Private-Value Auctions, Resale, and Common Value

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Abstract

We establish the bid-equivalence between an independent private-value (IPV) first-price auction model with resale and a model of first-price common-value auctions. The common value is defined by the transaction price when trade takes place. When there is no trade, the common value is defined through a monotonic extension of the resale price function. We show bid equivalence when (1) there are two bidders with a general resale mechanism; (2) there is one regular buyer and many speculators with a monopoly resale market; (3) there are two groups of identical bidders and the winning bidder in the resale stage conducts a second-price auction to sell the object to the losing bidders. The buyer-speculators model of auction with resale is bid-equivalent to the Wilson Drainage Tract common value model. We assume that the resale market satisfies a weak efficiency property, which means that trade must be efficient when the trade surplus is the highest possible under incomplete information. We show that when the weak efficiency property fails, the bid equivalence result may fail.

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1 Introduction

In this paper we study the way resale opportunities after the auction may affect bidders' behavior in the auction. When resale is allowed after the auction, we refer to it as an auction with resale, and represent it by a two-stage game. It is intuitively understood in the profession that resale is an important source of common- value among the bidders. In the survey for their book, Kagel and Levin (2002, page 2) said that "There is a common-value element to most auctions. Bidders for an oil painting may purchase for their own pleasure, a private-value element, but they may also bid for investment and eventual resale, reflecting the common-value element". Comments reflecting this conventional wisdom can also be found, for example, in Ashenfelter and Genesove (1992), Chakravorti et al (1995), Cramton (1995), Hausman and Wittman (1993), McAfee and McMillan (1987), McMillan (1994), Milgrom (1987), McMillan (1994), Rosenthal and Wang (1996), Rothkopf and Harstad (1994).

Earlier theoretical studies on the connection between resale and common value were carried out in Haile (2000,2001,2003), Gupta and Lebrun (1999), and Lebrun (2007). The idea was exploited in Gupta and Lebrun (1999) and Lebrun (2007) when the resale is a monopoly or monopsony market. Theoretical implications of resale opportunities on auctions have been studied in Bikhchandani and Huang (1989), Haile (2000, 2003) and Hafalir and Krishna (2008). Resale is sometimes assumed to have complete information to simplify the analysis as in Milgrom (1987), Kamien et al. (1989), Gupta and Lebrun (1999), Gale et al. (2000) and Haile (2003).

On the empirical side, Haile (2001) studied the empirical evidence of the effects of resale in the U.S. forest timber auctions. Haile, Hong, and Shum (2003) studied the U.S. forest lumber auctions and found the bidding data to conform to private-value auctions in some and common-value auctions in others. An explanation may be due to the presence or lack of resale. Hortascu and Kastl (2008) showed that bidding data for 3-month treasury bills are more like private-value auctions, but not so for 12-month treasure bills. The difference may be due to the relevance of resale in long term treasury bills. When an asset is held in a longer period, there may be a need for resale, and this may affect bidding behavior.

Restrictions on resale were often imposed by government agencies. In U.S. forest timber auctions, third-party transfers were prohibited except in approved circumstances. Forest Service policy identifying these circumstances included vague guidelines allowing transfers which "protect the interest of the United States" (U.S. Forest Service, 1976, 1981). However, resale in timber auctions can take place through subcontracting as described by Haile (2001). In spectrum auctions held by many governments, there were often restrictions on resale. For example, in the British 3-G spectrum auctions[1] of 2000, resale restrictions were

[1] The third-generation technology allows high speed data access to the internet. It was held on Mar 6, 2000, and concluded on April 27, 2000, raising 22.5 billion pounds (2.5% of the GNP of UK). This revenue is seven times the original estimate. Five licenses A,B,C,D,E

imposed despite economists' recommendation to the contrary. It is not clear why the restrictions were imposed. It is possible that the government may look bad when the bidders can turn around and resell for quick profits after the auction[2]. Bidders, however, find ways to circumvent such restrictions in the form of a change of ownership control. For example, a month after the British 3-G auction, Orange, the winner of the license E, was acquired by France Telecom, yielding a profit of 2 billion pounds to Vodafone[3]. The winner of the most valuable licence A is TIW (Telesystem International Wireless). In July 2000, Hutchison then sold 35% of its share in TIW to KPN and NTT DoCoMo with an estimated profit of 1.6 billion pounds[4].

Resale is sometimes conducted so that parties who did not or could not participate in the auction has a chance to acquire the object sold during the auction. This occurs for example in Bikhchandani and Huang (1989), where gains to resale trade arise due to the presence of new buyers in the secondary market. We will focus on resale among bidders of the original auction. There are situations in which resale opportunities to a third party only affect a bidder's valuation of the object, and thus can be indirectly represented by a change in valuations. In this case, our model may allow third party participation. However third party participation may give rise to issues that are not explored here.

By focusing on the resale among bidders, we study an interesting interaction between resale in the second stage and bidding behavior in the first stage. We will in fact show that in private-value auctions with resale, the bidding data will behave as if it is a common-value auction. This has been informally observed In Gupta and Lebrun (1999), and Lebrun (2007) in cases when the resale market is a simple monopoly or monopsony market. We will provide a theoretical examination for this intuition in more general resale environments. The main idea of the paper is that it is the resale price, not the private valuation, which determines the bidder behavior.

We describe the phenomenon by the term "bid-equivalence". It means that the bidding behavior of an independent private-value auction with resale is the same as a pure common-value auction in which the common-value is defined by the transaction price. The two auctions have the same equilibrium bid distributions. The concept of bid-equivalence is slightly weaker than the observational

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were offered. Licence A was available for bidding by non-incumbent operators only. A more detailed account is given in Klemperer (2004).

[2] Beyond the political and legal reasons, resale may facilitate collusions in the English auction as is shown in Garrat, Troger and Zheng (2007).

[3] Orange paid 4 billion pounds for the licence. In May of 2000, France telecom paid 6 billion pounds more than the price Mannesmann had paid for it in October 1999 before the auction. Orange was the number three UK mobile group at the time. The reason for the resale is due to a divestment agreement by Vodafone with the British government after acquiring Mannesmann.

[4] TIW was a Canadian company based in Montreal and largely owned by Hutchison Whampoa. The Hong Kong conglomerate) gained the upper hand when NTL Mobile, a joint venture of the UK cable operator and France Telecom, withdrew from the bidding. The price of the licence is 4.4 billion pounds. The profit is based on the implicit valuation of the license at 6 billion in the transaction.

equivalence used in Green and Laffont (1987). In observational equivalence, the payoffs of the bidders and the revenue of the auctioneer are the same for two different auctions. In our bid-equivalence, the payoffs of the bidders need not be the same in auctions with resale and common-value auctions. Laffont and Vuong (1996) showed that for any fixed number of bidders in a first-price auction, any symmetric affiliated values model is observationally equivalent to some symmetric affiliated private-values model. In a symmetric model, there is no incentive for resale. In our paper, we look at the asymmetric IPV auctions with resale. We show that when bidders anticipate trading activities after the auction, the equilibrium bidding behavior is the same as in a common-value auction. The gain from trade in our model is due to the asymmetric valuations of the bidders in the beginning of the first stage. In Haile's model, gain from resale is due to different valuations at the resale stage due to arrival of new information. A discussion of the bid equivalence in Haile's model is provided in section 6.

For the case of two-bidder auctions with resale, we establish a very general bid equivalence result. The framework is flexible enough to include a resale process in which bidders make sequential offers, or simultaneous offers in the resale stage. The main assumption is a weak efficiency property which says that trade should be efficient (occur with probability one) when the trade surplus is the highest possible. It is easily satisfied in most Bayesian bargaining equilibrium with sequential or simultaneous offers. It rules out the no-trade equilibrium in which there cannot be bid-equivalence. Our formulation of the weak efficiency property is equivalent to the sure-trade property in Hafalir and Krishna (2008)[5].

The bid equivalence result holds more generally with many bidders. However, the most general treatment cannot be offered in this paper due to complexity. For the multiple bidders case, it seems likely that there may be more restrictions on the resale mechanism for the bid equivalence result to hold. In particular the monopoly resale model may not work, as it violates the weak efficiency property when there are more than two buyers in the resale stage. A precise formulation of the resale mechanisms for the bid equivalence to hold is still in progress. With more than two bidders, a general existence result for the auctions with resale is still lacking in the literature. The model of auctions with resale with multiple asymmetric bidders is still not well-developed. For the bid equivalence result of this paper, we make two simplifying assumptions. First we assume that there are only two groups of bidders in the auction, and within each group, all bidders are identical. Secondly, in the resale stage, we assume that the winner of the auction sells the object by a second-price auction to the losers of the auction. One advantage of using the second-price auction resale mechanim is that it is efficient so that after the resale, there is no need

[5] Hafalir and Krishna (2008) use it to show the bidding symmetry property of weak and strong bidders in equilibrium. We use it to show the bid-equivalence result. In more general models (such as affiliated signals or more than three bidders), bid equivalence may hold even though the symmetry property typically fails.

for further resales. Although this is not fully general, experimental or empirical work is likely to be conducted under similar simplifications in the beginning. Our result will provide a good foundation for further such studies. This is done in section 7.

The bid-equivalence result may be somewhat surprising. One would expect that resale only contributes a common-value "component" to the bidding behavior, and there is still a private-value component. Our result however says that the bidding behavior is the same as if it is a pure common-value model. What happens to private-value component? The answer is that bid-equivalence is true only in equilibrium, hence the private-value is still relevant out-of-equilibrium. Furthermore, the private-value is incorporated in the definition of the common-value, and is therefore indirectly affecting the equilibrium bidding behavior. In equilibrium, the payoffs of bidders in the auction with resale is different from the payoffs in the common-value auction, even though the bidding behavior is the same. Therefore, it is not correct to say that we have equilibrium equivalence.

Auction with resale in general can be a very complicated game. The resale game may involve potentially complicated sequences of offers, rejections, and counter-offers. In-between the auction stage and the resale stage, there may be many possible bid revelation rules that affect the beliefs of the bargainers in the resale stage. We cannot deal with so many issues at the same time. For the bid revelation rule, we adopt the simplest framework of minimal information, i.e. that is there is no bid revelation in-between the two stages. Despite the lack of bid information, the bidders update their beliefs after winning or losing the auction. Since bidders with different valuations bid differently in the first stage, the updated beliefs depend on the bid in the first stage. For this reason, there are heterogeneous beliefs in the resale stage. Furthermore, a bidder may become a seller or a buyer in the resale stage depending on the bidding behavior in the first stage. This distinguishes the bargaining in the resale stage from the standard bargaining game in the literature in which there is a fixed seller or buyer, and the beliefs are homogeneous. The outcome of the bargaining in general is different. For instance, the updating may improve the efficiency of the bargaining compared to the standard homogenous model, as bargainers have better information.

One important implication of the bid equivalence result is that the equilibrium analysis of the two-stage auction with resale model is reduced to the simpler equilibrium analysis of the one-stage common-value model. The revenue of the auction with resale is completely determined by the common-value function, and the efficiency of the auction with resale is determined by the trading set as well as the common-value function. Applications of the bid equivalence result can be found in Cheng and Tan (2008) for the monopoly and monopsony cases and in Cheng and Tan (2009) for the general resale mechanism. Such applications enhance our understanding of the revenue impact of resale, and how the revenue ranking of different auctions may be affected by the resale opportunities.

The bid equivalence result also has important implications for the empirical and experimental study of private vs. common value auctions such as Haile, Hong, and Shum (2003) and Hortascu and Kastl (2008). It implies that the distinction between private and common value auctions is not purely a matter of which good is sold in the auctions. The existence of resale opportunities, and bidders' intention for resale are also important factors to consider. Some experimental work has been done on the asymmetric auctions with resale model (see Georganas and Kagel (2009)). It is an interesting question to ask whether bidders behave the same way in both models in practice or in experiments. Given the evidence that winner's curse is observed in many experiments in common-value auctions, it is interesting to see whether similar irrationality is observed in the auctions with resale. There is strong empirical support for the bidding behavior in the Wilson drainage track model. It is also promising that such support can be found in empirical studies of auctions with resale.

Section 2 presents the common value model. Section 3 describes the auction with resale model. Section 4 gives the bid equivalence result. Section 5 shows that the buyer-speculators model of auction with resale is bid equivalent to the Wilson tract model with one neighbor and many non-neighbors. Section 6 discuss the difference and similarities of our model and that of Haile. Section 7 gives a bid equivalence result for the many bidders case when the seller conducts a second-price auction to sell the object to the losing bidders in the resale stage.

2 The Common-Value Model

There are M risk neutral bidders in an auction for a single object. Each bidder receives a signal $s_i \in [0, s_i^*]$ about the unknown common value of the object. Let F_i be the cumulative distribution of the signal s_i. Assume that F_i has a continuous density function f_i. The (expected) common value conditional on the joint signal is denoted by $w(s_1, ..., s_M)$. All signals s_i are assumed to be independent from each other. We refer to the function w as the common value function. We will assume that the function w is continuously differentiable,non-decreasing in each t_i, $w(0,0) = 0$, and $w(t,t)$ is strictly increasing. Let $b_i(s_i)$ be a strictly increasing bidding strategy of bidder i, and ϕ_i^{-1} be its inverse. The definition of ϕ_i can be extended beyond the bidding range if we let $\phi(b) = s_i^*$ when b is outside the range. Let $\int\int_0^{z_j} |_{j \neq i}$ denote the integration over all signals except s_i, and the interval of integration for $s_j, j \neq i$ is $[0, z_j]$. Let s_{-i} denote the vector $s = (s_1, s_2, .., s_M)$ without the s_i component. Let $dF_{-i}(s_{-i}) = \prod_{j \neq i} dF_j(s_j)$. If a bidder i with signal s_i bids b, the expected profit in a first-price auction is

$$\pi_C = \int\int_0^{\phi_j(b)} |_{j \neq i} \, [w(s_i, s_{-i}) - b] dF_{-i}(s_{-i}).$$

We say that b is an optimal bid if it maximizes π_C. The bidding strategy profile $b_i(.), i = 1, 2, ..., M$ is called a Bayesian Nash equilibrium if each $b_i(s_i)$ is an optimal bid.

The signals can be normalized so that F_i is uniformly distributed over $[0, 1]$. This normalization will be done in the case of two-bidder model and we let t_i denote the normalized signal. Let $b_i(t_i)$ be the strictly increasing equilibrium bidding strategy of bidder i in the first-price common-value auction, and $\phi_i(b)$ be its inverse. In the two-bidder model, we have the following first order condition satisfied by the equilibrium bidding strategy

$$\frac{d \ln \phi_i(b)}{db} = \frac{1}{w(\phi_1(b), \phi_2(b)) - b} \text{ for } i = 1, 2. \tag{1}$$

with the boundary conditions $\phi_i(0) = 0, \phi_1^{-1}(1) = \phi_2^{-1}(1)$. The ordinary differential equation system with the boundary conditions determine the equilibrium bidding strategy profile.

Although there are well-known equilibrium existence theorems for the first-price common-value auction, we will give a constructive proof of the equilibrium in the two-bidder model. This proof will be useful for us in the bid equivalence result. The proof makes use of the single crossing properties. A function $h(t)$ satisfies the single crossing condition if $h(t) \geq 0$ when $h(t') \geq 0$, and $t > t'$. The function $h(t)$ satisfies the strict single crossing condition if $h(t) > 0$ when $h(t') \geq 0$, and $t > t'$. The function $g(b, t)$ satisfies the strict single crossing differences condition if for all $x' > x$, the function $h(t) = g(x', t) - g(x, t)$ satisfies the strict single crossing condition. The function $g(b, t)$ satisfies the smooth single crossing differences condition if it satisfies the single crossing differences condition, and for all b, $g_1(b, t) = 0, \delta > 0$, we must have $g_1(b, t + \delta) \geq 0$, and $g_1(b, t - \delta) \leq 0$.

Note that the equilibrium strategies in the following theorem depends only on the value of w on the diagonal $\{(t, t) : t \in [0, 1]\}$. Our approach shows that the equilibrium is not affected when w is modified outside the diagonal.

Theorem 1 *The pair of bidding strategies*

$$b_i(t_i) = \frac{1}{t_i} \int_0^{t_i} w(s, s) ds, t_i > 0, b_i(0) = 0$$

is an equilibrium in the first-price common-value auction with two bidders.

Proof. Since $w(s, s)$ is a continuous function, b_i is continuously differentiable

in $(0, 1]$. By the L'Hopital rule, $b_i(t_i) \to 0$ as $t_i \to 0$. Hence b_i is continuous on $[0, 1]$. Let $\phi_i = b_i^{-1}$ be the inverse bidding function. Then ϕ_i is also continuously differentiable in $(0, 1]$. We have

$$t_i b_i(t_i) = \int_0^{t_i} w(s, s) ds \tag{2}$$

Taking derivative of (2), we have

$$t_i b_i'(t_i) + b_i(t_i) = w(t_i, t_i).$$

Since $\phi_i'(b(t_i)) = \frac{1}{b_i'(t_i)}$, we have

$$\frac{t_i}{\phi_i(b_i(t_i))} = w(t_i, t_i) - b_i(t_i)$$

or

$$\frac{\phi_i(b)}{\phi_i'(b)} = w(\phi_1(b), \phi_2(b)) - b \text{ for all } b > 0,$$

and we know that the first-order conditions (1) are satisfied by $\phi_i, i = 1, 2$. We want to apply the Sufficiency Theorem (Theorem 4.2 of Milgrom (2004)). Let

$$g(b, t_1) = \int_0^{\phi_2(b)} (w(t_1, t_2) - b) dt_2$$

This function is continuously differentiable in (b, t_1). Let $b' > b$. We have

$$g(b', t_1) - g(b, t_1) = -(b' - b)\phi_2(b) + \int_{\phi_2(b)}^{\phi_2(b')} (w(t_1, t_2) - b') dt_2.$$

Since w is non-decreasing in t_1, the single crossing differences condition is satisfied. The partial derivative of g with respect to b is

$$g_1(b, t_1) = (w(t_1, \phi_2(b)) - b)\phi_2'(b) - \phi_2(b).$$

Since w is monotonic in t_1, the smooth single crossing differences condition is satisfied. The Sufficiency Theorem then says that $b_1(t_1)$ is an equilibrium strategy of bidder one. The proof for bidder two is the same. ■

3 Auctions with Resale

Our model begins with an independent private-value (IPV) auction with resale. A pure common-value auction will be constructed from the IPV auction and the resale process. Due to the complexity of the general multiple bidders case, we assume that there are two groups of bidders, group one and two with m, n members respectively. Each bidder has a private valuation whose cumulative distribution is $F_i(x)$ over $[0, a_i]$. Bidders' private valuations are independent from other. In addition, in each group all bidders have the same $F_i = F_1$ for group one and $F_i = F_2$ for group two. We assume that each F_i is continuously differentiable. Each bidder participates in a two stage game. In the first stage, a bidder makes a bid b in a first-price auction. We will only consider the case in which each bidder in the same group adopts the same strictly increasing bidding strategy in the first stage (except the buyer-speculators in section 6). Let $b_k, k = 1, 2$ be the strictly increasing bidding strategy of a group k bidder, and ϕ_k^{-1} be the inverse bidding strategy. Since the maximum bid for a different group may be different, we define $\phi_k(b) = a_k$, if b is outside the bidding range.

The first-price auction with resale is a two-stage game. The bidders participate in a standard sealed-bid first-price auction in the first stage. In the second stage, there is a resale game. At the end of the auction and before the resale stage, some information about the submitted bids may be available. The disclosed bid information in general changes the beliefs of the valuation of the other bidder. This may further change the outcome of the resale market. We shall adopt the simplest formulation in which no bid information is disclosed[6]. We call this the minimal information case. It should be noted that there is valuation updating even if there is no disclosure of bid information, as information about the identity of the winner alone leads to updating of the beliefs.

If a bidder i wins the auction, she may sell it to the losing bidders in the resale stage. If she loses the auction, she may buy it from the winning bidder. Her belief about group k bidders' valuations after winning the auction is F_k conditional on $[0, \phi_k(b)]$. If she loses the auction, the belief is F_k conditional on $[\phi_k(b), a_k]$. This belief system in the resale stage is common knowledge.

Since bidder i with different valuations bid differently in the first stage, they have different updated beliefs in the resale game. We refer to this as heterogeneous belief as a contrast to the standard multi-lateral trade model in which a trader's belief is independent of his valuation. We refer to this as homogeneous beliefs. The equilibrium behavior in the second stage resale game is therefore different from the standard bargaining model with homogeneous beliefs. Furthermore, each trader in the resale trade game may be a seller or a buyer depending on whether he wins or loses the auction in the first stage. This endogenous determination of the seller-buyer role is also absent from the standard bargaining model with homogeneous beliefs.

A trader's strategy in the resale game is therefore affected by her bid b_i in the first-stage auction, and whether she wins or loses the first-stage auction. She is not affected by b_j of the other bidders as they are not observable. We do not need to explicitly specify the resale game. Let $\sigma_k^w(., b_k(.))$ be the strategy in the second stage when the bidder is in group k, and wins the auction. If the bidder loses the auction, and another bidder of the same group wins the auction, we assume that there is no resale activity for the bidder, and a strategy need not be specified. Let $\sigma_k^l(., b_k(.))$ be the bidder's buying strategy in the second stage when the a member from another group wins. Each bidder has such a pair of strategies in the resale game. In equilibrium, we require that each $\sigma_i^w(x_i, b_i(x_i))$ is an optimal response to $\sigma_j^l(., b_j(.)), j \notin i$, and each $\sigma_i^l(x_i, b_i(x_i))$ is an optimal

[6] Although the equivalence result may be established in a broader context with disclosure of different bid information, it is sufficient to restrict ourselves to the resale market with no disclosure of bid information in this paper. Lebrun (2007) has shown that an equilibrium with bid disclosure is observationally equivalent to an equilibrium with no bid disclosure. Therefore the bid equivalence can be extended to the bid disclosure case. Haile (2001,2003) allowed bid disclosure in his model and this is discussed in section 6 below.

response to $\sigma_j^c(., b_j(.)), j \notin i, c = w$ for exactly one j from a different group. We call this a Bayesian Nash equilibrium of the resale game. According to the revelation principle, there is a direct trade mechanism $M^k, k = 1, 2$ in the resale stage in which truth-telling is incentive compatible and individually rational with the same equilibrium outcome. The mechanism M^k refers to the case when a group k members wins. We have suppressed the dependence on the bidding strategies $b_k, k = 1, 2$. We also need to consider the case when a bidder i deviates to the bid b_i, and let the resulting direct mechanism be denoted by $M^k(b_i)$.

In the direct trade mechanism, given the reported valuations, $x = (x_i)_{i=1}^{m+n}$, when bidder i in group one wins the auction, let $q^w(x, b_i)$ be the probability of trade, and $p^w(x, b_i)$ be the price of the transaction. When she loses the auction, let $q^l(x, b_i), p^l(x, b_i)$ be the probability of trade and transsaction price respectively. If she wins the auction, let $R_1(b)$ be the revenue from selling the object in the resale stage (when there is no resale, the revenue is assumed to be her own valuation). We have the expected revenue

$$R_1(b) = E\left[(p^w(x_i, x_{-i})q^w(x_i, x_{-i}) + (1 - q^w(x_i, x_{-i}))x_i | x_i, b\right].$$

If she loses the auction, the (net) profit from buying the object in the resale game is

$$R_2(b) = E\left[(x_i - p^l(x_i, x_{-i}))q^l(x_i, x_{-i}) | x_i, b\right].$$

The net profit from bidding b is given by

$$\pi_R = R_1(b) + R_2(b) - bF_1^{m-1}(\phi_1(b))F_2^n(\phi_2(b)). \tag{3}$$

Similarly, we can define $R_1(b), R_2(b)$, and the profit function of a group two bidder j bidding b given by

$$\pi_R = R_1(b) + R_2(b) - bF_1^m(\phi_1(b))F_2^{n-1}(\phi_2(b)).$$

The bid b is optimal if it maximizes π_R. The bidding strategy profile $b_1(.), b_2(.)$ in the first-stage auction is a perfect Bayesian Nash equilibrium if each $b_k(z)$ is an optimal bid for a bidder in group $k = 1, 2$ with valuation z. In this definition, the equilibrium property in the first stage relies on the mechanism $M^k(b_i)$ of the second stage, and at the same time, the equilibrium in the resale game also relies on the belief system genereted by the bidding strategies $b_k, k = 1, 2$.

For the rest of the section, we consider the two-bidder case. We find it useful to adopt a signal representation of the IPV model. This representation (called a distributional approach) is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004). This will allow us to apply the model to the case of discrete distributions, and make good use of the property of symmetric bid distributions in the two-bidder model.

In this representation, a bidder is described by an increasing valuation function $v_i(t_i) : [0, 1] \to [0, a_i]$, with the interpretation that $v_i(t_i)$ is the private valuation of bidder i when he or she receives the private signal t_i. We can normalize the signals so that both signals are uniformly distributed over $[0, 1]$. The two signals are assumed to be independent. The word "private" refers to the important property that bidder i's valuation is not affected by the signal t_j of the other bidder, while in the common-value model, this is not the case. The function $v_i(t_i)$ induces a distribution on $[0, a_i]$, whose cumulative probability distribution is given by $F_i(x_i) = v_i^{-1}(x_i)$. If $v_1(t) \leq v_2(t)$ for all t, we say that bidder one is the weak bidder, bidder two is the strong bidder, and the pair is a weak-strong pair of bidders[7]. This is equivalent to saying that F_1 is first-order stochastically dominated by F_2.

Let $q^w(x_1, x_2, b_1), p^w(x_1, x_2, b_1)$ be the allocation and pricing system when bidder one wins, and $q^l(x_1, x_2), p^l(x_1, x_2)$ when bidder one loses. We shall assume that the resale game satisfies the following weak efficiency property $q^w(x_1, \phi_2(b_1); b_1) = 1$ if either bidder one with valuation x_1 wins the auction and $x_1 \leq \phi_2(b_1)$, or bidder one loses the auction and $x_1 \geq \phi_2(b_1)$. Note that the pair $(x_1, \phi_2(b_1))$ has the highest trade surplus. Efficiency for the pair means that transaction must takes place for this pair. It is a weak efficiency property for the resale mechanism because it only requires efficiency for the pair with the highest trade surplus[8].

In equilibrium $b_1 = b_1(x_1)$, we can just write $q^w(x_1, x_2), p^w(x_1, x_2,)$ for $q^w(x_1, x_2, b_1), p^w(x_1, x_2, b_1)$ respectively. Furthermore, winning and losing cases are mutually exclusive, and the underlying mechanism is clearly understood. Therefore, we shall skip the superscript and just write $q(x_1, x_2), p(x_1, x_2,)$. Let Q be the set of (t_1, t_2) such that $q(x_1(t_1), x_2(t_2)) = 1$ and assume that $p(v_1(t_1), v_2(t_2))$ is nondecreasing in each variable, continuously differentiable on Q, and $p(v_1(t), v_2(t))$ is strictly increasing in t. The set Q determines whether there is resale, and who gets the good at the end. Therefore it is important for the study of efficiency properties of the auctions with resale.

Let b^* be the maximum bid of the bidders. By Proposition 3 (section 5.1) of Hafalir and Krishna (2008), the following first-order condition and symmetry property hold for the equilibrium of auction of resale.

Proposition 2 *If the inverse equilibrium bidding functions $\phi_i, i = 1, 2$ are differentiable in $(0, b^*]$, then the following first-order conditions are satisfied*

$$\frac{d \ln F_i(\phi_i(b)))}{db} = \frac{1}{p(\phi_1(b), \phi_2(b)) - b}, i = 1, 2, b \in (0, b^*]. \tag{4}$$

and we have $F_1(\phi_1(b)) = F_2(\phi_2(b))$ for all $b \in [0, b^*]$.

[7] Here we only require that F_1 is dominated by F_2 in the sense of the first order stochastic dominance. Note that this concept is weaker than that of Maskin and Riley (2000), in which conditional stochastic dominance is imposed.

[8] It can be shown that in a standard bilateral trade with incomplete information, an incentive efficient mechanism automatically satisfies the weak efficient property. The bargaining model here is not a standard one due to heterogeneous beliefs. However, we believe that the same result should hold in this case as well.

In most interesting bilateral trade models, such as $k-$double auctions or sequential bargaining models, it is often the case that trade takes place with probability 0 or 1. For the standard bilateral trade model, Myerson and Satterthwaite (1983) and Satterthwaite and Williams (1989) have shown that trade occurs with probability 1 or 0 when they are incentive efficient. In this case, Q is the set of pairs for which trade occurs for sure, and outside Q there is no trade. The description of the resale process includes most of the well-known equilibrium models of bilateral bargaining between the seller and the buyer. The following examples illustrate some of the most often used models of bilateral trade.

Example 3 *Monopoly or Monopsony Resale*

There is a weak bidder (one) and a strong bidder (two). Assume that the winner of the auction is the monopolist seller in the resale game. This is the monopoly resale model (with commiment). The monopolist makes a take-it-or-leave-it offer, and the transaction price is the optimal monopoly price. Bidder one with signal t_1 has the valuation $v_1(t_1)$ and bids b in the first stage. When she wins the auction, she believes that bidder two's valuation is F_2 over $[0, \phi_2(b)]$. If $\phi_2(b) \geq v_1(t_1)$, bidder one makes an optimal offer to bidder two. Assume that there is a uniquely determined optimal offer (equilibrium) price $p(t_1, b)$. Then trade takes plance if and only if $v_2(t_2) \geq p(t_1, b)$. If she loses the auction, and $\phi_2(b) \leq v_1(t_1)$, then bidder two is the monopolist, and makes offers to bidder one. Let $p(t_2, b)$ be the optimal monopoly offer of bidder two, then trades takes place if and only if $v_1(t_1) \leq p(t_2, b)$. In equilibrium, only bidder one makes offers, and we have $Q = \{(t_1, t_2) : t_1 \geq t_2, v_2(t_2) \geq p(t_1, b_1(t_1))\}$. Hence trade occurs if and only if $(t_1, t_2) \in Q$, and the trading price is the optimal offer price $p(t_1, b_1(t_1))$. Note that the weak efficiency property is satisfied becasue an optimal monopolist price does not exceed the highest valuation buyer, so that trade occurs with probability 1 when the buyer has the highest valuation $\phi_2(b_1(t_1))$.

Similarly, in a monopsony resale mechanism with a take-it-or-leave-it offer by the buyer, the buyer chooses an optimal monopsony price higher than the lowest possible valuation of the seller. The offer is accepted when the seller has the lowest valuation, hence the weak efficiency property also holds, and the transaction price is the optimal monopsony price. In equilibrium only bidder two makes offers.

Another variation is to designate one of the bidder, say bidder one, as the offer-maker. When it is not a weak-strong pair, bidder one may become a seller or a buyer depending on the realized signals. Thus it is a mixture of the monopoly and the monopsony market. The choice of the offer-maker or the market type affects the bargaining power of the bidders and the outcome of the resale. In the case of the monopoly market mechanism, the choice of the offer-maker is not fixed in the beginning, and is contingent on the outcome of the auction.

Example 4 *Endogeneous Determination of Buyer-Seller Role*

To illustrate the endogenous nature of the seller or buyer role in the resale, let the bidders' valuation distributions be given by $F_1(x) = x$,

$$\begin{aligned} F_2(y) &= 2y^2 \text{ when } y \in [0, 0.5] \\ &= 4y - 2y^2 - 1 \text{ when } y \in [0.5, 1] \end{aligned}$$

The resale is a monopoly. In equilibrium, if $x < 0.5$, bidder one bids $b_1(x)$ and wins the auction, he is the seller. The optimal monopoly price is

$$p = \frac{2x + \sqrt{4x^2 + 6x}}{6}.$$

Trade occurs if $y \geq p$ or

$$t_2 \geq 2\left(\frac{2x + \sqrt{4x^2 + 6x}}{6}\right)^2$$

There is no resale, if he loses the auction. If $x > 0.5$, and bidder one wins the auction, there is no resale. If he loses the auction, he is a buyer. the optimal offer by buyer two with valuation $y = F_2^{-1}(t)$ is

$$p = \frac{1}{2}(t+1) - \frac{1}{4}\sqrt{2 - 2t}.$$

There is trade if $x \geq p$. The graph is plotted in Figure 2. The trading set Q is the union of the two areas bounded by the straight line and the red curve.

The folliowing illustrates a monopoly resale model when the monopolist cannot credibly commit to her offers in the first period, and engage in sequential offers to sell the good with a delay cost.

Example 5 *Resale Markets with Sequential Offers*

Consider a bargaining model with two rounds of offers by the seller. Assume that signals are independent, and we have a weak-strong pair. The seller with the signal t_1 and the valuation $v_1(t_1)$ makes an offer P_1 in the first period. This offer is either accepted or rejected, with the threshold of acceptance represented by Z, *i.e.* a buyer accepts the first offer if and only if his or her valuation is above Z. If the first offer is accepted, the game ends. If it is not accepted, the seller makes a second offer P_2 which is a take-it-or-leave-it offer. Let $P_1(t_1), P_2(t_1), Z(t_1)$ denote the equilibrium first-period, second-period prices and threshold level in this bargaining problem. Given the reported $(v_1(t_1), v_2(t_2))$, bidder one makes the first offer if she wins the auction. There is no trade if $v_2(t_2) < P_2(t_1)$. Trade occurs (with probability one) with the transaction price $p(v_1(t_1), v_2(t_2)) = P_1(t_1)$ if $v_2(t_2) \geq Z(t_1)$, and the transaction price $p(v_1(t_1), v_2(t_2)) = \delta P_2(t_2)$ if $P_2(t_1) \leq v_2(t_2) < Z(t_1)$. The set Q is

$$Q = \{(t_1, t_2) : t_1 \geq t_2, v_2(t_2) \geq Z(t_1) \text{ or } P_2(t_1) \leq v_2(t_2) < Z(t_1)\}$$

There is no trade when the pair is not in Q. The weak efficiency property is satisfied because we must have $Z(t_1) < v_2(t_2)$, and we have $p(v_1(t_1), v_2(t_2)) = P_1(t_1)$. The weak efficiency property holds in a monopoly resale mechanism with many rounds of offers from the seller, if the equilibrium first offer is lower than the highest valuation of the buyer. This is true if the monopolist has a strictly positive payoff in the equilibrium.

In general, offers can come from both the seller and the buyer. This is typically modeled by a double auction. The following linear auction model has an interesting constrast with standard double auction model wtih homogeneous beliefs.

Example 6 *Resale by a Double Auction*

The resale market allows simultaneous offers made by both the buyer and the seller similar to a double auction game. Assume that the signals are independent and $v_1(t) = t, v_2(t) = 2t$ so that $F_1(x) = x, F_2(x) = \frac{x}{2}$. The first stage is a first-price auction. In the resale game, let p_s, p_b be the offer price by the seller and buyer respectively. The transaction takes place if and only if $p_s \leq p_b$, and the transaction price is given by

$$p = \frac{p_s + p_b}{2}.$$

Let the inverse bidding strategy in the first-price auction with resale be ϕ_1, ϕ_2 and in equilibrium we have $\phi_2(b) = 2\phi_1(b)$ by Proposition 2. To find an equilibrium with linear strategies in the resale game, let $p_s(v_1) = c_1 v_1 + d_1, p_b(v_2) = c_2 v_2 + d_2$ be the equilibrium strategies as functions of valuations. Bidder one with valuation v_1 chooses $p \leq 2c_2 v_1 + d_2$ to maximize

$$\int_{\frac{p-d_2}{c_2}}^{2v_1} \left[\frac{p + c_2 v_2 + d_2}{2} - v_1 \right] dv_2.$$

The derivative of the payoff with respect to p is given by

$$\begin{aligned} & -\frac{p - v_1}{c_2} + \frac{1}{2} \int_{\frac{p-d_2}{c_2}}^{2v_1} dv_2 \\ = \quad & \frac{1}{c_2}\left(-\frac{3}{2}p + (1 + c_2) v_1 + \frac{1}{2} d_2\right), \end{aligned}$$

which is decreasing in p. Therefore the payoff function is concave. The first-order condition of optimality gives us

$$p_s(v_1) = \frac{2}{3}(1 + c_2) v_1 + \frac{1}{3} d_2. \tag{5}$$

For the bidder two with valuation v_2, the price offer $p \geq \frac{v_2}{2} c_1 + d_1$ maximizes

$$\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} \left[v_2 - \frac{c_1 v_1 + d_1 + p}{2} \right] dv_1.$$

The first-order condition for the optimal offer is

$$\frac{v_2 - p}{c_1} - \frac{1}{2}\int_{\frac{v_2}{2}}^{\frac{p-d_1}{c_1}} dv_1 = 0$$

or

$$v_2 - p - \frac{c_1}{2}(\frac{p - d_1}{c_1} - \frac{v_2}{2}) = 0,$$

and we have the optimal offer of the buyer

$$p_b(v_2) = \frac{4 + c_1}{6} v_2 + \frac{1}{3} d_1.$$

To be an equilibrium, we must have

$$d_1 = \frac{1}{3} d_2, d_2 = \frac{1}{3} d_1,$$

$$c_1 = \frac{2}{3}(1 + c_2), c_2 = \frac{4 + c_1}{6}.$$

Solving the equations, we have

$$d_1 = d_2 = 0, c_1 = \frac{5}{4}, c_2 = \frac{7}{8}.$$

Since the maximum bid must be the same for the two bidders, we have the (piecewise) linear equilibrium in the resale game is then given by

$$\begin{aligned} p_s(v_1) &= \frac{5}{4} v_1, v_1 \in [0, 1], \\ p_b(v_2) &= \frac{7}{8} v_2 \text{ for } v_2 \leq \frac{10}{7}, \\ &= \frac{5}{4} \text{ for } v_2 \in (\frac{10}{7}, 2]. \end{aligned}$$

The transaction price in the direct mechanism corresponding to this resale game equilibrium is given by

$$\begin{aligned} p(v_1(t_1), v_2(t_2)) &= \frac{1}{2}(\frac{5}{4} v_1(t_1) + \frac{7}{8} v_2(t_2)) = \frac{5}{8} t_1 + \frac{7}{8} t_2 \text{ if } v_2(t_2) \leq \frac{10}{7}, \\ &= \frac{5}{8} t_1 + \frac{5}{8} \text{ if } v_2(t_2) > \frac{10}{7}. \end{aligned}$$

Here $Q = \{(t_1, t_2) : t_1 \geq t_2, \min(\frac{7}{8} v_2(t_2), \frac{5}{4}) \geq \frac{5}{4} v_1(t_1)\} = \{(t_1, t_2) : t_1 \geq t_2, , \min(t_2, \frac{5}{7}) \geq \frac{5}{7} t_1\}$, or $Q = \{(t_1, t_2) : t_1 \geq t_2 \geq \frac{5}{7} t_1\}$. Trade occurs with probability one if and only if $(t_1, t_2) \in Q$, and there is no trade outside Q. Trade occurs if and only if $2v_1 \geq v_2 \geq \frac{10}{7} v_1$.

From Proposition 2, we get the equilibrium bidding strategies:

$$\begin{aligned} b_1(t_1) &= \frac{1}{t_1}\int_0^{t_1} w(t,t)dt = \frac{1}{t_1}\int_0^{t_1}(\frac{5}{8}t+\frac{7}{8}t)dt = \frac{3}{4}t_1, \text{ for } t_1 \le \frac{5}{7} \\ &= \frac{1}{t_1}\int_0^{\frac{5}{7}}\frac{3}{2}tdt + \frac{1}{t_1}\int_{\frac{5}{7}}^{t_1}(\frac{5}{8}t+\frac{5}{8})dt = \frac{5}{8}(1+\frac{1}{2}t_1-\frac{5}{14t_1}), \text{ for } t_1 \ge \frac{5}{7} \end{aligned}$$

and the same formula applies to $b_2(t_2)$. Hence

$$\phi_1(b) = \phi_2(b) = \frac{4}{3}b \text{ for } b \le \frac{15}{28}.$$

Remark 7 *With homogeneous beliefs of the traders, the optimal offer functions are* $p_s(v_1) = \frac{2}{3}v_1 + \frac{1}{2}$, $v_1 \in [0,1]$ *for and seller, and* $p_b(v_2) = \frac{2}{3}v_2 + \frac{1}{6}, v_2 \in [\frac{1}{2}, \frac{3}{2}]; = \frac{1}{2}$ *when* $v_2 \le \frac{1}{2}; = \frac{7}{6}$ *when* $v_2 \ge \frac{3}{2}$ *for the buyer. Trade offers if and only if* $v_2 \ge v_1 + \frac{1}{2}$. *Since* $v_2 \ge v_1 + \frac{1}{2}$ *implies* $v_2 \ge \frac{10}{7}v_1$, *trade is less efficient in the homogeneous case. This is because the updating of beliefs improves efficiency of trade in our model.*

Remark 8 *Another important difference from the standard double auction bilateral trade model is that here we have the boundary conditions* $p_s(0) = p_b(0) = 0$. *It can be shown that any differentiable equilibrium of the k-double auction must satisfy*

$$p_s'(0) = \frac{5}{4}, p_b'(0) = \frac{7}{8}.$$

Therefore there is only one differentiable strictly monotone equilibrium by the theory of differential equations. In the standard double auction model with homogeneous beliefs, Satterthwaite and Williams (1989, Th. 3.2) have shown that there is a two-parameter family of such equilibria.

In Figure 1, we graph the double auction resale model with heterogeneous beliefs. The thick line is the set of pairs with the highest possible trade surplus. The area above the thick line has zero probability in the resale stage because of heterogeneous beliefs. There is no trade below the thin line. The set Q is the region bounded by the thick line and the thin line. The dash line corresponds to the set of pairs in which both bidders have the same valuation. Efficient trade means that trade should occur for all pairs bounded by the thick line and the dash line. The weak efficiency property only requires that trade takes place on the thick line. Only the transaction price defined on the thick line is used in the bidding strategy.

Figure 1

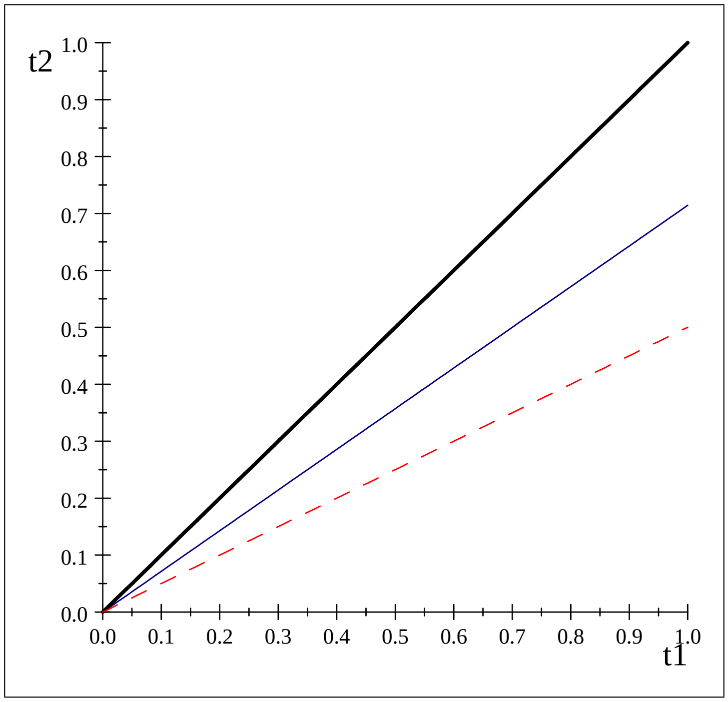
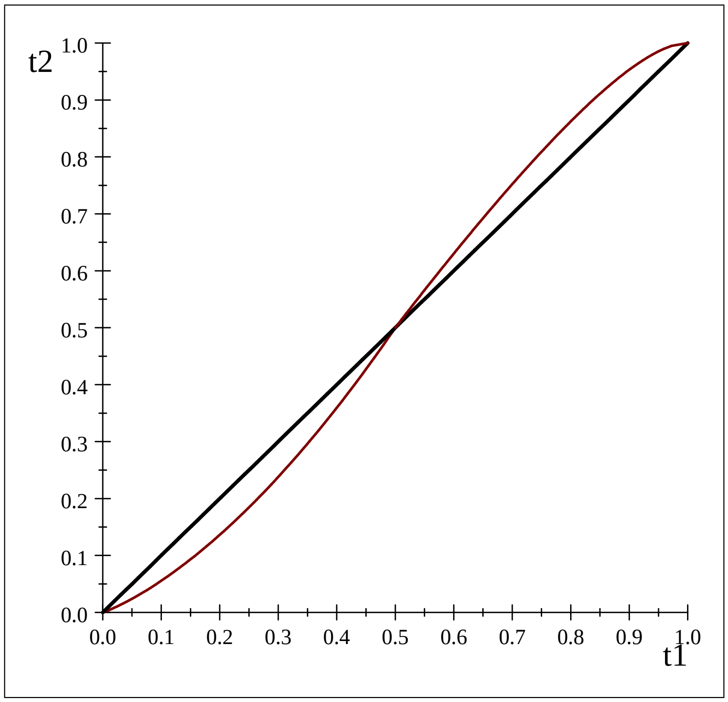


Figure 2



4 Bid Equivalence with two bidders

We assume that the pricing function $p(v_1(t_1), v_2(t_2))$ can be extended from Q to the set of all pairs (t_1, t_2) monotonically and smoothly so that the extension is continuously differentiable on the unit square. This extended function is denoted by $w(t_1, t_2)$. Consider a common-value auction in which the common value is defined by $w(t_1, t_2)$. From Theorem 1, we know that the equilibrium is the same for all different smooth monotonic extensions of p, as the bidding strategy only depends on the value of w on the diagonal $\{(t, t) : t \in [0, 1]\}$ which belongs to Q by the weak efficiency property.

The following bid equivalence result applies to all such extensions. Intuitively this says that the common value is defined by the transaction price in the resale game when trade occurs with probability one. When there is possibility of no trade, the common value can be defined arbitrarily without affecting the equilibrium.

Theorem 9 *Assume that there is no disclosure of bid information in between the auction stage and the resale stage, and the resale process satisfies the weak efficiency property. Assume that there is a strictly monotone differentiable equilibrium bidding strategy profile $b_i(t), i = 1, 2$ in the auction with resale. Let $w(t_1, t_2)$ be a monotonic extension of the the pricing function $p(v_1(t_1), v_2(t_2))$ on Q, Then $b_i(t), i = 1, 2$ is also an equilibrium of the common-value auction with the common value function w, and we have bid-equivalence between the auction with resale and the common-value auction.*

Proof of Theorem 9:

From Proposition 2, the solution of the first-order condition gives us the symmetric bidding strategy

$$b_i(t_i) = \frac{1}{t_i} \int_0^{t_i} p(v_1(s), v_2(s)) ds, t_i > 0, b_i(0) = 0. \tag{6}$$

This strategy only depends on the value of w on the diagonal (t, t), hence is the same for all monotonic smooth extension of p. According to Theorem 1, the equilibrium bidding strategy in (6) is an equilibrium of the common-value auction.

For the proof in Theorem 1 to work, we require that the monotonic extension to be smooth. It is convenient to allow the extension to have kinks. Because the equilibrium is unaffected by different smooth extensions, we can use those smooth extensions to approximate a monotonic extension with possible kinks and the equilibrium property holds in the limit. This means that the bid equivalence result still holds even if we allow the extension to have kinks.

Kinked extensions are easier to describe. The following method is one way to extend the definition of w to all pairs. For (t_1, t_2) outside Q, $t_2 < t_1$, let $k_1(t_1) = \inf\{t_2 : (t_1, t_2) \in Q\}$. we define $w(t_1, t_2) = w(t_1, k(t_1))$ for $(t_1, t_2) \notin Q$. For $t_2 > t_1$, we can just let $w(t_1, t_2) = w(t_2, t_2)$. In this extension, we have kinks on the boundary of Q. We have the property $w(t_1, t_2) \geq x$ when $v_1(t_1) = v_2(t_2) = x$. We can use the resale game example with simultaneous offers in the last section to illustrate the extension. On Q, we have

$$\begin{aligned} w(t_1, t_2) &= \frac{5}{8}t_1 + \frac{7}{8}t_2 \text{ if } t_2 \leq \frac{5}{7}, \\ &= \frac{5}{8}t_1 + \frac{5}{8} \text{ if } t_2 > \frac{5}{7}. \end{aligned}$$

For $(t_1, t_2) \notin Q, t_2 < \frac{5}{7}t_1$, we define $w(t_1, t_2) = w(t_1, \frac{5}{7}t_1) = \frac{5}{4}t_1$. If $t_2 > t_1$, we let $w(t_1, t_2) = w(t_1, t_1) = \frac{3}{2}t_1$ when $t_1 \leq \frac{5}{7}$, and $w(t_1, t_1) = \frac{5}{8}t_1 + \frac{5}{8}$ when $t_1 > \frac{5}{7}$. When $v_1(t_1) = v_2(t_2) = x$, we have $t_2 = \frac{1}{2}t_1$, and in this case, we have $w(t_1, t_2) = \frac{5}{4}t_1 = \frac{5}{4}x > x$.

Another alternative definition of the common value outside Q is to let $k_2(t_2) = \sup\{t_1 : (t_1, t_2) \in Q\}$, and then define $w(t_1, t_2) = w(k_2(t_2), t_2)$. In this definition, we may have $w(t_1, t_2) \leq x$ when $v_1(t_1) = v_2(t_2) = x$. For the same example, when $(t_1, t_2) \notin Q, t_2 < \frac{5}{7}t_1$, we define $w(t_1, t_2) = w(\frac{7}{5}t_2, t_2) = \frac{7}{4}t_2$ when $t_2 \leq \frac{5}{7}$. and $w(\frac{7}{5}t_2, t_2) = \frac{7}{8}t_2 + \frac{5}{8}$ when $t_2 > \frac{5}{7}$. When $v_1(t_1) = v_2(t_2) = x$, we have $t_2 = \frac{1}{2}t_1$, and in this case, we have $w(t_1, t_2) = \frac{7}{4}t_2 = \frac{7}{8}x < x$ for $t_2 \leq \frac{5}{7}$.

In Figure 2, the extension of the price function outside Q can be defined as follows. The common value at any point on the lower part can be defined by going up vertically until a boundary point of Q is reached, and the transaction price at that boundary point is the common value. At any point on the upper half, the common value can be defined by going right horizontally until a boundary point of Q is reached, and the transaction price at that boundary point is the common value.

In a bilateral trade equilibrium, trade need not occur with probability one even when the trade surplus is the highest possible amount. In this case, the weak efficiency property fails, and the bid equivalence result need not hold. We shall give such an example with discrete valuations. Note that the above bid equivalence result can be easily extended to the case of discrete valuations. To see this, assume that there are two bidders in an independent private-value asymmetric auction. Bidder one valuation is either 0 or 1 with probability 0.7 for 0. Bidder two valuation is either 0 or 2 with probability 0.4 for 0. We call bidder one the weak bidder, and bidder two the strong bidder. We can adopt a signal representation of the IPV model. This representation is called a distributional approach) is first proposed in Milgrom and Weber (1985) and discussed extensively in Milgrom (2004). In this representation, both bidders are described by step functions $v_i(t_i) : [0,1] \rightarrow \{0,1\}$,

$$v_1(t_1) = 0, \text{ when } t_1 \in [0, 0.7); = 1 \text{ when } t_1 \in [0.7, 1]$$

$$v_2(t_2) = 0, \text{ when when } t_1 \in [0, 0.5); = 1 \text{ when } t_1 \in [0.5, 1]$$

with the interpretation that $v_i(t_i)$ is the private valuation of bidder i when he or she receives the private signal t_i. The two signals are assumed to be independent.

The bidding strategies are mixed strategies which can be represented by $b_i(t_i), i = 1, 2$. The inverse bidding function $\phi_i(b)$ is the cumulative bid distribution of bidder $i, i = 1, 2$. The beliefs after the first-stage auction is described conditional distribution of the signal t_j over $[0, \phi_j(b)]$ when bidder i bids b and wins the auction. When bid i loses the auction after bidding b, the belief about bidder two is described by the conditional distribution of t_j over $[\phi_j(b), 1]$. In the space of valuations, such distributions are also discrete distributions. In the bilateral trade, a direct mechanism is a report of the signal rather valuation. A direct trade mechanism then is represented by the probability of trade $q(t_1, t_2)$, and the trade price $p(t_1, t_2)$. Thus we can adopt the model with a continuum of valuations to the model with a continuum of signals. The first-order condition of the equilibrium in the auction with resale holds except at t_i when there is a change in valuation. The symmetry property continues to hold with the same proof. The trading price on Q can be extended monotonically to the whole range to define a common value function. The equilibrium in the common-value auction satisfies the same first-order condition except a finite number of points. The equilibrium bidding strategy is given by the same formula, except that it is piecewise continuously differentiable rather than continuously differentiable. There is a bid equivalence between the equilibrium in the auction with resale and the common value auction. For the example in the last paragraph, when the resale is a monopoly, the optimal monopoly price is 2 when bidder one wins the auction. There is no resale if bidder two has valuation 0, or if bidder two wins the object and has valuation 2. The payoff of bidder one with valuation 1 bidding $b > 0$ is

$$(\phi_2(b) - 0.4)(2 - b) + 0.4(1 - b) = \phi_2(b)(2 - b) - 0.4.$$

The first-order condition for this maximization problem is

$$\phi_2'(b)(2 - b) - \phi_2(b) = 0,$$

or

$$\frac{\phi_2'(b)}{\phi_2(b)} = \frac{1}{2 - b}. \tag{7}$$

For a bidder one with valuation 0, the payoff from bidding $b > 0$ is

$$(H_2(b) - 0.4)(2 - b) + 0.4(0 - b) = H_2(b)(2 - b) - 0.8,$$

and the first-order condition is also (7). We have the same first-order condition for a bidder one with different valuations, because the resale opportunity has changed his valuation to 2 rather than 0 or 1.

For bidder two with valuation 2, there is no profit in the resale whether or not there is resale. The payoff from bidding $b > 0$ is

$$H_1(b)(2-b),$$

and the first-order condition is also given by

$$\frac{\phi_1'(b)}{\phi_2(b)} = \frac{1}{2-b}.$$

From the boundary conditions, $\phi_1(0) = 0.4 = \phi_2(0)$, we get equilibrium[9] inverse bidding functions in the auction with resale:

$$\phi_1(b) = \phi_2(b) = \frac{0.8}{2-b} \text{ for } b \in [0, 1.2].$$

and the equilibrium biding strategies

$$\begin{aligned} b_i(t) &= 2 - \frac{0.8}{t} \text{ when } t \geq 0.4 \\ &= 0 \text{ when } t \leq 0.4. \end{aligned}$$

We have the set of trade

$$Q = \{(t_1, t_2) : t_1 < 0.4\} \cap [0.4, 1] \times [0.4, 1] \cap \{(t_1, t_2) : t_1 \geq t_2, 0 \leq t_i \leq 1, i = 1, 2\}.$$

and the pricing function on $Q : p(t_1, t_2) = 0$ if $t_1 < 0.4; = 2$ if $(t_1, t_2) \in [0.4, 1] \times [0.4, 1]$. The common value function is a monotonic extension of p. For instance, we can have the extension $w(t_1, t_2) = 0$ if $t_1 < 0.4; = 2$ if $t_1 \geq 0.4$ for all pairs in $[0, 1] \times [0, 1]$. The equilibrium bidding strategy of the common value function is

$$\begin{aligned} b_i(t) &= \frac{1}{t} \int_0^t p(s, s) ds = \frac{2}{t}(t - 0.4) = 2 - \frac{0.8}{t} \text{ if } t \geq 0.4 \\ &= 0 \text{ if } t < 0.4, \end{aligned}$$

which is the same as the bidding strategy of the auction with resale.

Our counter example is a variation of the above example in which the resale market is a monopoly with trade impediments. This means that the winner of the auction makes offers, but trade can only occur with probability 0.5 when the buyer accepts the offer. There is no trade when the buyer does not accept the offer. With this trade mechanism, the optimal offer by the seller is the same as the monopoly price without restriction. Hence the optimal offer price is 2

[9] This equilibrium is unique for any tie-breaking rule adopted. In auctions with resale, bidder one with valuation 0 may bid positive amount because of future resale.

whenever there is a positive probability that the buyer has valuation 2. Let ϕ_i denote the equilibrium cumulative bid distribution of bidder i. The weak bidder with valuation 1 chooses b to maximize the following payoff

$$0.5(\phi_2(b)-0.4)(2-b)+0.5(\phi_2(b)-0.4)(1-b)+0.4(1-b)=\phi_2(b)(1.5-b)-0.2.$$

The first-order condition is

$$\frac{\phi_2'(b)}{\phi_2(b)}=\frac{1}{1.5-b}. \tag{8}$$

Bidder one with valuation 0 chooses b to maximize

$$0.5(\phi_2(b)-0.4)(2-b)+0.5(\phi_2(b)-0.4)(0-b)+0.4(0-b)=\phi_2(b)(1-b)-0.4$$

and the first-order condition is

$$\frac{\phi_2'(b)}{\phi_2(b)}=\frac{1}{1-b}. \tag{9}$$

Bidder two with valuation 2 chooses b to maximize the payoff

$$\phi_1(b)(2-b)$$

with the first-order condition

$$\frac{\phi_1'(b)}{\phi_1(b)}=\frac{1}{2-b}. \tag{10}$$

Let b^* be the common maximum bid of both bidders. We have the following boundary conditions

$$\phi_2(0)=0.4, \phi_2(b^*)=\phi_1(b^*)=1.$$

Note that the first-order conditions of the two bidders are not symmetric, and we don't expect the equilibrium bid distributions of the two bidders to satisfy the symmetry property. If the symmetry property fails, then the equilibrium of the auction with this restricted monopoly resale market cannot be bid equivalent to any equilibrium of a common-value auction. To find the equilibrium, let $\phi_1(c)=0.7$. We have

$$\phi_1(b)=\frac{A}{2-b} \text{ for } b\in[0,b^*],$$

where $A=2-b^*, 0.7(2-c)=2-b^*$. For bidder two, we have

$$\phi_2(b)=\frac{B}{1.5-b} \text{ for } b\in[c,b^*],$$

hence $B=1.5-b^*$. For $b\in[0,c]$, we must have $\phi_2(0)=0.4$, hence

$$\phi_2(b)=\frac{0.4}{1-b} \text{ for } b\in[0,c].$$

We must have

$$\phi_2(c) = \frac{0.4}{1-c} = \frac{B}{1.5-c}$$

or

$$\frac{0.4(1.5-c)}{1-c} = B = 1.5 - b^* = 1.5 - (2 - 0.7(2-c)) = 0.9 - 0.7c$$

We have the following quadratic equation in c :

$$(0.9 - 0.7c)(1-c) - 0.4(1.5-c) = 0.7c^2 - 1.2c + 0.3 = 0$$

and the solution is $c = 0.303\,86$. From this number, we have $b^* = 0.687\,30, A = 1.31270, B = 0.81270$. Hence we have the following equilibrium cumulative bid distributions when ties are broken in favor of bidder one:

$$\phi_1(b) = \frac{1.3127}{2-b}, b \in [0, 0.6873],$$

$$\begin{aligned}
\phi_2(b) &= \frac{0.4}{1-b}, b \in [0, 0.30386] \\
&= \frac{0.8127}{1.5-b}, b \in [0.30386, 0.6873].
\end{aligned}$$

Clearly, this equilibrium does not satisfy the symmetry property, and the bid equivalence result fails.

5 Buyer-Speculators Model of Auction with Resale

A special interesting case is worth mentioning here. Assume that group one has only one member, and group two consists of n speculators. We refer to the only group one member as bidder one and call it a regular buyer. A speculator in group two has no value for the object, but participates in the auction for resale. We call this the buyer-speculators model of auction with resale. Assume that the resale market is a monopoly. We shall extend our bid equivalence result in the last section to the buyer-speculators model. This means that the equilibrium bidding behavior of the buyer-speculators model is the same as the Wilson Drainage Tract common value Model in which there is one neighbor and n non-neighbors. The common value is the defined by the monopoly price offered by the winning speculator.

Consider an auction with resale model with n speculators indexed by $i = 2, ..., n+1$.and one regular buyer, Bidder one has a valuation function $v(t)$

over $[0,1]$. Let $b_i(t)$ be the strictly increasing bidding strategy of bidder i and $\phi_i = b_i^{-1}$ be the inverse. There is no resale if bidder one wins the auction, and resale will occur if one of the speculators wins the auction. If bidder one with valuation $v(t)$ bids b, and loses the auction, the speculator i with highest bid wins the auction, and sells the object to bidder one. The price depends on $\phi_1(b)$ because the winning speculator believes that bidder one valuation is distributed over $[0, \phi_1(b)]$. Let $p(x)$ be the optimal monopoly price when the speculator makes offers to bidder one with valuation distributed over $[0, x]$. Let b^* be the maximum bid of all the speculators, $G(b) = \prod_{\iota=2}^{m+1} \phi_i(b)$. The profit of bidder one can be written as

$$\pi_R = \int_b^{b^*} [v(t) - p(v(\phi_1(c)))]\, dG(c) + G(b)(v(t) - b).$$

The first-order condition is

$$G'(b)[p(v(\phi_1(b))) - b] - G(b) = 0,$$

or

$$\frac{d \ln G(b)}{db} = \frac{1}{p(v(\phi_1(b))) - b}. \tag{11}$$

For a speculator $i = 2$, the profit from bidding b is

$$\prod_{j=3}^{n} \phi_j(b) \left\{ [\phi_1(b) - F(p(v(\phi_1(b))))] p(v(\phi_1(b))) - \phi_1(b) b \right\}.$$

Since the speculator uses a mixed strategy, it is profit is a constant which is zero when $b = 0$. Therefore the speculator has zero profit, and we have

$$[\phi_1(b) - F(p(v(\phi_1(b))))] p(v(\phi_1(b))) - \phi_1(b) b = 0. \tag{12}$$

Let $R(b) = [\phi_1(b) - F(p(v(\phi_1(b))))] p(v(\phi_1(b)))$. By the envelop theorem, we have

$$R'(b) = \phi_1'(b) p(v(\phi_1(b)))$$

Taking the derivative of (12), we get

$$\phi_1'(b)[p(v(\phi_1(b))) - b] - \phi_1(b) = 0,$$

or

$$\frac{d \ln \phi_1(b)}{db} = \frac{1}{p(v(\phi_1(b))) - b}. \tag{13}$$

From the first-order conditions, we know that $\phi_1(b) = G(b)$. From this, we have the bidding strategy

$$b_1(t) = \frac{1}{t} \int_0^t p(v(s)) ds.$$

This is an equilibrium in the Wilson tract model with one neighbor and many non-neighbors, and the common value is given by $p(v(t_1))$ (see, for example, Milgrom (2004), Theorem 5.3.2). Therefore we have bid-equivalence between the auction with resale and the Wilson tract model.

It is an interesting question to ask whether bidders behave the same way in both models in practice or in experiments[10]. Given the evidence that winner's curse is observed in many experiments in common-value auctions, it is interesting to see whether similar irrationality is observed in the auctions with resale. There is strong empirical support for the bidding behavior in the Wilson track model. It is also promising that such support can be found in empirical studies of auctions with resale.

6 Haile's Resale Model

In the auctions with resale model of Haile (2000,2001,2003), bidders receive signals in the first stage which are independent from each other, but are affiliated with their private values. In other words, before bidding, there is incomplete information about private valuation in the first stage. This uncertainty about private valuation is resolved in the second stage before resale occurs. The reason for resale is due the different realized private valuations in the second stage. This model is appropriate when there is dynamic discovery of information over time. In our model, bidders know their private valuations before the bidding takes place in the first stage. Thus asymmetry about valuation starts in the beginning, and there is incentive for trade because the allocation of the object need not be efficient in the first stage auction.

The advantage of Haile's auctions with resale model is that in the first stage sealed-bid auctions, the bidding strategies are symmetric and easier to solve when the resale market is relatively simple. The disadvantage is that the updating of belief is more complicated. Bidders need to update their beliefs about the signals received by their opponents in the first stage, as well as the realized private valuation (the use value in Haile's terminology) in the second stage. The updating is simplified if the private valuations become common knowledge in the second stage. In our model, it is difficult to get explicit solutions to the asymmetric bidding behavior in the first stage in general even if the resale outcome is very simple. The updating of beliefs is simpler because it only involves a change in the support of the private valuation, and the updated belief about the opponents' private valuation is just the conditional distribution on the new support. When there are two bidders, we do get an explicit solution to the equilibrium bidding behavior in the first stage. Therefore the two ways of modeling auctions with resale are complementary to each other.

We assume that there is no disclosure of bid information in our model, while Haile assumed that all bids are disclosed and in his section 3 even the private

[10] Experiments based on the auction with resale model of Haile have been done in Lange, List, and Price (2004) for first-price auctions and Georganas (2008) for English auctions.

valuations are made common knowledge. In Haile's model, the disclosure of bids and valuations simplifies the updating process, while in our model, the bid disclosure complicates the analysis, as it may eliminate the strictly increasing equilibrium bidding strategies (see Lebrun (2007) for an analysis with bid disclosure). At the conceptual level, the different assumptions on bid disclosure only lead to different updating and beliefs between the two stages. We take the simplest information assumption to focus on the idea of bid equivalence. In Haile's model, bid equivalence holds in a trivial way, as explained below. Furthermore, in his model, the connection between resale price and common value is quite weak.

There is no heterogeneity of beliefs in the resale stage in the Haile model. This is because all bids are announced in his model, and beliefs in the second stage are only affected by the signals in the first stage. When the bids are announced after the first-stage auction, the signals can be inferred, and become common knowledge in the resale stage. Therefore a bidder's belief about the opponent is independent of his revealed private valuation in the second stage. There is no signaling in our model as the deviation from bidding is not detected when the bids are not disclosed. In Haile's model, signaling is a feature of his model.

We now give a brief description of the Haile (2003) model with two bidders when the first stage is a first-price auction, and the resale mechanism is an optimal auction (OA) conducted by the winner of the first stage auction. Each bidder receives a signal s_i in the first stage, and the realized private valuation in the second stage is denoted by v_i which is distributed according to a cumulative probability distribution $G(v_i|s_i)$ with the support $[0, 1]$. The signal s_i and v_i are affiliated, but both are independent from s_j, v_j of the other bidder. Let $F(s_i)$ be the cumulative probability distribution of each s_i. Let $\phi_i = b_i^{-1}$ be the inverse of the bidding strategy in the first stage. In the second stage, all bids are announced, and bidder i knows $\phi_j(b_j) = s_j$, hence believes that bidder j valuation has the distribution $G(v_j|s_j)$. When bidder i wins the auction, he makes an optimal monopoly offer to bidder j. Assume that there is complete information in the second stage (as in section 3 of Haile (2003)). For each realized $v_i \in [0, 1]$, all surplus is extracted from bidder j, and the revenue from the resale is $\int_0^1 (v_i + \int_{v_i}^1 v_j dG(v_j|s_j)$. Hence the expected revenue from the resale market for bidder i with the signal s_i is $w_O(s_i, s_j) = \int_0^1 \left[\int_0^1 (v_i + \int_{v_i}^1 v_j dG(v_j|s_j)\right] dG(v_i|s_i)$. In the symmetric model of Haile, the bid equivalence holds in a trivial way. Any symmetric equilibrium is an equilibrium of some common value auction. We can just define the common value as $w(s_i, s_j) = w_O(\min(s_i, s_j), (s_i, s_j))$ for example. This common value is stochastically related to the transaction price as well as the private valuation (when there is no trade), so that the connection between transaction price and common value is very weak. In our model, $s_i = v_i$, and the common value is defined by the optimal monopoly price $p(s_i, s_j)$ whenever trade is realized for the pair. Therefore, the bid equivalence result has clear meaning in our model. Furthermore, it is more useful because the bid equivalence leads to identical bid distributions for asymmetric bidders.

In the Haile model with different auctions in the first stage or resale mechanisms in the second stage, the same comments above apply. We still have bid equivalence, but the common value is only weakly connected to the resale price, and in general there is a combination of private and market valuations involved in the first-stage equilibrium bidding strategy. For our model, only the transaction price is involved in the equilibrium bidding strategy. Haile (2000,2003) also studied the implications of bargaining power and competition in auctions with resale. Similar issues are investigated in Cheng and Tan (2008,2009) in our framework.

7 Extension to the m-bidder Model

In this section, we shall assume that the winner of the first stage auction uses a second-price auction in the second stage to sell the object to the losing bidders. The seller in the second-stage auction reserves the right to keep the object to herself if the second highest bid is below her valuation. This amounts to a reservation price equal to the valuation of the seller in the second stage. Other reservation prices can be considered, and many of our results hold with more general resale mechanisms. However, we shall deal with the simpler and important case of the second-price auctions here. An important property of the second-price auction is that it has the efficiency property which is stronger than any weak efficiency conditions needed for resale mechanism in the m bidder case. Another reason for considering the resale by the second price auction is that the common value function can be defined without reference to the bidding strategy in the first stage auction. This is not the case in Theorem 9. A comprehensive treatment of the multiple bidder case is beyond the scope of this paper.

First we shall establish an important relationship between the common-value auction and auction with resale in the symmetric model with many bidders. It says that the profit functions of the two auctions differ only by a constant (not depending on bids). Since the optimal bid is determined by the maximization of the profit function, the bid equivalence result follows easily from it. This property holds for asymmetric auctions, and will be shown for the case of two groups of identical bidders. In fact, there is in some sense a strong tautological element to the bid equivalence result.

There are $n \geq 3$ identical bidders. The cumulative distribution of the valuations of each bidder is denoted by $F(x)$ over $[0, a]$. Given a vector of valuations $x = (x_1, x_2, ..., x_n)$, let x_{-i} denote the vector x without the x_i component. Let $x_{-1,i}, i \neq 1$ denote a vector without the component in 1 and i. Let $dG_{-i}(x_{-i}) = \prod_{k \neq i} dF_1(x_k)$. Similarly, we let $dG_{-1,i}(x_{-1,i}) = \prod_{k \neq 1,i} dF_1(x_k)$. Let $q(x)$ denote the second highest value among x_i.

Consider the common-value model in which $s_i = x_i$. Define the common-value function $w(x) = q(x)$. We shall consider only symmetric equilibrium in which all bidders use the same bidding strategy. Let $\phi(b)$ be the inverse of the strictly increasing bidding strategies $b(x_i)$ of each bidder. We have the following proposition.

Proposition 10 *Given $\phi(.)$ of other bidders, and a bidder i with valuation x_0 bidding b, let $\pi_C(b), \pi_R(b)$ be the profit function in the common-value auction and the auction with resale (in the first stage) respectively. Then $\pi_R(b) - \pi_C(b)$ is independent of b, and is given by*

$$\pi_R(b) - \pi_C(b) = \iint_0^{x_0} [x_0 - \max(x_{-i})]\, dG_{-i}(x_{-i}).$$

Proof. Assume that bidder one has valuation x_0, and bid b. The profit in the common-value auction is

$$\pi_C = \iint_0^{\phi(b)} q(x_0, x_{-1}) dG_{-1}(x_{-1}) - bF^{n-1}(\phi(b)).$$

Now consider the auction with resale. First assume that $\phi(b) < x_0$. When bidder one wins the auction, there is no resale, and the revenue in the auction with resale is

$$R_1 = x_0 F^{n-1}(\phi(b)).$$

If bidder one loses the auction, she buys from the winning bidder only if the winning bidder has valuation lower than x_0,and she has the highest valuation among the losers. Therefore, in this case the revenue is

$$R_2 = (n-1) \int_{\phi(b)}^{x_0} \left[\iint_0^{x_i} [x_0 - q(x_0, x_{-1,i})]\, dG_{-1,i}(x_{-1,i}) \right] dF(x_i)$$

$$= x_0 F^{n-1}(x_0) - x_0 F^{n-1}(b) - \iint_0^{x_0} q(x_0, x_{-1}) dG_{-1}(x_{-1}) + \iint_0^{\phi(b)} q(x_0, x_{-1}) dG_{-1}(x_{-1}).$$

Hence

$$\pi_R - \pi_C = x_0 F^{n-1}(x_0) - \iint_0^{x_0} q(x_0, z) dG_{-1}(x_{-1}) = \iint_0^{x_0} [x_0 - \max(x_{-1})] dG_{-1}(x_{-1}).$$

Now assume that $\phi(b) \geq x_0$. If bidder one wins the auction, there is resale if one of the losing bidder has higher valuation than x_0, and the revenue is

$$R_1 = x_0 F^{n-1}(x_0) + (n-1) \int_{x_0}^{\phi(b)} \left[\iint_0^{x_i} q(x_0, x_{-1}) dG_{-1,i}(x_{-1,i}) \right] dF\,(x_i)$$

$$= x_0 F^{n-1}(x_0) + \iint_0^{\phi(b)} q(x_0, x_{-1}) dG_{-1}(x_{-1}) - \iint_0^{x_0} q(x_0, x_{-1}) dG_{-1}(x_{-1}).$$

When bidder one loses the auction, there is no resale and there is no revenue. Hence

$$\pi_R - \pi_C = x_0 F^{n-1}(x_0) - \iint_0^{x_0} q(x_0, x_{-1}) dG_{-1}(x_{-1}) = \iint_0^{x_0} [x_0 - \max(x_{-1})] dG_{-1}(x_{-1}).$$

Hence in either case, the Proposition holds. ■

From the Proposition, we have the following bid equivalence result.

Theorem 11 *Assume that there are n identical bidders. In the resale stage, the winner of the auction holds a second price auction to sell the object to the losing bidders. Then a strictly increasing bidding strategy $b(.)$ is a symmetric equilibrium of the common-value auction if and only if it is a symmetric equilibrium bidding strategy of the first stage of the auction with resale.*

Proof. Let $b(.)$ be a strictly increasing symmetric equilibrium bidding strategy of the common-value auction. Since $b(x_i)$ maximizes π_C, it also maximizes π_R, therefore, $b(x_i)$ is also an optimal bid in the auction with resale when all other bidders adopt the bidding strategy $b(.)$. The converse holds by the same argument. ■

Now we want to show the same result for asymmetric auctions with two group of bidders when the bidders in each group are identical in their valuations. Note that this proof is more general than the special arguments in the buyer-speculators model.

There are two groups of bidders, group one and two with m, n members respectively. Within each group, all bidders are identical. Consider the common-value model in which $s_i = x_i$ for group one and $s_j = y_j$. Define the common-value function $w(x, y) = q(x, y)$ as the second highest valuation of all bidders. We want to establish the bid equivalence between the common-value auction defined by this common value function and the auctions with resale model described above. We shall consider only equilibrium in which all bidders in the same group use the same bidding strategy. Let $\phi_1(b), \phi_2(b)$ be the inverse of the strictly increasing bidding strategies $b_1(x_i), b_2(x_j)$ of group one, two bidders respectively. We have the following bid equivalence result.

For convenience, we shall use the index i for group one, and index j for group two. Let x be a vector with valuations for group one and y be a vector of valuations for group two, and $z = (x, y)$ for a mixed vector. For any vector (possibly incomplete list) of valuations z, let $\max(z), q(z)$ be the highest, and second highest valuation among z_k. Let x_{-i}, y_{-j} denote the vector x, y without the x_i, y_j component respectively. Let $x_{-1,i}$ denote a vector without the component in 1 and i.

Let $dG_{-i}(x_{-i}) = \prod\limits_{k \neq i} dF_1(x_k), dH(y) = \prod\limits_{j} dF_2(y_j)$. Similarly, we let $dG_{-1,i}(x_{-1,i}) = \prod\limits_{k \neq 1,i} dF_1(x_k), dH_{-j}(y_{-j}) = \prod\limits_{k \neq j} dF_2(y_k)$. We have the following result similar to Proposition 10 for the symmetric case. The proof uses a similar idea, and is in the appendix.

Proposition 12 *Given $\phi_1(.), \phi_2(.)$, and a bidder i in group one with valuation x_0 bidding b, let $\pi_C(b), \pi_R(b)$ be the profit function of bidding b in the common-value auction and in the first stage of the auction with resale respectively. Then $\pi_R(b) - \pi_C(b)$ is independent of b, and is given by*

$$\pi_R(b) - \pi_C(b) = \iint_0^{x_0} [x_0 - \max(x_{-i}, y)]\, dG_{-i}(x_{-i}) dH(y).$$

Similar results hold for any bidder in group two.

By the same simple arguments of the symmetric case, we have the following bid equivalence result.

Theorem 13 *Assume that there are two groups of bidders. In each group, all bidders are identical. In the resale stage, the winner of the auction holds a second price auction to sell the object to the losing bidders. Then a strictly increasing pair of bidding strategies $b_1(.), b_2(.)$ is an equilibrium of the common-value auction if and only if it is an equilibrium pair of bidding strategies of the first stage of the auction with resale.*

Appendix

Proof of Proposition 12:

Without loss of generality, we can assume that bidder one in group one has valuation x_0. To simplify notations, let $z = (x_{-1}, y)$. For the common-value auction, the net profit from bidding b is

$$\pi_C = \iint_0^{\phi_1(b)} \iint_0^{\phi_2(b)} q(x_0, z) dG_{-1}(x_{-1}) dH(y) - bF_2^n(\phi_2(b)) F_1^{m-1}(\phi_1(b)). \quad (14)$$

For the auctions with resale model, to write down the formula for the profit function, we need to divide into different cases. Consider the case $\min(\phi_1(b), \phi_2(b)) \geq x_0$. The revenue from the resale game is given by

$$R_1 = x_0 F_1^{m-1} F_2^n(x_0) + (m-1) \int_{x_0}^{\phi_1(b)} \left(\iiint_0^{x_i} q(x_0, z) dG_{-1,i}(x_{-1,i}) dH(y) \right) dF_1(x_i)$$

$$+ n \int_{x_0}^{\phi_2(b)} \left(\iiint_0^{y_j} q(x_0, z) dG_{-1}(x_{-1}) dH_{-j}(y_{-j}) \right) dF_2(y_j)$$

$$= x_0 F_1^{m-1} F_2^n(x_0) + \iint_0^{\phi_1(b)} \iint_0^{\phi_2(b)} q(x_0, z) dG_{-1}(x_{-1}) dH(y) - \iint_0^{x_0} q(x_0, z) dG_{-1}(x_{-1}) dH(y)$$

$$= x_0 F_1^{m-1} F_2^n(x_0) - \iint_0^{x_0} \max(z) dG_{-1}(x_{-1}) dH(y) + \iint_0^{\phi_1(b)} \iint_0^{\phi_2(b)} q(x_0, z) dG_{-1}(x_{-1}) dH(y) \quad (15)$$

If she loses the auction, there is no resale, and $R_2 = 0$.

From (3) and (14), we have

$$\pi_R - \pi_C = \iint_0^{x_0} [x_0 - \max(z)] \, dG_{-1}(x_{-1}) dH(y). \quad (16)$$

Now consider the case $\max(\phi_1(b), \phi_2(b)) < x_0$. There is no resale if she wins the auction, hence

$$R_1 = x_0 F_1^{m-1}(\phi_1(b)) F_2^n(\phi_2(b)).$$

If she loses the auction, she buys from the winning bidder only if she has the highest valuation among the losers. The revenue is

$$R_2 = (m-1) \int_{\phi_1(b)}^{x_0} \left[\iiint_0^{x_i} [x_0 - q(x_0, z)] \, dG_{-1,i}(x_{-1,i}) dH(y) \right] dF_1(x_i)$$

$$+n\int_{\phi_1(b)}^{x_0}\left[\iint_0^{y_j}[x_0-q(x_0,z)]\,dG_{-1}(x_{-1})dH_{-j}(y_{-j})\right]dF_2(y_j)$$

$$=x_0F_1^{m-1}(x_0)F_2^n(x_0)-x_0F_1^{m-1}(\phi_1(b))F_2^n(\phi_2(b))+\int\int_0^{\phi_1(b)}\int\int_0^{\phi_2(b)}q(x_0,z)dG_{-1}(x_{-1})dH(y)$$

$$-\iint_0^{x_0}q(x_0,z)dG_{-1}(x_{-1})dH(y)$$

$$=\iint_0^{x_0}[x_0-\max(z)]dG_{-1}(x_{-1})dH(y)+\int\int_0^{\phi_1(b)}\int\int_0^{\phi_2(b)}[q(x_0,z)-x_0]dG_{-1}(x_{-1})dH(y)$$

Therefore we have

$$\pi_R=\iint_0^{x_0}[x_0-\max(z)]dG_{-1}(x_{-1})dH(y)+\int\int_0^{\phi_1(b)}\int\int_0^{\phi_2(b)}[q(x_0,z)-b]dG_{-1}(x_{-1})dH(y)$$

and we have (16) in this case as well.

Now consider the case $\phi_1(b)<x_0\leq\phi_2(b)$. In this case when bidder one wins the auction, she only sells to the bidders in group two. If she loses the auction, she only buys it from bidders in group one. Hence we have

$$R_1=x_0F_1^{m-1}(\phi_1(b))F_2^n(x_0)+nF_1^{m-1}(\phi_1(b))\int_{x_0}^{\phi_2(b)}\left(\iint_0^{y_j}q(x_0,y)dH_{-j}(y_{-j})\right)dF_2(y_j),$$

$$=x_0F_1^{m-1}(\phi_1(b))F_2^n(x_0)+n\int_{x_0}^{\phi_2(b)}\left(\int\int_0^{\phi_1(b)}\iint_0^{y_j}q(x_0,z)dG_{-1}(x_{-1})dH_{-j}(y_{-j})\right)dF_2(y_j)$$

$$=x_0F_1^{m-1}(\phi_1(b))F_2^n(x_0)+\int\int_0^{\phi_1(b)}\int\int_0^{\phi_2(b)}q(x_0,z)dG_{-1}(x_{-1})dH(y)$$

$$-\int\int_0^{\phi_1(b)}\iint_0^{x_0}q(x_0,z)dG_{-1}(x_{-1})dH(y)$$

$$R_2=(m-1)\int_{\phi_1(b)}^{x_0}\left[\iint_0^{x_i}[x_0-q(x_0,z)]\,dG_{-1,i}(x_{-1,i})dH(y)\right]dF_1(x_i)$$

$$=x_0F_2^n(x_0)[F_1^{m-1}(x_0)-F_1^{m-1}(\phi_1(b))]-(m-1)\int_{\phi_1(b)}^{x_0}\left[\iint_0^{x_i}q(x_0,z)dG_{-1,i}(x_{-1,i})dH(y)\right]dF_1(x_i)$$

$$= x_0 F_2^n(x_0)[F_1^{m-1}(x_0) - F_1^{m-1}(\phi_1(b))] - \iint_0^{x_0} q(x_0, z) dG_{-1}(x_{-1}) dH(y)$$

$$+ \int_0^{\phi_1(b)}\!\!\int \iint_0^{x_0} q(x_0, z) dG_{-1}(x_{-1}) dH(y)$$

We have

$$\pi_R = x_0 F_1^{m-1}(x_0) F_2^n(x_0) + \int_0^{\phi_1(b)}\!\!\int \int_0^{\phi_2(b)}\!\!\int q(x_0, z) dG_{-1}(x_{-1}) dH(y)$$

$$- \iint_0^{x_0} \max(z) dF_1^{m-1}(x_{-1}) dF_2^n(y) - bF_1^{m-1}(\phi_1(b)) F_2^n(\phi_2(b))$$

and we again have (16) in this case as well. The last case $\phi_2(b) < x_0 \leq \phi_1(b)$ is completely similar. In all cases, the difference between the two profit functions is independent of b. The above proof is applicable to any bidder in group one or two.

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