we can write

$$\begin{pmatrix} \frac{\partial \lambda(\theta,U)}{\partial \theta} & \frac{\partial \lambda(\theta,U)}{\partial \mathrm{vec} U} \end{pmatrix} = M(U) \Delta(\theta) N(U)$$

where $M(U)$ and $N(U)$ are, respectively, an $n_\Lambda \times n_\Lambda$ diagonal matrix and an $(n_\theta + n_Z^2) \times (n_\theta + n_Z^2)$ diagonal matrix defined as:

$$M(U) \equiv \begin{pmatrix} \mathrm{Id}_{n_K^2} & & & & & \\ & U' \otimes \mathrm{Id}_{n_Z} & & & & \\ & & \mathrm{Id}_{n_K n_W} & & & \\ & & & U' \otimes \mathrm{Id}_{n_Z} & & \\ & & & & U' \otimes U^{-1} & \\ & & & & & U^{-1} \otimes U^{-1} \end{pmatrix}$$

$$N(U) \equiv \begin{pmatrix} \mathrm{Id}_{n_\theta} & \\ & U^{-1\prime} \otimes \mathrm{Id}_{n_Z} \end{pmatrix}$$

Since U is full rank, both $M(U)$ and $N(U)$ are full rank, so we have that

$$\mathrm{rank} \begin{pmatrix} \frac{\partial \lambda(\theta,U)}{\partial \theta} & \frac{\partial \lambda(\theta,U)}{\partial \mathrm{vec} U} \end{pmatrix} = \mathrm{rank}\, \Delta(\theta)$$

Note that if the rank of $\Delta(\theta)$ remains constant in a neighborhood of θ_0, then the rank of the partial derivatives of λ remains constant in a neighborhood of (θ_0, Id).

Step 2. We now establish the necessary (order) condition. Counting the number of linearly independent rows in the matrix $\Delta(\theta)$ in (C.9), it is easy to show that:

$$\mathrm{rank}\Delta(\theta) \leqslant n_Y(n_Z + n_K) + n_Z^2 + \frac{n_Z(n_Z+1)}{2}$$

where we have used the fact that $\Sigma(\theta)$ is always symmetric so:

$$\mathrm{rank} \begin{pmatrix} \frac{\partial \mathrm{vec} \Sigma(\theta)}{\partial \theta} & -(\mathrm{Id}_{n_Z^2} + T_{n_Z,n_Z})(\Sigma(\theta) \otimes \mathrm{Id}_{n_Z}) \end{pmatrix} \leqslant \frac{n_Z(n_Z+1)}{2}.$$

Hence, a necessary condition for (9) to hold is that:

$$n_\theta + n_Z^2 \leqslant n_Y(n_Z + n_K) + n_Z^2 + \frac{n_Z(n_Z+1)}{2} \qquad \text{i.e.} \qquad n_\theta \leqslant n_Y(n_Z + n_K) + \frac{n_Z(n_Z+1)}{2}$$

□

Appendix D Proofs of Section 4

Proof of Proposition 3 To begin, note that the matrix of partial derivatives of χ can be written as:

$$\begin{pmatrix} \frac{\partial \chi(\theta,U)}{\partial \theta} & \frac{\partial \chi(\theta,U)}{\partial \text{vec}U} \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi(\theta)}{\partial \theta} & 0_{n_\varphi \times n_Z^2} \\ \frac{\partial \lambda(\theta,U)}{\partial \theta} & \frac{\partial \lambda(\theta,U)}{\partial \text{vec}U} \end{pmatrix}$$

Using the same reasoning as in the proof of Proposition 2, the above can further be written as:

$$\begin{pmatrix} \frac{\partial \chi(\theta,U)}{\partial \theta} & \frac{\partial \chi(\theta,U)}{\partial \text{vec}U} \end{pmatrix} = \begin{pmatrix} \text{Id}_{n_\varphi} & 0_{n_\varphi \times n_\Lambda} \\ 0_{n_\Lambda \times n_\varphi} & M(U) \end{pmatrix} \Delta_\varphi(\theta) N(U)$$

where the $n_\Lambda \times n_\Lambda$ matrix $M(U)$ and $(n_\theta + n_Z^2) \times (n_\theta + n_Z^2)$ matrix $N(U)$ are as in the proof of Proposition 2, and where we have let:

$$\Delta_\varphi(\theta) \equiv \begin{pmatrix} \frac{\partial \chi(\theta,\text{Id})}{\partial \theta} & \frac{\partial \chi(\theta,\text{Id})}{\partial \text{vec}U} \end{pmatrix} = \begin{pmatrix} \frac{\partial \varphi(\theta)}{\partial \theta} & & 0_{n_\varphi \times n_Z^2} \\ & \Delta(\theta) & \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial \varphi(\theta)}{\partial \theta} & 0_{n_\varphi \times n_Z^2} \\ \frac{\partial \text{vec}P(\theta)}{\partial \theta} & 0_{n_K^2 \times n_Z^2} \\ \frac{\partial \text{vec}Q(\theta)}{\partial \theta} & \text{Id}_{n_Z} \otimes Q(\theta) \\ \frac{\partial \text{vec}R(\theta)}{\partial \theta} & 0_{n_K n_W \times n_Z^2} \\ \frac{\partial \text{vec}S(\theta)}{\partial \theta} & \text{Id}_{n_Z} \otimes S(\theta) \\ \frac{\partial \text{vec}\Psi(\theta)}{\partial \theta} & \text{Id}_{n_Z} \otimes \Psi(\theta) - \Psi(\theta)' \otimes \text{Id}_{n_Z} \\ \frac{\partial \text{vec}\Sigma(\theta)}{\partial \theta} & -(\text{Id}_{n_Z^2} + T_{n_Z,n_Z})(\Sigma(\theta) \otimes \text{Id}_{n_Z}) \end{pmatrix}$$

It then follows that:

$$\text{rank} \begin{pmatrix} \frac{\partial \chi(\theta,U)}{\partial \theta} & \frac{\partial \chi(\theta,U)}{\partial \text{vec}U} \end{pmatrix} = \text{rank}\Delta_\varphi(\theta)$$

from which the rank condition follows. For the order condition, note that we again have:

$$\text{rank} \begin{pmatrix} \frac{\partial \chi(\theta,U)}{\partial \theta} & \frac{\partial \chi(\theta,U)}{\partial \text{vec}U} \end{pmatrix} \leqslant n_\varphi + n_Y(n_Z + n_K) + n_Z^2 + \frac{n_Z(n_Z+1)}{2}$$

so a necessary condition for identifiability of θ under the a priori restrictions (11) is:

$$n_\theta + n_Z^2 \leqslant n_\varphi + n_Y(n_Z + n_K) + n_Z^2 + \frac{n_Z(n_Z+1)}{2} \quad \text{i.e.} \quad n_\varphi \geqslant n_\theta - \left[n_Y(n_Z + n_K) + \frac{n_Z(n_Z+1)}{2} \right]$$

□

Appendix E Proofs of Section 6

Proof of Proposition 6 We follow the structural identification approach Glover (1973) which is based on the Positive Real Lemma of Anderson (1969), Glover (1973), and Glover and Willems

(1974). Consider a minimal linear dynamical system relating the $(n_Z + n_K)$ state vector $S_t = (Z_t', K_t')'$, to the n_Y observable vector $Y_t = (K_t', W_t')'$ and a latent n_Z shock $\widetilde{\epsilon}_t$ given by:

$$\begin{aligned} S_{t+1} &= \widetilde{A}(\theta)S_t + \widetilde{B}(\theta)\widetilde{\epsilon}_t \\ Y_t &= \widetilde{C}(\theta)S_t + \widetilde{D}(\theta)\widetilde{\epsilon}_t \end{aligned} \tag{E.1}$$

with $\widetilde{\epsilon}_t \sim WN(0, \mathrm{Id}_{n_Z})$. Let $\widetilde{H}(z;\theta)$ be the corresponding transfer function, i.e. $Y_t = \widetilde{H}(z;\theta)\widetilde{\epsilon}_t$ with

$$\widetilde{H}(z;\theta) = \widetilde{C}(\theta)[z\mathrm{Id} - \widetilde{A}(\theta)]^{-1}\widetilde{B}(\theta) + \widetilde{D}(\theta). \tag{E.2}$$

Corollary 4.5 Glover (1973) states that in the system (E.1) θ_1 and θ_0 are observationally equivalent if and only if there exists a symmetric $(n_Z + n_K) \times (n_Z + n_K)$ matrix $\mathcal{M}$ and a full rank $(n_Z + n_K) \times (n_Z + n_K)$ matrix $\mathcal{T}$ such that:

$$\begin{aligned} \mathcal{M} - \widetilde{A}(\theta_1)\mathcal{M}\widetilde{A}(\theta_1)' &= \widetilde{B}(\theta_1)\widetilde{B}(\theta_1)' - \mathcal{T}\widetilde{B}(\theta_0)\widetilde{B}(\theta_0)'\mathcal{T}' \\ \widetilde{C}(\theta_1)\mathcal{M}\widetilde{C}(\theta_1)' &= \widetilde{D}(\theta_1)\widetilde{D}(\theta_1)' - \widetilde{D}(\theta_0)\widetilde{D}(\theta_0)' \\ \widetilde{C}(\theta_1)\mathcal{M}\widetilde{A}(\theta_1)' &= \widetilde{D}(\theta_0)\widetilde{B}(\theta_0)'\mathcal{T}' - \widetilde{D}(\theta_1)\widetilde{B}(\theta_1)' \\ \widetilde{A}(\theta_1)\mathcal{T} &= \mathcal{T}\widetilde{A}(\theta_0) \\ \widetilde{C}(\theta_0) &= \widetilde{C}(\theta_1)\mathcal{T} \end{aligned}$$

In order to apply Corollary 4.5 in Glover (1973), we first need to map our set up, which is given by (B.1) with his setup, which is given by (E.1). Comparing (E.1) with (B.1), it is clear that we can go from one representation to the other by simply letting:

$$\widetilde{A}(\theta) = A(\theta), \quad \widetilde{B}(\theta) = B(\theta)L(\theta), \quad \widetilde{C}(\theta) = C(\theta), \quad \widetilde{D}(\theta) = 0, \quad \text{and} \quad \widetilde{\epsilon}_t = L(\theta)^{-1}\epsilon_{t+1}, \tag{E.3}$$

where $L(\theta)$ is the Cholesky triangle of $\Sigma(\theta)$, i.e. $\Sigma(\theta) = L(\theta)L(\theta)'$. Under Assumptions 1, 2, 3, 4(i) and 4(ii)', Lemma B-3 still applies and the A, B, C representation in (B.1) is minimal. Thus, Corollary 4.5 applied to our setup becomes:

$$\mathcal{M} - A(\theta_1)\mathcal{M}A(\theta_1)' = B(\theta_1)\Sigma(\theta_1)B(\theta_1)' - \mathcal{T}B(\theta_0)\Sigma(\theta_0)B(\theta_0)'\mathcal{T}' \tag{E.4}$$

$$C(\theta_1)\mathcal{M}C(\theta_1)' = 0 \tag{E.5}$$

$$C(\theta_1)\mathcal{M}A(\theta_1)' = 0 \tag{E.6}$$

$$A(\theta_1)\mathcal{T} = \mathcal{T}A(\theta_0) \tag{E.7}$$

$$C(\theta_0) = C(\theta_1)\mathcal{T} \tag{E.8}$$

where

$$A(\theta) \equiv \begin{pmatrix} \Psi(\theta) & 0 \\ Q(\theta) & P(\theta) \end{pmatrix}, \; B(\theta) \equiv \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix}, \; C(\theta) \equiv \begin{pmatrix} 0 & \mathrm{Id} \\ S(\theta) & R(\theta) \end{pmatrix}.$$

Partitioning the matrix $\mathcal{T}$ as

$$\mathcal{T} = \begin{pmatrix} T_4 & T_3 \\ T_2 & T_1 \end{pmatrix}$$

and writing the above equalities for $A(\theta_1)\mathcal{T} = \mathcal{T}A(\theta_0)$ and $C(\theta_0) = C(\theta_1)\mathcal{T}$ gives:

$$T_4 = T, \quad T_3 = 0, \quad T_2 = 0, \quad T_1 = \mathrm{Id},$$
$$\Psi(\theta_1)T = T\Psi(\theta_0), \quad P(\theta_1) = P(\theta_0), \quad Q(\theta_1)T = Q(\theta_0), \quad R(\theta_1) = R(\theta_0), \quad S(\theta_1)T = S(\theta_0)$$

where T is a full rank $n_Z \times n_Z$ matrix. Similarly, partitioning the matrix $\mathcal{M}$ as

$$\mathcal{M} = \begin{pmatrix} M_1 & M_2 \\ M_2' & M_4 \end{pmatrix}$$

with $M_1 = M_1'$ and $M_4 = M_4'$, and writing the above equalities for $C(\theta_1)\mathcal{M}C(\theta_1)' = 0$ gives:

$$\begin{aligned} 0 &= M_4 \\ 0 &= M_2'S(\theta_1)' + M_4R(\theta_1)' \\ S(\theta_1)M_1S(\theta)_1' + R(\theta_1)M_2'S(\theta_1)' + 0 &= S(\theta_1)M_2R(\theta_1)' + R(\theta_1)M_4R(\theta_1)' \end{aligned}$$

from which it follows that

$$M_4 = 0, \quad M_2'S(\theta_1)' = 0, \quad S(\theta_1)M_1S(\theta)_1' = 0. \tag{E.9}$$

Using the above with the equalities for $C(\theta_1)\mathcal{M}A(\theta_1)' = 0$ gives:

$$M_2'\Psi(\theta_1)' = 0, \quad M_2'Q(\theta_1)' = 0, \quad S(\theta_1)M_1\Psi(\theta_1)' = 0, \quad S(\theta_1)M_1Q(\theta_1)' = 0 \tag{E.10}$$

Finally, combining all of the above with (E.4) gives:

$$M_1 - \Psi(\theta_1)M_1\Psi(\theta_1)' = \Sigma(\theta_1) - T\Sigma(\theta_0)T', \quad M_2 - \Psi(\theta_1)M_1Q(\theta_1)' = 0, \quad Q(\theta_1)M_1Q(\theta_1)' = 0. \tag{E.11}$$

Now, $M_2'S(\theta_1)' = 0$, $M_2'Q(\theta_1)' = 0$, and $M_2'\Psi(\theta_1)' = 0$ together imply:

$$M_2' \begin{pmatrix} S(\theta)' & Q(\theta)' & \dots & (\Psi(\theta)')^{n_Z+n_K-2}Q(\theta)' & (\Psi(\theta)')^{n_Z+n_K-1}S(\theta)' \end{pmatrix}' = 0.$$

Since by Assumption 4(ii)' the second matrix is of full rank n_Z, the above implies that $M_2 = 0$. Gathering all of the above, and letting $M \equiv M_1$ gives:

$$\begin{aligned}
P(\theta_1) &= P(\theta_0) \\
Q(\theta_1) &= Q(\theta_0)T^{-1} \\
R(\theta_1) &= R(\theta_0) \\
S(\theta_1) &= S(\theta_0)T^{-1} \\
\Psi(\theta_1) &= T\Psi(\theta_0)T^{-1} \\
M - \Psi(\theta_1)M\Psi(\theta_1)' &= \Sigma(\theta_1) - T\Sigma(\theta_0)T' \\
Q(\theta_1)MQ(\theta_1)' &= 0 \\
Q(\theta_1)M\Psi(\theta_1)' &= 0 \\
S(\theta_1)M\Psi(\theta_1)' &= 0 \\
S(\theta_1)MQ(\theta_1)' &= 0 \\
S(\theta_1)MS(\theta_1)' &= 0
\end{aligned}$$

where T is full rank and $M = M'$. Letting $U = T^{-1}$ then gives the desired result. □

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