

# Identification and Estimation of Nonstationary Dynamic Discrete Choice Models

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## Abstract

Under common assumptions for dynamic discrete choice models with general forms of non-stationarity, we prove a novel Markovian property that allows us to bypass the state transition distribution in the identification and estimation of flow utility parameters. We show that under mild additional assumptions, the identification of the flow utility parameters amounts to the unique solution of a linear system of equations, which depends on a testable rank condition. The theoretical and empirical implications of the rank condition in our model are thoroughly discussed. We propose a three-step conditional-choice-probability-based semiparametric estimator that circumvents estimation of and simulating from the state transition distribution. Simulation experiments show that the rank condition is easy to fulfill, and our estimator gives comparable finite sample performance as the Hotz-Miller estimators but is computationally much less demanding. The asymptotic distribution of the estimator is provided, and the sensitivity of the estimator to the key additional assumption is also examined.

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# 1 Introduction

Structural dynamic discrete choice (DDC) models are a powerful framework of analysis in many fields – e.g., industrial organization (Barwick and Pathak, 2015), labor markets (Rust and Phelan, 1997), agricultural economics (Scott, 2014) – where the inter-temporal preference of forward-looking agents needs to be explicitly modeled in estimation and counterfactual prediction. Since the seminal work by Rust (1987), various estimation methods have been proposed to reduce computational complexity (Hotz and Miller, 1993; Hotz et al., 1994; Aguirregabiria and Mira, 2002, 2007; Bunting and Ura, 2025), relax distributional assumptions on the unobserved state variables (Norets and Tang, 2014; Chiong, Galichon and Shum, 2016; Buchholz, Shum and Xu, 2021; Berry and Compiani, 2023), or exploit finite dependence properties (Arcidiacono and Miller, 2011, 2019; Kalouptsi, Scott and Souza-Rodrigues, 2021b). Estimators of DDC models typically use estimated transition distributions of the observed state variables, or even simulated draws from them, as inputs, with Adusumilli and Eckardt (2019) being a notable exception.

State transition distributions can be regarded as known for identification purpose, but to estimate or simulation from them can be difficult in practice. Nonparametric estimation of state transition distributions usually suffers from slow rate of convergence, especially when the state variables have a large dimension or contain continuous variables, because they are conditional distributions of all the observed state variables in one period given all the observed state variables in the previous period. As a pragmatic alternative, researchers routinely postulate parametric specifications, but it is not always obvious which parametric specification is appropriate, and ad hoc choices rarely have sound economic justification. Simulation from or numerical integration based on estimated state transitions adds further computation burden, regardless whether the state transitions are nonparametric or parametric. Moreover, estimating state transitions might be unnecessary for counterfactual analysis, because certain counterfactual behaviors are identified without restricting state transition (i.e., identified for all state transitions, see Aguirregabiria, 2010; Arcidiacono and Miller, 2020, and reference therein for details), and certain counterfactual involves new state transition that is different from the one in the data used for estimating the structural parameters. Even for the counterfactuals that require state transition estimates as inputs, a state-transition-free estimator can still be useful in guiding the comparison and selection among different specifications of the state transition distributions.

The main practical contribution of this paper is to develop a novel semiparametric estimator of flow utility parameters (and discount factor) that bypasses the estimation and simulation of state transition distributions. Instead, our estimator bases off a linear system that characterizes the optimal decision rule and has the values of certain conditional mean functions as inputs. It proceeds in three simple steps: (i) nonparametrically estimate the conditional choice probabilities (CCPs); (ii) nonparametrically estimate relevant conditional means, which have the CCPs as the

dependent variables; (iii) plug the estimated CCPs and conditional means into the linear system to estimate the parameters of interest via linear minimum distance (MD). The conditional means have a smaller dimension and better finite sample properties than the conditional densities, and our estimator does not involve any simulation or iteration. The length of the panel data can be short ( $T \geq 2$ ), the observed state variables can be continuous, and the data horizon could be shorter than the decision horizon (“short panel” studied in [Arcidiacono and Miller, 2020](#)). In our simulation experiments (Section 5), our estimator is approximately 100-200 times faster than the two-step CCP estimator ([Hotz et al., 1994](#), HMSS) with nonparametrically estimated state transition distributions.

Another practical advantage is that our estimator does not need to distinguish between stationary and nonstationary DDC models *ex ante*. Stationarity means that the value and the policy functions are both time-invariant. Nonstationarity, therefore, could arise from time-varying primitives, such as time-varying flow utility (i.e., preference) and time-varying state transition distributions (i.e., environment). Even if none of the model primitives is time-varying, the value and the policy functions could still be time-varying simply due to finite decision horizon.

Fundamental to our approach is a Markovian property for the observed state variables (without conditioning on the choice), which holds under common assumptions (Assumptions 1 to 3) that are often made in the DDC literature (see, e.g., a summary by [Aguirregabiria and Mira, 2010](#)). We may not be the first one to recognize this Markovian property, but we are the first to articulate it under clear and coherent assumptions and to take advantage of it in identification and estimation. Intuitively, the Markovian property means that the current state variables are less informative than those in a future period that is closer to the target future period. This, combined with the law of iterated expectations, is the key to simplifying a representation of the optimal decision rule that involves iterated conditional means given the state variables in all future periods to a representation given the state variables in the current period only. Such a simplification eliminates the need for estimation of and/or recursive simulation from the state transition distributions as is typically required in evaluating the objective functions for various estimators of DDC models.

In our identification analysis, we start with this simplified representation and transform it into a partially linear system under a linear flow utility assumption (Assumption 5) that is weaker than typical in the DDC literature.<sup>1</sup> If one is willing to make an additional mild assumption about the data-terminal-period integrated value function (Assumption 6), then we show that the system can be further transformed into a linear one in the parameters of interest, enabling simple and constructive sufficient identification conditions, which further enable our easy-to-implement estimator. Sensitivity to this additional assumption is examined in the paper.

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<sup>1</sup>We avoid arbitrary “normalization” of the expected flow utility for one choice. [Norets and Tang \(2014\)](#); [Aguirregabiria and Suzuki \(2014\)](#); [Chou \(2016\)](#); [Kalouptsi, Scott and Souza-Rodrigues \(2021a\)](#) demonstrate the bias induced by such a “normalization”.

**Relation to literature.** Given the well-known result that the unrestricted flow utility of DDC models is non-identified (Rust, 1994; Magnac and Thesmar, 2002) from choice data, Hotz and Miller (1993)’s inversion theorem lays the foundation for studying point identification conditions, including this paper. Magnac and Thesmar (2002) consider identifying the difference in continuation values and discuss the identifying power of absorbing state, exclusion restriction and parametric restriction. Arcidiacono and Miller (2011) introduce the “finite dependence” property, a generalization of renewal choice (Rust, 1987) and terminal choice (Bajari et al., 2016). For short panels, Arcidiacono and Miller (2020) exploit a stronger “single action finite dependence” property. Abbring and Daljord (2020) show that exclusion restriction generally does not identify the discount factor and can be rejected by the data. This paper explores an alternative set of point identification conditions for flow utility (and discount factor), in a setting general enough to allow for stationary or nonstationary models, as well as long or short panels. Our conditions do not require the finite dependence property or any exclusion restriction, although they may assist identification if hold. In this sense, this paper complements existing literature on identification.

Rust (1987)’s seminal work on the estimation of DDC models reduces the computational burden of full solution maximum likelihood estimator (MLE) by nesting fixed point algorithm inside the likelihood function (NFXP). Following Hotz et al. (1994), whose pioneering HMSS estimator reduces computational burden even more by solving not the dynamic programming problem but instead a moment criterion based on CCP, the inversion theorem of Hotz and Miller (1993) and CCP have become essential to almost all subsequent non-maximum-likelihood estimators. Aguirregabiria and Mira (2002) show the asymptotic equivalence between NFXP and HMSS by considering a class of estimators (NPL) that nests a pseudo likelihood inside a fixed point problem based on CCP. In a game setting with discrete state variables, Pesendorfer and Schmidt-Dengler (2008) show that HMSS and NPL fall into a same class of least squares estimators, with the weights to individual equilibrium conditions being different. Srisuma and Linton (2012) generalize Pesendorfer and Schmidt-Dengler (2008)’s approach to allow for continuous state variables. Exploiting the finite dependence property (Arcidiacono and Miller, 2019), an analogy to continuous choice models (Aguirregabiria and Magesan, 2013), or both (Kalouptsi, Scott and Souza-Rodrigues, 2021b), various estimators with further reduction in computational burden were proposed. Bunting and Ura (2025) study an index invertibility property that, if holds in DDC models, helps speed up M estimators. Drawing on their respective novel insights, Chiong, Galichon and Shum (2016); Buchholz, Shum and Xu (2021) relax the distributional assumptions on utility shocks. Our semi-parametric estimator also builds on Hotz and Miller (1993)’s inversion theorem and uses estimated CCP as input; it accommodates continuous state variables, short panel and nonstationarity under a same framework. More importantly, the above estimators all require estimated state transition distributions (or even simulation from or numerical integration of them) as inputs to evaluate their respective objectives, while our estimator completely bypasses the state transitions and does not

involve any simulation or iteration. Our approach complements existing ones in greatly simplifying the evaluation of various quantities used in the estimation. The closest paper to ours, in this practical sense, is [Adusumilli and Eckardt \(2019\)](#), which focuses on stationary models. In another related paper, [Buchholz, Shum and Xu \(2021\)](#) propose to estimate the parameters of the linear-in-parameters flow utility functions using a linear-in-parameters moment condition. We also derive a linear-in-parameters moment condition that is exactly linear for long panels, and is approximately linear for short panels if we use a linear-in-parameters approximation of the terminal condition. The derivation and motivation of the linear moment conditions is nevertheless different. [Buchholz, Shum and Xu \(2021\)](#) derive their moment condition without making an assumption on the distribution of the utility shocks, but their estimation method requires non-parametric estimation of the state transition density (see [Buchholz, Shum and Xu, 2021](#), p. 320). Our linear moment condition does not require estimation of the state transition density, although it does require a parametric choice of the distribution of the utility shocks. The estimation of the recursive state transition densities is particularly onerous in finite horizon DDC models. [Buchholz, Shum and Xu \(2021\)](#) only consider the infinite horizon stationary case.

Recent literature on DDC models took on several new directions, including identifying various counterfactual objects ([Aguirregabiria, 2010](#); [Aguirregabiria and Suzuki, 2014](#); [Arcidiacono and Miller, 2020](#); [Kalouptsi, Scott and Souza-Rodrigues, 2021a](#)), unobserved heterogeneity ([Kasahara and Shimotsu, 2009](#); [Arcidiacono and Miller, 2011](#); [Hu and Shum, 2012](#); [Chou, Derdenger and Kumar, 2019](#); [Higgins and Jochmans, 2023](#)), mixed choices ([Blevins, 2014](#)) and partial identification ([Norets and Tang, 2014](#); [Berry and Compiani, 2023](#); [Kalouptsi et al., 2024](#)). Whether our approach is useful for studying these important aspects is left for future research. For conciseness, we also focus on single-agent DDC models, but extending our approach to dynamic discrete games is possible.

**Notation.** “ $f$ ” is a generic symbol for (conditional) probability density/mass functions. “ $\equiv$ ” means that the object on its left-hand side is defined as the expression on its right-hand side.

**Plan of paper.** Section 2 sets up a dynamic discrete choice model that permits general non-stationarity and proves the Markovian property. Section 3 transforms the identification of the flow utility parameters into the solution of a linear system under common and mild new assumptions, analyzes the identification of the linear system, and provides a three-step CCP-based semiparametric estimator of the flow utility parameters that bypasses the state transition distribution estimation. Section 4 discusses the bias induced by relaxing the new assumption made in Section 3 and how to reduce it. Section 5 compares the finite sample performance of our estimator and two versions of the HMSS estimator in simulation experiments. We make concluding remarks in Section 6. We focus on binary choice in the main text, and all our results can be generalized to multinomial choice models at notational expense (Section G). All the proofs and certain related issues are in the appendices.

## 2 Model Setup and Markovian Property

Section 2.1 introduces the model and basic assumptions and lists a few well-known results on which our analysis is based on. Section 2.2 proves the novel Markovian property and its immediate implications.

### 2.1 A Nonstationary Dynamic Binary Choice Model

Each agent makes a binary choice  $a_t \in \{0, 1\}$  in each of a number of periods, denoted by  $t \in \mathcal{T} \equiv \{T_{start}, T_{start} + 1, \dots, T_{end}\}$ , where  $T_{end} = \infty$  is allowed. Let  $u_t(a_t, s_t) + \varepsilon_{a_t t}$  denote the flow utility that has an additively separable form, where the expected flow utility  $u_t(a_t, s_t)$  depends on the choice  $a_t$  and  $s_t$ , a  $d_s \times 1$  vector of observed state variables, through an unknown function  $u_t$ , and  $\varepsilon_{a_t t}$  ( $a_t = 0, 1$ ) are unobserved (to researchers) scalar flow utility shocks. Let  $\varepsilon_t \equiv (\varepsilon_{0t}, \varepsilon_{1t})'$  denote the unobserved state variables and  $\Omega_t \equiv (s_t', \varepsilon_t')$ . The primitives of this model consist of the flow utility functions  $u_t(a, s)$ , the state transition distribution functions  $f_t(\Omega_{t+1} | \Omega_t)$ , and the initial distribution of  $\Omega_{T_{start}}$ .<sup>2</sup>

We maintain the following Assumptions 1 to 3 in this paper.

**Assumption 1** (Controlled Markov process). *For all  $t$  and  $j \in \mathbb{N}^+$  such that  $t, t + 1$  and  $t - j$  all belong to  $\mathcal{T}$ , assume that  $\Omega_{t+1} \perp (\Omega_{t-j}, a_{t-j}) | (\Omega_t, a_t)$ .*

**Assumption 2** (Flow utility shocks). *For all  $t$  such that  $t$  and  $t - 1$  both belong to  $\mathcal{T}$ , assume: (i)  $\varepsilon_t \perp s_t$ ; (ii)  $\varepsilon_t \perp s_{t-1}$ ; (iii)  $\varepsilon_t$  is serially independent; and (iv)  $\varepsilon_{0t} \perp \varepsilon_{1t}$ .*

**Assumption 3** (Conditional independence). *For all  $t$  such that  $t$  and  $t + 1$  both belong to  $\mathcal{T}$ , assume that  $s_{t+1} \perp \varepsilon_t | (s_t, a_t)$ .*

Assumptions 1 to 3 are common assumptions in the literature of DDC models and have been made, explicitly or implicitly, in many applications. In particular, Assumptions 2(i) and 2(ii) are implied by Assumption IID in the survey by Aguirregabiria and Mira (2010) and are necessary for the conditional independence assumption in (Rust, 1987, p. 1011).<sup>3</sup> Moreover, Assumptions 2(iii), 2(iv) and 3 correspond to Assumptions IID, CLOGIT and CI-X, respectively, in the survey by Aguirregabiria and Mira (2010).

Under Assumption 1, the agents observe  $\Omega_t$  and choose  $a_t$  to maximize their expected discounted lifetime payoff  $\mathbb{E}(\sum_{j=0}^{T_{end}-t} \beta^j (u_{t+j}(a_{t+j}, s_{t+j}) + \varepsilon_{a_{t+j} t+j}) | \Omega_t, a_t)$  in period  $t$ , where  $\beta \in (0, 1)$  is the

<sup>2</sup>Under Assumption 1 below, the distribution of  $\Omega_{T_{start}}$  is insignificant for the analysis of the model. If there is unobserved heterogeneity, not captured by  $\varepsilon_t$ , among the agents (we will consider this in a separate paper), then the initial distribution becomes important.

<sup>3</sup>From the phrasing of Assumption IID in Aguirregabiria and Mira (2010), it is not obvious that Assumptions 2(i) and 2(ii) are implied, but they claim in the paragraph after Assumption DIS that, using our notation, Assumptions CI-X and IID together imply that  $f(s_{t+1}, \varepsilon_{t+1} | a_t, s_t, \varepsilon_t) = f_\varepsilon(\varepsilon_{t+1}) f_s(s_{t+1} | a_t, s_t)$ , which does not hold without Assumptions 2(i) or 2(ii).

discount factor, and we use  $a_t^o(s_t, \varepsilon_t)$  to denote the optimal decision rule. The probability of choosing choice 1 in period  $t$  conditional on  $s_t = s$  (i.e., the CCP function) is obtained by taking  $a_t^o(s_t, \varepsilon_t)$  and integrating out  $\varepsilon_t$ :

$$p_t(s) \equiv \Pr(a_t = 1 \mid s_t = s) = \int a_t^o(s, \varepsilon_t) f_\varepsilon(\varepsilon_t) d\varepsilon_t. \quad (1)$$

Note that the subscript  $t$  emphasizes that the CCP might be a different function in every period, and this is a result of the general nonstationarity (see Remark 1 below) allowed by the model in this paper. In the rest of this paper, we will suppress the argument of the CCP functions and use  $p_t$  to denote  $p_t(s_t)$  for notational conciseness, whenever it is not confusing. The *integrated* value function in period  $t$  is the discounted lifetime payoff just before  $\varepsilon_t$  realizes if the agent follows the optimal decision rule in the current and all future periods:

$$\bar{V}_t(s_t) \equiv \mathbb{E} \left( \sum_{j=0}^{T_{end}-t} \beta^j [a_{t+j}^o(s_{t+j}, \varepsilon_{t+j})(u_{t+j}(1, s_{t+j}) + \varepsilon_{1t+j}) + (1 - a_{t+j}^o(s_{t+j}, \varepsilon_{t+j}))(u_{t+j}(0, s_{t+j}) + \varepsilon_{0t+j})] \mid s_t \right). \quad (2)$$

By Bellman's principle, it can be shown to satisfy the following recursive relation:

$$\bar{V}_t(s_t) = U_t^o(s_t) + \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t) \quad (3)$$

for  $t$  such that  $t$  and  $t + 1$  both belong to  $\mathcal{T}$ , where  $U_t^o(s_t) \equiv \mathbb{E}(a_t^o(s_t, \varepsilon_t)(u_t(1, s_t) + \varepsilon_{1t}) + (1 - a_t^o(s_t, \varepsilon_t))(u_t(0, s_t) + \varepsilon_{0t}) \mid s_t)$  is the *integrated optimal* flow utility function. Define the choice-specific conditional value function as the current flow utility of any choice  $a \in \{0, 1\}$ , without the shock  $\varepsilon_{at}$ , plus the expected future discounted payoff if following the optimal decision rule from period  $t + 1$  onward:

$$v_{at}(s_t) \equiv u_t(a, s_t) + \beta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t, a_t = a), \text{ where } a = 0 \text{ or } 1. \quad (4)$$

It is a well-known result (Hotz and Miller, 1993; Arcidiacono and Miller, 2011, Lemma 1) that for each  $a \in \{0, 1\}$ , there exists a real-valued function  $\psi_{at}$  of the CCP only such that

$$\psi_{at}(p_t) = \bar{V}_t(s_t) - v_{at}(s_t). \quad (5)$$

Intuitively,  $\psi_{at}$  adjusts for the fact that choice  $a$  might not be the optimal choice today; it equals to the conditional mean of  $\varepsilon_{at}$  given choice  $a$  being optimal, so the functional form of  $\psi_{at}$  only depends on the distribution of  $\varepsilon_{at}$ . Taking the difference of eq. (5) between  $a = 1$  and  $a = 0$  and plugging in eq. (4), we get

$$\psi_{0t}(p_t) - \psi_{1t}(p_t) = v_{1t}(s_t) - v_{0t}(s_t) = u_t(1, s_t) - u_t(0, s_t) + \beta \Delta \mathbb{E}(\bar{V}_{t+1}(s_{t+1}) \mid s_t). \quad (6)$$

In eq. (6) and in the rest of this paper, we use

$$\Delta \mathbb{E}(h_\tau \mid s_t) \equiv \mathbb{E}(h_\tau \mid s_t, a_t = 1) - \mathbb{E}(h_\tau \mid s_t, a_t = 0) \quad (7)$$

to denote the difference in the conditional means of any future random variable  $h_\tau$  given  $s_t$  and between  $a_t = 1$  and  $a_t = 0$  ( $\tau > t$ ); they reflect the forward-looking aspects of the agent’s decision problem. Throughout this paper, we maintain that the agents form rational expectation; that is, the operator  $\mathbb{E}$  refers to both mathematical expectation (under the optimal decision rule) and the agents’ expectation.

**Remark 1** (Nonstationarity). *Nonstationarity in the context of DDC models means that the agent’s decision problem in a period  $t$  is not an identical copy of that in other periods  $t' \neq t$ , so that the CCPs in eq. (1) and the value functions (both  $\bar{V}_t(\cdot)$  and  $v_{at}(\cdot)$  in eqs. (2) to (6)) can be time-varying. The model in this paper permits several sources of nonstationarity: (i) the function  $u_t(a, s)$  that determines the non-stochastic part of the flow utility may vary with  $t$ ; (ii) the state transition density  $f_t(\Omega_{t+1}|\Omega_t)$  may vary with  $t$ ; and (iii) even when both  $u_t$  and  $f_t$  are time-invariant, nonstationarity could arise simply because the decision terminal period  $T_{end}$  may be finite and/or unknown. Any of these sources could result in time-varying CCPs and/or time-varying value functions. While the approach in this paper applies to stationary models (see Remark 7 below for elaboration), certain widely used methods for stationary models, such as solving/approximating the value functions from eq. (3) (or other recursive relation derived from the Bellman’s principle, e.g., see [Srisuma and Linton, 2012](#); [Barwick and Pathak, 2015](#); [Adusumilli and Eckardt, 2019](#)), are not applicable to nonstationary models.*

Equations (3) to (6) are well-known, important implications of [Hotz and Miller \(1993\)](#)’s inversion theorem and the basis for many approaches to identification and estimation of DDC models (e.g., [Aguirregabiria, 2010](#); [Aguirregabiria and Mira, 2002, 2010](#)), because they concisely relate various objects concerning the optimal decision rule. Developments that combine them with new insights include: [Srisuma and Linton \(2012\)](#) utilize the integral equation representation of eq. (3), and [Buchholz, Shum and Xu \(2021\)](#) further allow for unspecified distribution of  $\varepsilon_{at}$  by expressing its quantile function as the solution to an integral equation; [Chiong, Galichon and Shum \(2016\)](#) recognize the duality of an expression related to eq. (5); [Arcidiacono and Miller \(2011\)](#); [Kalouptsi, Scott and Souza-Rodrigues \(2021b\)](#) express the second term on the right-hand side of eq. (6) in terms of the optimal choices and state transitions in consecutive future periods and study the choice pairs that cancel out after finite periods.

## 2.2 Markovian Property

We also build on equations (3) and (6), but with the Markovian property, articulated below, we can “telescope” a version of eq. (6) involving iterated conditional means, which serves as the basis of our state-transition-distribution-free approach to identification and estimation.

**Theorem 1** (Markovian observed state variables). *Under Assumptions 1 and 2(i)-(iii),  $s_t$  is a first*

order Markov process; that is,  $s_{t+1} \perp\!\!\!\perp s_{t-j} \mid s_t$  for  $t$  and  $j \in \mathbb{N}^+$  such that  $t, t+1$  and  $t-j$  all belong to  $\mathcal{T}$ .

What makes DDC models dynamic is the dependence of future evolution of the state variables  $\Omega_{t+j}$  on the choice  $a_t$ , so this Markov property may seem odd at first sight. In fact, this Markovian property alone does not imply Assumption 1, nor it is implied by Assumption 1 alone. They are simply two separate statements. To show the importance of this Markovian property, we need the following additional auxiliary lemma.

**Lemma 1** (Conditional independence). *For  $t$  and  $j \in \mathbb{N}^+$  such that  $t, t+1$  and  $t+j$  all belong to  $\mathcal{T}$ , suppose  $g(\cdot)$  is a measurable function of  $s_{t+j}$ . Then, under Assumptions 1 and 2(i)-(iii),*

$$\mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t) = \mathbb{E}(g(s_{t+j}) \mid s_t, a_t).$$

Now we go back to eq. (6) and see how the Markovian property in Theorem 1 helps. Increase  $t$  in eq. (3) by increment of 1 until some  $T^* \in \mathcal{T}$  and recursively plug into eq. (6), we get

$$\begin{aligned} \psi_{0t}(p_t) - \psi_{1t}(p_t) &= u_t(1, x_t) - u_t(0, x_t) + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \Delta \mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(U_\tau^o(s_\tau) \mid s_{\tau-1}) \cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t) \\ &\quad + \beta^{T^*-t} \Delta \mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(\mathbb{E}(\bar{V}_{T^*}(s_{T^*}) \mid s_{T^*-1}) \mid s_{T^*-2}) \cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t). \end{aligned} \quad (8)$$

Equation (8) is similar to eq. (14) in Corollary 4 of Arcidiacono and Miller (2020), although the two equations use different expressions for the last two iterative terms, where the expectations are taken over the distributions of  $s_{t+j}$  induced by choosing the action according to  $p_{t+j}(\cdot)$  and the state transitions conditional on  $s_t$ . The integral equation approach in Srisuma and Linton (2012) uses an equation<sup>4</sup> that has (two layers of) iterative conditional expectations like eq. (8). To apply Theorem 1 to simplify eq. (8), notice that the last two terms in eq. (8) has the following common iterative form (recall eq. (7)):

$$\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(g(s_{t+j}) \mid s_{t+j-1}) \cdots \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t), \quad (9)$$

where  $t$  and  $j \in \mathbb{N}^+$  are such that  $t, t+1, \dots, t+j$  all belong to  $\mathcal{T}$ , and  $g(\cdot)$  is a measurable function of  $s_{t+j}$ . Theorem 1 enables us to simplify eq. (9):

$$\begin{aligned} &\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(g(s_{t+j}) \mid s_{t+j-1}) \cdots \mid s_{t+4}) \mid s_{t+3}) \mid s_{t+2}) \mid s_{t+1}) \mid s_t, a_t) \\ &= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(g(s_{t+j}) \mid s_{t+j-1}) \cdots \mid s_{t+4}) \mid s_{t+3}) \mid s_{t+2}, s_{t+1}) \mid s_{t+1}) \mid s_t, a_t) \\ &= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(g(s_{t+j}) \mid s_{t+j-1}) \cdots \mid s_{t+4}) \mid s_{t+3}) \mid s_{t+1}) \mid s_t, a_t) \\ &= \mathbb{E}(\mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(g(s_{t+j}) \mid s_{t+j-1}) \cdots \mid s_{t+4}) \mid s_{t+3}, s_{t+1}) \mid s_{t+1}) \mid s_t, a_t) \\ &= \mathbb{E}(\mathbb{E}(\mathbb{E}(\cdots \mathbb{E}(g(s_{t+j}) \mid s_{t+j-1}) \cdots \mid s_{t+4}) \mid s_{t+1}) \mid s_t, a_t) \end{aligned}$$

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<sup>4</sup>The equation between eq. (5) and eq. (6) in Srisuma and Linton (2012).

$$\begin{aligned}
&= \dots \\
&= \mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t),
\end{aligned} \tag{10}$$

where the first and the third equalities hold by  $s_{t+3} \perp\!\!\!\perp s_{t+1} \mid s_{t+2}$  and  $s_{t+4} \perp\!\!\!\perp s_{t+1} \mid s_{t+3}$ , respectively, both implied by Theorem 1, and the second and the fourth equalities hold by the law of iterated expectations. The rest of eq. (10) holds by repeatedly applying such an argument. Then, Lemma 1, eq. (8) and eq. (10) together lead to the following result.

**Theorem 2** (Telescoping). *Under Assumptions 1 to 3, for  $t, T^* \in \mathcal{T}$  and  $t < T^*$ ,*

$$\begin{aligned}
\psi_{0t}(p_t) - \psi_{1t}(p_t) &= u_t(1, x_t) - u_t(0, x_t) + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \Delta \mathbb{E}(U_\tau^o(s_\tau) \mid s_t) \\
&\quad + \beta^{T^*-t} \Delta \mathbb{E}(\bar{V}_{T^*}(s_{T^*}) \mid s_t).
\end{aligned} \tag{11}$$

We believe we are the first to clearly articulate the Markovian property of  $s_t$  (Theorem 1) and to exploit it to bypass state transition in identification and estimation of flow utility in DDC models characterized by Assumptions 1 to 3.<sup>5</sup> Intuitively, the Markovian property means that none of the observed state variables  $s_{t-j}$  in earlier periods provides additional information about  $s_{t+1}$  beyond what is contained in  $s_t$ . Since the law of iterated expectations essentially states that only the least informative information set matters, such Markovian property is the key to telescoping eq. (8), an equation that characterizes the optimal decision rule and involves iterative conditional means, to eq. (11), which only has simple conditional mean differences.

Without such telescoping, a moment-based estimator off eq. (8), such as the HMSS estimator, would typically require simulating whole sequences  $a_t, s_{t+1}, a_{t+1}, \dots, s_{T^*}, a_{T^*}$  to evaluate the moment. For each period in the sequence,  $a_t$  is drawn from a binary distribution with estimated CCP  $\hat{p}(s_t)$ , and  $s_t$  is drawn from the distribution characterized by the estimated state transition distribution  $\hat{f}_{s_t \mid s_{t-1}, a_{t-1}}$ . This brings about several implementation issues, especially when  $s_t$  has a large dimension or contains continuous variables. (i) A parametric specification is often used when nonparametric estimation of  $f_{s_t \mid s_{t-1}, a_{t-1}}$  is difficult (see, for example, Li and Racine, 2007, Chapter 5), but it is not always obvious what parametric specification is appropriate, or how sensitive the estimates or the counterfactuals are to such a specification.<sup>6</sup> (ii) Although the simulation draws need only to be made once, the numerical integration based on them must be performed for every hypothesized parameter value. Whether the state transition is parametrically or non-parametrically estimated, a large simulation may be necessary to deliver good approximation of

<sup>5</sup>Eq. (8) in Buchholz, Shum and Xu (2021) is similar to eq. (11) here since it also has simple conditional mean difference components that suggest a Markovian property, although they arrived at it via an integral equation representation in a stationary model. Besides, their representation is hard to generalize to multinomial choice models and they need to estimate the state transition due to unrestricted shock distributions.

<sup>6</sup>Other estimator reviewed in Section 1 that use estimated state transition as input suffer from the same issue.

the moment, demanding huge computational and memory resources. (iii) If  $f_{s_t|s_{t-1}, a_{t-1}}$  varies with  $t$ ,<sup>7</sup> then the choice of  $T^*$ , the end period of the simulated sequences, can be tricky. On the one hand, if one chooses  $T^*$  to be before the data terminal period, then  $f_{s_t|s_{t-1}, a_{t-1}}$  and  $p_t$  for  $t \leq T_{tr}$  can all be estimated from the data. The problem arises when only short panel data are available (i.e.,  $T^*$  very small), in this case the impact of the truncation is not negligible, so the estimate of  $\mathbb{E}(\bar{V}_{t+1}(s_{t+1}) | s_t, a_t)$  can be substantially biased, leading to biased parameter estimates. On the other hand, if one chooses  $T^*$  to be after the data terminal period, then they must make additional assumptions regarding how the state variables and the flow utility parameters will evolve beyond the data horizon. Unfortunately, there is no agreed-upon way of making such assumptions, and it is unclear how large the bias is.

In contrast, taking advantage of the Markovian property of  $s_t$ , eq. (11) points to the possibility of bypassing the estimation and simulation of the state transition. In Section 3, we will exploit the implications of eq. (11) for the estimation and identification of the flow utility parameter. We will demonstrate that if one is willing to make a few mild additional assumptions, then a vastly simpler estimator with an easy-to-verify identification condition is available. We conjecture that progress could be made in relaxing some of these additional assumptions and in utilizing the Markovian property to simplify some existing estimators, but we leave this conjecture for future inquiry.

### 3 Identification and Estimation

Section 3.1 transforms eq. (11) into a linear system under a few mild additional assumptions. (Among them, only Assumption 6 is new to the DDC literature.) Section 3.2 provides testable condition for identifying the flow utility parameter from the linear system and discusses its economic and empirical content, special cases and caveats. Section 3.3 prescribes our simple state-transition-free three-step CCP-based semiparametric estimator for the flow utility parameter.

#### 3.1 Transformation into a Linear System

**Assumption 4** (Logit). *For all  $t \in \mathcal{T}$ , assume  $\varepsilon_{0t}$  and  $\varepsilon_{1t}$  both follow a type I extreme value distribution.*

Assumption 4 postulates a logit model and corresponds to Assumption CLOGIT in Aguirregabiria and Mira (2010). Under Assumption 4,  $\psi_{0t}(p_t) - \psi_{1t}(p_t) = \ln(p_t/(1-p_t))$  in eq. (6), eq. (8) and eq. (11).<sup>8</sup>

<sup>7</sup>This generally implies that  $p_t$  also varies with  $t$  (i.e., a nonstationary model).

<sup>8</sup>Under Assumption 4,  $\psi_0(p_t) = \gamma - \ln(1-p_t)$  and  $\psi_1(p_t) = \gamma - \ln(p_t)$ , where  $\gamma$  is the Euler's constant. If  $\varepsilon_{at}$  follows another distribution, then  $\psi_a$  takes a different form, but it will still be a known function of the CCP only, and its functional form only depends on the distribution of  $\varepsilon_t$ .

**Assumption 5** (Linear expected flow utility). *Let  $x_t$  denote a  $d_x \times 1$  subvector of  $s_t$ , with  $d_x \leq d_s$ . For each  $t \in \mathcal{T}$ , assume  $u_t(0, x_t) = x_t' \delta_{0,t}$  and  $u_t(1, x_t) = x_t' \delta_{1,t}$  for some  $\delta_{0,t}$  and  $\delta_{1,t}$ . We normalize  $\delta_{0,1}$  to an arbitrary  $d_x \times 1$  vector of constants, denoted by  $c$ .*

Assumption 5 allows  $x_t$  to be  $s_t$  itself or a proper subvector of  $s_t$ . In the latter case, the coordinates of  $s_t$  that are not in  $x_t$ , denoted by a  $d_z \times 1$  vector  $z_t$  (with  $d_z \equiv d_s - d_x$ ), are observed state variables that do not directly affect the current flow utility but may affect the future lifetime payoff through their impact on the distribution of  $s_{t+1}$ .  $z_t$  are referred to as “excluded variables” in the DDC literature, and their function in our approach will be discussed in Remark 6 below.<sup>9</sup>

It is a common assumption in the DDC literature to assume that  $u_t(0, x_t) = 0$  for all  $x_t$  values and all  $t \in \mathcal{T}$  (or equivalently,  $\delta_{0,t} = 0$  for all  $t \in \mathcal{T}$ ), but we refrain from making such an assumption because it has been illustrated to result in substantial bias in counterfactual analysis (Aguirregabiria and Suzuki, 2014; Norets and Tang, 2014; Chou, 2016). More importantly, such a strong assumption is unnecessary, because in contrast to static binary choice models, dynamic models can leverage inter-temporal variation in  $x_t$  to separate  $u_t(0, x_t)$  and  $u_t(1, x_t)$ , in addition to the difference between the two.<sup>10</sup> A normalization for the data initial period, however, is necessary.<sup>11</sup>

The linear specification of  $u_t(a, x_t)$ , although common in the DDC literature, may appear restrictive. This concern could be partially alleviated as the state variables themselves and various functions of them (e.g., power series) are allowed to be included in  $x_t$ .<sup>12</sup> Under Assumption 5, the identification of the flow utility function, the key primitive structural object of this model, boils down to identifying  $\delta_{0,t}$  and  $\delta_{1,t}$  for  $t \in \mathcal{T}$ . Under Assumptions 1 to 5, the following result links the optimal flow utility function  $U_t^o(s_t)$  in eq. (11) with these parameters:

$$\begin{aligned} U_t^o(s_t) &= p_t u_t(1, x_t) + (1 - p_t) u_t(0, x_t) + \gamma - p_t \ln p_t - (1 - p_t) \ln(1 - p_t) \\ &= p_t x_t' \delta_{1,t} + (1 - p_t) x_t' \delta_{0,t} + \gamma - p_t \ln(p_t) - (1 - p_t) \ln(1 - p_t), \end{aligned} \quad (12)$$

where  $\gamma$  is the Euler’s constant (not to be confused with  $\gamma^K$  defined in Assumption 6 below), the

<sup>9</sup> An example of excluded variable  $z_t$  is spouse’s education attainment in the problem of labor force participation  $a_t$ . Given spouse’s current income (a coordinate of  $x_t$ ), spouse’s education attainment does not directly affect current utility, but it is likely to affect how spouse’s income evolve in the future, hence affecting the discounted future payoff.

<sup>10</sup> Remark 5 below is also related to this.

<sup>11</sup> One could normalize  $\delta_{1,t}$  to an arbitrary constant vector  $c$  instead, but it will not affect the identification conditions or the values of the parameters  $\delta$  and  $\gamma^K$  discussed in Section 3.2 below. Moreover, because this arbitrary value will be canceled in the difference in the data initial period flow utility between the two choices and does not affect future payoff, it will not affect the counterfactual analysis either. In this sense, it is a real normalization.

<sup>12</sup> In fact, it is possible to relax Assumption 5 to something similar to Assumption 6 below. That is, suppose there exists a  $K_u \times 1$  vector of known functions of  $x$ , denoted by  $q^{K_u}(x) \equiv (q^{K_u,1}(x), \dots, q^{K_u,K_u}(x))'$ , and for each  $t \in \mathcal{T}$ , there exist  $K_u \times 1$  unknown vectors of parameters  $\delta_{0,t}^{K_u}$  and  $\delta_{1,t}^{K_u}$ , such that  $u_t(0, x_t) = q^{K_u}(x_t)' \delta_{0,t}^{K_u}$  and  $u_t(1, x_t) = q^{K_u}(x_t)' \delta_{1,t}^{K_u}$ . The identification of  $\delta_{0,t}^{K_u}$  and  $\delta_{1,t}^{K_u}$  follows the same argument as for  $\delta_{0,t}$  and  $\delta_{1,t}$  in Section 3.2 below, and the bias resulting from the violation of this specification can be quantified in the same way as in Section 4.1 below, both straightforwardly.

first equality holds by eq. (3.8) and eq. (4.12) of [Hotz and Miller \(1993\)](#), and the second equality holds by plugging in the expression of  $u_t(1, x_t)$  and  $u_t(0, x_t)$  in [Assumption 5](#).

We reparameterize  $\Delta_t \equiv \delta_{1,t} - \delta_{0,t}$  for  $t = 1, \dots, T^*$ , then  $\delta_{1,t} = \delta_{0,t} + \Delta_t$ . Plugging the expression of  $U_t^o(s_t)$  in eq. (12) into eq. (11), replacing  $\psi_{0t}(p_t) - \psi_{1t}(p_t)$  with  $\ln(p_t/(1-p_t))$ , rearranging and defining

$$\Delta \bar{x}_{1,t}^\tau \equiv \Delta \mathbb{E}(p_\tau x_\tau | s_t) \quad (13a)$$

$$\Delta \bar{x}_{0,t}^\tau \equiv \Delta \mathbb{E}((1-p_\tau)x_\tau | s_t), \quad (13b)$$

$$\Delta \bar{x}_t^\tau \equiv \Delta \mathbb{E}(x_\tau | s_t), \quad (13c)$$

$$y_{T^*-1} \equiv \ln\left(\frac{p_{T^*-1}}{1-p_{T^*-1}}\right), \text{ and,} \quad (13d)$$

$$y_t \equiv \ln\left(\frac{p_t}{1-p_t}\right) + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \Delta \bar{\eta}_t^\tau \quad (13e)$$

for  $t = 1, \dots, T^* - 2$  and  $\tau > t$ , where

$$\begin{aligned} \Delta \bar{\eta}_t^\tau &\equiv \Delta \mathbb{E}(\eta_\tau | s_t), \\ \eta_\tau &\equiv p_\tau \ln(p_\tau) + (1-p_\tau) \ln(1-p_\tau), \text{ for } \tau > t, \end{aligned} \quad (14)$$

we get the following equations

$$y_{T^*-1} = x'_{T^*-1} \Delta_{T^*-1} + \beta \Delta \mathbb{E}(\bar{V}_{T^*}(s_{T^*}) | s_{T^*-1}), \text{ and} \quad (15a)$$

$$\begin{aligned} y_t &= x'_t \Delta_t + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \Delta \bar{x}_t^{\tau'} \delta_{0,\tau} + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \Delta \bar{x}_{1,t}^{\tau'} \Delta_\tau \\ &\quad + \beta^{T^*-t} \Delta \mathbb{E}(\bar{V}_{T^*}(s_{T^*}) | s_t) \end{aligned} \quad (15b)$$

for  $t = 1, \dots, T^* - 2$ . Note that  $c$ , the arbitrary normalized value of  $\delta_{0,1}$ , does not show up in eq. (15).

We assume that the discount factor  $\beta$  is known for conciseness and focus on the identification of  $\Delta_t$  and  $\delta_{0,t}$ ,<sup>13</sup> the key structural parameters of the model that specify the flow utility functions. Although  $y_t$ ,  $\Delta \bar{x}_t^\tau$  and  $\Delta \bar{x}_{1,t}^\tau$  may appear to have complicated expressions, by their definitions in eqs. (13) to (14), they can be regarded as known for identification purpose, since all their components are either observed or identified from the data. In terms of estimation, they are all conditional mean difference functions between the two choices, with the “dependent variable” involving the CCPs that can be estimated in a prior step. The only outstanding complication now is the unknown integrated value function  $\bar{V}_{T^*}(s_{T^*})$  in the last terms of eqs. (15a) to (15b).

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<sup>13</sup>Remark 8 below shows that the identification only needs minor modification to account for unknown  $\beta$ , and the corresponding estimation method follows straightforwardly by slightly modifying the estimator in [Section 3.3](#) (omitted in this paper). Alternatively,  $\beta$  can be identified and estimated from a secondary data source, which is also common in the DDC literature.

**Remark 2** (Infeasibility of [Robinson \(1988\)](#) approach). *Although eq. (15) is a partially linear system, the approach developed by [Robinson \(1988\)](#) might not be feasible to identify or to estimate the linear parameters  $\Delta_t$  ( $t = 1, \dots, T^* - 1$ ) and  $\delta_{0,t}$  ( $t = 2, \dots, T^* - 1$ ). This is because the “non-linear terms” in eq. (15) are unknown functions of  $s_t$ , which include all the regressors in the “linear terms” ( $x_t$ , as well as  $z_t$  that show up in  $\Delta \bar{x}_t^T$  and  $\Delta \bar{x}_{1,t}^T$ ) as subsets ( $t = 1, \dots, T^* - 1$ ). This could lead to the failure of the key condition (3.5) in [Robinson \(1988\)](#). Moreover, eq. (15) imposes additional restrictions that  $\Delta \mathbb{E}(\bar{V}_{T^*}(s_{T^*}) | s_t)$  ( $t = 1, \dots, T - 1$ ) is a different unknown function of  $s_t$  for each  $t$ , and yet the unknown inner function  $\bar{V}_{T^*}(s_{T^*})$  remains invariant.*

Now it is a good time to emphasize an important distinction between *decision horizon*  $\mathcal{T} \equiv \{T_{start}, T_{start} + 1, \dots, T_{end}\}$  (introduced at the beginning of [Section 2.1](#)) and *data horizon*, denoted as  $\mathcal{T}_{data} \equiv \{1, \dots, T\}$ . The data horizon, available for researchers to observe, may only be a subset of the decision horizon – that is,  $T \leq T_{end}$  or  $T_{start} \leq 1$ . [Arcidiacono and Miller \(2020\)](#) refer to the scenario  $T < T_{end}$  as “short panel”, although  $T$  may be a large number; they call the  $T = T_{end}$  scenario “long panel”. All the results we derive so far hold for  $\forall T^* \in \mathcal{T}$ ; in particular, eq. (15) still holds if  $T^*$  is replaced by  $T$ . The following assumption about the data-terminal-period integrated value function  $\bar{V}_T(s_T)$ , new to the literature but mild, will help further transform eq. (15) into a linear system, for which we are equipped with many tools from the canon.

**Assumption 6** (Data-terminal-period integrated value function). *Assume that there exists a  $K \times 1$  vector of known functions of  $s$ , denoted by  $q^K(s) \equiv (q^{K,1}(s), \dots, q^{K,K}(s))'$ , and a  $K \times 1$  unknown vector of parameters  $\gamma^K \in \mathbb{R}^K$ , such that*

$$\bar{V}_T(s_T) = q^K(s_T)' \gamma^K. \quad (16)$$

**Remark 3** (Long panel). *Assumption 6 holds in an important special case – when the data terminal period is also the decision terminal period (i.e.,  $T = T_{end}$ ), or long panel. In this case, the integrated value function  $\bar{V}_T(s_T)$  consists of the current flow utility only and by eq. (12), it is*

$$\bar{V}_T(s_T) = U_T^o(S_T) = x_T' \delta_{0,T} + p_T x_T' \Delta_T + \gamma - \eta_T, \quad (17)$$

where  $\eta_T$  is defined in eq. (14), which is a function of  $p_T = \exp(x_T' \Delta_T) / (1 + \exp(x_T' \Delta_T))$  due to  $T = T_{end}$ . Note that  $\Delta_T$  can be identified and estimated using only the cross-sectional data on the terminal period choices and state variables using standard static logit approach, therefore,  $\Delta_T$ ,  $p_T$  and  $\eta_T$  can all be identified and easily estimated. In summary, this is a special case where [Assumption 6](#) holds exactly with  $K = 2d_x + 1$ ,  $q^K(x_T, z_T) = (x_T', p_T x_T', \gamma - \eta_T)'$  and  $\gamma^K = (\delta'_{0,T}, \Delta'_T, 1)'$ .

[Assumption 6](#) could be thought of as simply an alternative to other identification restrictions in the literature (e.g., exclusion restriction, a specific parametric restriction, or single action finite

dependence restriction in Arcidiacono and Miller, 2020) that, if holds, also helps achieve point identification. Assumption 6 could also be interpreted as approximating the  $\bar{V}_T(s)$  function using a series of basis functions  $q^K(s)$ , such as power series. This is in the same spirit as Keane and Wolpin (1994); Geweke and Keane (2000), who use polynomials to approximate the “future component” (i.e., next-period value function) in likelihood-based estimators of DDC structural parameters. In the rest of Section 3, we proceed with Assumption 6 holding exactly, but we will discuss this approximation perspective in Section 4. It is hard to find general primitive conditions for Assumption 6, nor it is testable (like many identification restrictions). We leave the identifiability question of  $\Delta_t$  and  $\delta_{0,t}$  from eq. (15) for future inquiry.

Assuming Assumption 6 holds, then eq. (15) can be substantially simplified to a system of equations linear in  $\Delta_t$  ( $t = 1, \dots, T - 1$ ),  $\delta_{0,t}$  ( $t = 2, \dots, T - 1$ ) and  $\gamma^K$ :

$$y_{T-1} = x'_{T-1} \Delta_{T-1} + \beta \Delta \bar{q}'_{T-1} \gamma^K, \text{ and} \quad (18a)$$

$$y_t = x'_t \Delta_t + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}'_{t,\tau} \delta_{0,\tau} + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}'_{1,t} \Delta_\tau + \beta^{T-t} \Delta \bar{q}'_t \gamma^K \quad (18b)$$

for  $t = 1, \dots, T - 2$ , where

$$\Delta \bar{q}'_t \equiv \Delta \mathbb{E}(q^K(s_T) | s_t), \quad (19)$$

whose randomness comes from both  $x_t$  and  $z_t$ . Like  $\Delta \bar{x}'_t$ ,  $\Delta \bar{x}'_{1,t}$  and  $y_t$ , it is straightforward to identify and estimate  $\Delta \bar{q}'_t$ .

Equation (18) is the linear system that serves as the basis of our subsequent identification analysis (Section 3.2) and estimation (Section 3.3). The parameters of interest are  $\delta \equiv (\Delta'_1, \delta'_{0,2}, \Delta'_2, \dots, \delta'_{0,T-1}, \Delta'_{T-1})'$  and  $\gamma^K$ .<sup>14</sup>

### 3.2 Identification of the Linear System

This subsection gives the sufficient and necessary condition for the identification of  $\delta$  and  $\gamma^K$  from the linear system eq. (18). We also recognize that  $\delta$  and  $\gamma^K$  are likely to be over-identified and give a sufficient condition for that. The available data are a panel of state-variable-and-choice pairs  $D_{i,t} \equiv (a_{i,t}, s'_{i,t})' = (a_{i,t}, x'_{i,t}, z'_{i,t})'$  for  $N$  agents and  $T$  periods ( $i \in \{1, \dots, N\}$  and  $t \in \mathcal{T}_{data} = \{1, \dots, T\}$ ).<sup>15</sup> Throughout this paper, we treat  $T$  as fixed and allow  $N$  to approach infinity. Let  $D_i \equiv (D'_{i,1}, \dots, D'_{i,T})'$ .

<sup>14</sup>If  $T < T_{end}$ ,  $\delta_{1,T}$  and  $\delta_{0,T}$  cannot be identified without making assumptions regarding how the state variables and the flow utility parameters will evolve beyond the data horizon, because there are no data to distinguish the flow utility in period  $T$  from the expected future payoffs after  $T$ . If  $T = T_{end}$ , then Remark 3 above shows that  $\delta_{1,T}$  and  $\delta_{0,T}$  are subvectors of  $\gamma^K$ .

<sup>15</sup>Some papers (e.g., Arcidiacono and Miller, 2019) only require cross-sectional data, provided that the state variables in two adjacent periods are both observed.

**Assumption 7** (Sampling scheme). *Assume:  $D_i$  is independent across  $i$ .*

The following features of eq. (18) are useful for the identification analysis: (i) the “dependent variables”  $y_t$  and the “independent variables”  $x_t$ ,  $\Delta\bar{x}_t^\tau$ ,  $\Delta\bar{x}_{1,t}^\tau$  and  $\Delta\bar{q}_t^K$  (the latter three are all forward-looking terms) can be regarded as known for the identification purpose, so the only unknowns are the “coefficients”  $\delta$  and  $\gamma^K$ ; (ii) eq. (18) does not contain any “error” terms, but the identification of  $\delta$  and  $\gamma^K$  still boils down to a Jacobian rank condition derived from eq. (18), just like the usual linear regression model; (iii) for a fixed  $t$ , eq. (18) only contains contemporary  $\Delta_t$  and future  $\Delta_\tau$  and  $\delta_{0,\tau}$  ( $\tau > t$ ), so it has a “block-triangular” structure that is useful in certain scenarios. We will first focus on two illuminating special cases and then extend to the general case.

We start with the special case where  $T = 2$ . In this case, eq. (18b) does not exist,  $\theta_2 \equiv (\Delta'_1, \gamma^{K'})'$  is the only parameter, and eq. (18a) having a unique solution  $\theta_2$  is equivalent to the system of  $d_x + K$  equations with  $d_x + K$  unknowns

$$\mathbb{E}(X_{2,1}y_1) = \mathbb{E}(X_{2,1}X'_{2,1})\theta_2$$

having a unique solution,<sup>16</sup> where the  $d_x + K$  vector  $X_{2,1} \equiv (x'_1, \beta\Delta\bar{q}_1^{K'})'$ . The sufficient and necessary condition of this is that the square Jacobian matrix  $L_2 \equiv \mathbb{E}(X_{2,1}X'_{2,1})$  – i.e., the second moment matrix of  $X_{2,1}$  – has full rank. Note that this condition is equivalent to the second moment matrix of  $(x'_1, \Delta\bar{q}_1^{K'})'$  having rank  $d_x + K$  since  $\beta$  is a constant. There is no over-identification opportunity when  $T = 2$ .

Next, we consider the case where  $T = 3$ , which clearly illustrates the sources of identification and over-identification of the linear system for  $T > 2$  cases. Define the  $3d_x + K$  vector  $X_{3,1} \equiv (x'_1, \beta\Delta\bar{x}_1^{2'}, \beta\Delta\bar{x}_{1,1}^{2'}, \beta^2\Delta\bar{q}_1^{K'})'$  and the  $d_x + K$  vector  $X_{3,2} \equiv (x'_2, \beta\Delta\bar{q}_2^{K'})'$ . Let  $\theta_{3,1} \equiv (\Delta'_1, \delta'_{0,2})'$ ,  $\theta_{3,2} \equiv (\Delta'_2, \gamma^{K'})'$  and  $\theta_3 \equiv (\theta'_{3,1}, \theta'_{3,2})'$ . In this case, eq. (18) having a unique solution  $\theta_3$  is equivalent to the following system of  $4d_x + 2K$  equations with  $3d_x + K$  unknowns,

$$\begin{bmatrix} \mathbb{E}(X_{3,2}y_2) \\ \mathbb{E}(X_{3,1}y_1) \end{bmatrix} = L_3\theta_3, \text{ with } L_3 \equiv \begin{bmatrix} 0_{(d_x+K) \times 2d_x} & \mathbb{E}(X_{3,2}X'_{3,2}) \\ & \mathbb{E}(X_{3,1}X'_{3,1}) \end{bmatrix}, \quad (20)$$

having a unique solution. The sufficient and necessary condition of this is that the Jacobian matrix  $L_3$  has full column rank (i.e.,  $3d_x + K$ ). Note that the first  $d_x + K$  rows of eq. (20) correspond to period  $t = 2$ , and  $\mathbb{E}(X_{3,2}X'_{3,2})$  is the corresponding square Jacobian matrix; the last  $3d_x + K$  rows correspond to period  $t = 1$ , and  $\mathbb{E}(X_{3,1}X'_{3,1})$  is the corresponding square Jacobian matrix. Because  $L_3$  has more rows ( $4d_x + 2K$ ) than columns ( $3d_x + K$ ), so there might be multiple ways for it to achieve full column rank. Moreover, the equations corresponding to  $t = 1$  contain all the parameters and will identify them all if  $\mathbb{E}(X_{3,1}X'_{3,1})$  has full rank (i.e.,  $3d_x + K$ ). Furthermore, the equations corresponding to  $t = 2$  contain  $\theta_{3,2}$  and will identify it if  $\mathbb{E}(X_{3,2}X'_{3,2})$  has full rank (i.e.,  $d_x + K$ );

<sup>16</sup>This equivalence can be easily shown by the usual identification argument of linear regression models.

and then by the block-triangular feature of  $L_3$ , the equations corresponding to  $t = 1$  will identify  $\theta_{3,1}$  if a weaker rank condition on  $\mathbb{E}(X_{3,1}X'_{3,1})$  holds. Finally, note that  $\theta_3$  contains  $\delta_{0,t}$  in addition to  $\Delta_t$ , which is another advantage of using  $T > 2$  periods of data.

To formally state the identification condition for general  $T$ , we first define

$$\underbrace{X_{T,T-1}}_{(d_x+K) \times 1} \equiv (x'_{T-1}, \beta \Delta \bar{q}_{T-1}^{K'})' \quad \text{and} \quad (21a)$$

$$\underbrace{X_{T,t}}_{((2T-2t-1)d_x+K) \times 1} \equiv (x'_t, \beta \Delta \bar{x}_t^{t+1'}, \beta \Delta \bar{x}_{1,t}^{t+1'}, \dots, \beta^{T-1-t} \Delta \bar{x}_t^{T-1'}, \beta^{T-1-t} \Delta \bar{x}_{1,t}^{T-1'}, \beta^{T-t} \Delta \bar{q}_t^{K'})' \quad (21b)$$

for  $T \geq 2$  and  $t = 1, 2, \dots, T-2$ , and let

$$\underbrace{\tilde{L}_{T,t}}_{((2T-2t-1)d_x+K) \times ((2T-2t-1)d_x+K)} \equiv \mathbb{E}(X_{T,t}X'_{T,t}) \quad \text{and} \quad (22a)$$

$$\underbrace{L_{T,t}}_{((2T-2t-1)d_x+K) \times ((2T-3)d_x+K)} \equiv \begin{bmatrix} 0_{((2T-2t-1)d_x+K) \times (2(t-1)d_x)} & \tilde{L}_{T,t} \end{bmatrix} \quad (22b)$$

for  $T \geq 2$  and  $t = 1, \dots, T-1$ . We stack  $L_{T,t}$  for  $t = T-1, \dots, 1$  (in that order) to get the  $((T-1)^2 d_x + (T-1)K) \times ((2T-3)d_x + K)$  block-triangular Jacobian matrix  $L_T$ :

$$L_T = \begin{bmatrix} 0_{(d_x+K) \times 2d_x} & 0_{(d_x+K) \times 2(T-5)d_x} & 0_{(d_x+K) \times 2d_x} & 0_{(d_x+K) \times 2d_x} & \mathbb{E}(X_{T,T-1}X'_{T,T-1}) \\ 0_{(3d_x+K) \times 2d_x} & 0_{(3d_x+K) \times 2(T-5)d_x} & 0_{(3d_x+K) \times 2d_x} & & \mathbb{E}(X_{T,T-2}X'_{T,T-2}) \\ 0_{(5d_x+K) \times 2d_x} & 0_{(5d_x+K) \times 2(T-5)d_x} & & & \mathbb{E}(X_{T,T-3}X'_{T,T-3}) \\ \vdots & & & \ddots & \\ 0_{((2T-5)d_x+K) \times 2d_x} & & & \mathbb{E}(X_{T,2}X'_{T,2}) & \\ & & & \mathbb{E}(X_{T,1}X'_{T,1}) & \end{bmatrix}. \quad (23)$$

Using this notation, eq. (18) having a unique solution  $\theta_T \equiv (\delta', \gamma^{K'})'$  is equivalent to the following system of  $(T-1)^2 d_x + (T-1)K$  equations with  $(2T-3)d_x + K$  unknowns

$$\begin{bmatrix} \mathbb{E}(X_{3,2}y_2) \\ \mathbb{E}(X_{3,1}y_1) \end{bmatrix} = L_3 \theta_3, \quad \text{with } L_3 \equiv \begin{bmatrix} 0_{(d_x+K) \times 2d_x} & \mathbb{E}(X_{3,2}X'_{3,2}) \\ & \mathbb{E}(X_{3,1}X'_{3,1}) \end{bmatrix}$$

having a unique solution.

**Theorem 3** (Identification and over-identification of  $\delta$  and  $\gamma^K$ ). *Under Assumptions 1 to 7, (i) the parameters  $\delta$  and  $\gamma^K$  are identified if and only if the  $L_T$  matrix defined in eq. (23) has full column rank (i.e.,  $(2T-3)d_x + K$ ); (ii) if more than one matrix that consists of  $(2T-3)d_x + K$  distinct rows from  $L_T$  has full rank, then the parameters  $\delta$  and  $\gamma^K$  are over-identified.*

Similar to the  $T = 3$  case, the key sources of over-identification is the multitude of equations for periods  $t = 1, \dots, T-1$ , the linearity of the flow utility (recall Assumption 5), and the presence of the forward-looking terms in eq. (18).

**Corollary 1.** *Under Assumptions 1 to 7,  $\delta$  and  $\gamma^K$  are identified if the second moment matrix of  $(x'_1, \Delta\bar{x}_1^{2'}, \Delta\bar{x}_{1,1}^{2'}, \dots, \Delta\bar{x}_1^{\tau'}, \Delta\bar{x}_{1,1}^{\tau'}, \dots, \Delta\bar{x}_1^{T-1'}, \Delta\bar{x}_{1,1}^{T-1'}, \Delta\bar{q}_1^{K'})'$  has full rank.*

**Corollary 2.** *Under Assumptions 1 to 7,  $\delta$  and  $\gamma^K$  are identified if: (i) the second moment matrix of  $(x'_{T-1}, \Delta\bar{q}_{T-1}^{K'})'$  has full rank (i.e.,  $d_x + K$ ); and (ii) the second moment matrix of  $(x'_t, \Delta\bar{x}_t^{t+1'})'$  has full rank (i.e.,  $2d_x$ ) for all  $t = 1, \dots, T - 2$ .*

Corollary 1 recognizes that the equations corresponding to  $t = 1$  contain all the parameters and will identify them all if  $\mathbb{E}(X_{T,1}X'_{T,1})$  has full rank (i.e.,  $(2T-3)d_x + K$ ). This observation, when  $T$  is large, can be utilized to reduce the sensitivity of our estimator to Assumption 6 (we will elaborate this in Section 4.2 below). Corollary 2 takes advantage of the block-triangular structure of  $L_T$  to sequentially identify the parameters: condition (i) ensures that  $\Delta_{T-1}$  and  $\gamma^K$  are identified, which can therefore be regarded as known for subsequent analysis; condition (ii) then sequentially ensures the identification of  $\Delta_t$  and  $\delta_{0,t+1}$  for  $t = T-2, \dots, 1$  (in that order). This feature, as will be detailed in Remark 5 below, is useful in dealing with time-invariant variables in  $x_t$ .

The rank conditions in Theorem 3 and Corollaries 1 to 2 are testable. For a general matrix  $A$ , a given integer  $r$ , and the hypotheses  $H_0 : \text{rank}(A) = r$  vs.  $H_1 : \text{rank}(A) > r$ , Camba-Méndez and Kapetanios (2009); Robin and Smith (2000) summarized and proposed statistical tests of the rank of  $A$  using its estimate  $\hat{A}$ , all assuming that the entries of  $\hat{A}$  are root- $n$  consistent and asymptotically normal. These tests can be modified to empirically test the rank conditions in Theorem 3 and Corollaries 1 to 2 since all the entries of  $L_T$  are either means of directly observed variables or values of conditional mean difference functions that can be shown to be root- $n$  consistent and asymptotically normal with asymptotic variance derived using Newey (1994). The in-depth inquiry into such tests, however, is outside the scope of this paper.

In the rest of this subsection, we analyze when the rank conditions hold or fail in various scenarios. We hope to highlight that the Markovian property of  $s_t$ , combined with Assumptions 4 to 6, greatly simplifies the identification analysis of DDC models.

**Remark 4** (Rank condition). *The rank conditions in Corollary 2 is especially easy to understand. Recall eq. (13c) that  $\Delta\bar{x}_t^{t+1} = \Delta\mathbb{E}(x_{t+1} | s_t)$ , so condition (ii) in Corollary 2 requires the variables in the vector  $(x'_t, \Delta\mathbb{E}(x_{t+1} | s_t))'$  not to be perfectly linearly correlated among each other.*

*As for condition (i) in Corollary 2, we have to distinguish between  $T = T_{end}$  and  $T < T_{end}$ . For the case  $T = T_{end}$ , it is without loss of generality that we discuss  $T = T_{end} = 2$  in detail. In this case, we have  $q^K(x_2, z_2) = (x'_2, p_2x'_2, \gamma - \eta_2)'$  and  $\gamma^K = (\delta'_{0,2}, \Delta'_2, 1)'$ . Note that  $p_2x'_2\Delta_2 + \gamma - \eta_2$  is known as discussed in Remark 3, so  $\Delta\mathbb{E}(p_2x'_2\Delta_2 + \gamma - \eta_2 | s_1)$  can be moved to the left-hand side, and eq. (18a) becomes*

$$y_1 - \beta\Delta\mathbb{E}(p_2x'_2\Delta_2 + \gamma - \eta_2 | s_1) = x'_1\Delta_1 + \beta\Delta\mathbb{E}(x_2 | s_1)'\delta_{0,2}. \quad (24)$$

So, condition (i) in Corollary 2 boils down to the variables in the vector  $(x'_1, \Delta\mathbb{E}(x_2|s_1))'$  not being perfectly linearly correlated among each other, which is essentially the same as condition (ii). Note that the randomness in  $\Delta\mathbb{E}(x_{t+1}|s_t)$  comes solely from  $s_t$  and that  $x_t$  is a subvector of  $s_t$ , so this condition requires that none of the variables in  $\Delta\mathbb{E}(x_{t+1}|s_t)$  is a purely linear function of  $x_t$  alone.

For the case  $T < T_{end}$ , it is hard to analyze condition (i) explicitly since  $\bar{V}_T(s_T)$  is unknown. In Section 5 we will demonstrate using Monte Carlo that condition (i) in Corollary 2 can easily be satisfied, with or without excluded variables  $z_t$ .

**Remark 5** (Rank condition fails for certain  $x_t$ ). The rank condition can be shown to fail if  $x_t$  contains variables whose future evolution is not affected by the current choice  $a_t$ . Time-invariant variables and exogenously evolving state variables are two empirically important examples. Without loss of generality, we assume that only the first  $d_{x^\circ}$  coordinates of  $x_t$ , denoted by  $x_t^\circ$ , are either time-invariant or exogenously evolving, and use  $x_t^*$  to denote the last  $d_{x^*} = d_x - d_{x^\circ}$  coordinates. So, we can partition  $x_t = (x_t^\circ, x_t^*)'$ ,  $\Delta\bar{x}_t^\tau = (\Delta\bar{x}_t^{\circ\tau}, \Delta\bar{x}_t^{*\tau})'$ ,  $\delta_{0,t} = (\delta_{0,t}^\circ, \delta_{0,t}^*)'$ , and so on. For conciseness, we also assume that none of the functions in  $\Delta\mathbb{E}(q^K(s_t)|s_{T-1})$  is a function of  $x_{T-1,1}$  alone so that  $\Delta_{T-1}$  and  $\gamma^K$  are identified.

The key problem is that conditional mean functions  $\mathbb{E}(x_\tau^\circ | x_t, z_t, a_t)$  (for  $\tau > t$ ) do not vary with  $a_t$ . If  $x_t^\circ$  remains time-invariant for each agent for the entire sample duration (e.g., intercept, birthplace, ethnicity), then  $\Delta\bar{x}_t^{\circ\tau} \equiv \Delta\mathbb{E}(x_\tau^\circ | x_t, z_t) = 0$  because  $x_\tau^\circ = x_1^\circ$  for both  $a_t = 1$  and  $a_t = 0$  and all  $\tau, t$  and  $x_t, z_t$  values. If the evolution of  $x_t^\circ$  is exogenous and not affected by the choice (e.g., age, market price), then  $\Delta\bar{x}_t^{\circ\tau} \equiv \Delta\mathbb{E}(x_\tau^\circ | x_t, z_t) = 0$  as well because  $\mathbb{E}(x_\tau^\circ | x_t, z_t, a_t = 1) = \mathbb{E}(x_\tau^\circ | x_t, z_t, a_t = 0)$ .

The consequence is best understood via the sequential approach suggested by Corollary 2. Consider eq. (18b) for  $t = T - 2$  first, which becomes

$$y_{T-2} - \beta\Delta\bar{x}_{1,T-2}^{T-1'}\Delta_{T-1} - \beta^2\Delta\bar{q}_{T-2}^{K'}\gamma^K = x'_{T-2}\Delta_{T-2} + \beta\Delta\bar{x}_{T-2}^{\circ T-1'}\delta_{0,T-1}^\circ + \beta\Delta\bar{x}_{T-2}^{* T-1'}\delta_{0,T-1}^*, \quad (25)$$

where the left-hand side is known under condition (i) in Corollary 2. Since  $\Delta\bar{x}_{T-2}^{\circ T-1} = 0$ ,  $\delta_{0,T-1}^\circ$  cannot be identified from eq. (25), which although holds regardlessly. All the coordinates of  $\Delta_{T-2}$  and  $\delta_{0,T-1}^*$ , however, can be identified using eq. (25), provided that all the coordinates of  $x_{T-2}$  (including  $x_{T-2}^\circ$ ) and  $\Delta\bar{x}_{T-2}^{* T-1}$  have cross-sectional variation.

Then proceed to eq. (18b) for  $t = T - 3$ , which becomes

$$\begin{aligned} & y_{T-3} - \beta\Delta\bar{x}_{1,T-3}^{T-2'}\Delta_{T-2} - \beta^2\Delta\bar{x}_{1,T-3}^{T-1'}\Delta_{T-1} - \beta^2\Delta\bar{x}_{T-3}^{\circ T-1'}\delta_{0,T-1}^\circ - \beta^3\Delta\bar{q}_{T-3}^{K'}\gamma^K \\ & = x'_{T-3}\Delta_{T-3} + \beta\Delta\bar{x}_{T-3}^{\circ T-2'}\delta_{0,T-2}^\circ + \beta\Delta\bar{x}_{T-3}^{* T-2'}\delta_{0,T-2}^*, \end{aligned} \quad (26)$$

where the left-hand side is known. Note that even though  $\delta_{0,T-1}^\circ$  is not identified, it does not matter since  $\Delta\bar{x}_{T-3}^{\circ T-1} = 0$ . This means that we are free to normalize  $\delta_{0,T-1}^\circ$  to any real value, and it will not affect the identification and estimation of the other parameters.<sup>17</sup> By the same argument as above,

<sup>17</sup>For the counterfactual scenario where  $x_t^\circ$  remains time-invariant (even if it differs from the value in the data) or

$\delta_{0,T-2}^\circ$  is not identified, but all the coordinates of  $\Delta_{T-3}$  and  $\delta_{0,T-2}^*$  can be identified using eq. (26), provided that all the coordinates of  $x_{T-3}$  and  $\Delta \bar{x}_{T-3}^{*T-2}$  have cross-sectional variation.

This argument proceeds recursively for all earlier periods, and the takeaways can be summarized as follows. First,  $\Delta_t$  for  $t = 1, \dots, T-1$  are identified even in the presence of time-invariant or exogenously evolving state variables, and this is because only contemporaneous cross-sectional variation is required to identify the difference in flow utility between the two choices (recall static binary choice models). Second,  $\delta_{0,t}^\circ$  (for  $t = 2, \dots, T-1$ ) cannot be identified if  $x_t^\circ$  is time-invariant or evolves exogenously, but their values do not affect the identification and estimation of the other parameters, and therefore they can be normalized to arbitrary values. Third,  $\delta_{0,t}^*$  (for  $t = 2, \dots, T-1$ ), the parameters that correspond to time-varying or endogenously evolving variables  $x_t^*$  can be identified. The minor modification needed to accommodate time-invariant or exogenously evolving state variables in identification and estimation is elaborated in Appendix C.

It is worth pointing out that time-varying variables do not have to change value for every agent in every period. Provided that  $x_{t-1}$  has cross-sectional variation, the identification of  $\delta_{0,t}^*$  ( $t = 2, \dots, T-1$ ) only requires  $x_t^*$  to change value differently between the two choices for some agent in every period so that  $\Delta \mathbb{E}(x_t^* | s_{t-1})$  are non-zeros at the population level. This allows  $x_t^*$  to be both intermittently time-varying variables (e.g., marital status, highest degree) and continually time-varying variables (e.g., disposable income).

The non-identification of  $\delta_{0,t}^\circ$  is intuitive – the future evolution of  $x_\tau^\circ$  is not affected by the choice, so their impact on the utility cannot be identified from choice data. Such non-identification is an intrinsic result given the model setup and primitives, not due to our approach. In fact, our approach makes it easier to see when the identification of  $\delta_{0,t}$  would fail.

**Remark 6** (Excluded variables). When  $z_t$  is empty,  $\Delta \mathbb{E}(x_{t+1} | s_t) = \Delta \mathbb{E}(x_{t+1} | x_t)$  is a function of  $x_t$  only. The requirement of Corollary 2 (discussed in Remark 4) that the variables in the vector  $(x_t', \Delta \mathbb{E}(x_{t+1} | s_t))'$  are not perfectly linearly correlated among each other can still hold if  $\Delta \mathbb{E}(x_{t+1} | x_t)$  exhibits enough non-linearity. When  $z_t$  is non-empty, this requirement is easier to fulfill. In this sense, excluded variables  $z_t$  only play an auxiliary role.<sup>18</sup>

When  $z_t$  is needed for identification, even one excluded variable might be able to restore the rank condition for multiple  $x_t$  if it introduces enough non-linearity to  $\Delta \mathbb{E}(x_{t+1} | x_t, z_t)$ . Consider a situation where  $\Delta \mathbb{E}(x_{t+1} | x_t, z_t)$  contains an additive component that is linear in  $x_t$ , denoted as  $\rho_1 x_t$ , with  $\rho_1$  being a  $d_x \times d_x$  matrix of constants.<sup>19</sup> Suppose  $z_t$  is non-empty and there exists 

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 evolves exogenously (even if the evolution rule differs from that in the data), this normalization does not affect the counterfactual analysis either.

<sup>18</sup>Note that if some  $z_t$  affects the evolution of  $s_\tau$  ( $\tau > t$ ) differently between  $a_t = 1$  and  $a_t = 0$ , then  $z_t$  should be included as part of  $s_t$  even if the vector  $(x_t', \Delta \mathbb{E}(x_{t+1} | x_t))'$  satisfies the rank requirement, because omitting  $z_t$  in this scenario will result in biased CCP (i.e., omitted variable bias).

<sup>19</sup>If  $z_t$  is empty, then  $\Delta \mathbb{E}(x_{t+1} | x_t, z_t) = \rho_1 x_t$ , and the rank condition in Corollary 2 fails.

a  $d_x \times 1$ -vector-valued function of  $d_z$  arguments and a  $d_x \times d_x$  matrix of constants, denoted as  $\ell(\cdot) \equiv (\ell_1(\cdot), \dots, \ell_{d_x}(\cdot))'$  and  $\rho_2$ ,<sup>20</sup> respectively, such that

$$\underbrace{\begin{bmatrix} x_t \\ \Delta \mathbb{E}(x_{t+1} | x_t, z_t) \end{bmatrix}}_{2d_x \times 1} = \underbrace{\begin{bmatrix} I_{d_x} & 0_{d_x \times d_x} \\ \rho_1 & \rho_2 \end{bmatrix}}_{2d_x \times 2d_x} \underbrace{\begin{bmatrix} x_t \\ \ell(z_t) \end{bmatrix}}_{2d_x \times 1}, \quad (27)$$

then the rank condition in Corollary 2 holds if: (a) the second moment matrix of  $(x'_t, \ell(z_t))'$  is invertible; and (b)  $\rho_2$  has full rank (i.e.,  $d_x$ ).<sup>21</sup>

To illustrate these two points, we conduct simulation experiments in Section 5 below using two different DGPs that satisfy the requirement of Corollary 2. DGP 1 has one excluded variable and two time-varying state variables (i.e.,  $d_z = 1$ ,  $d_{x^*} = 2$  and  $d_{x^\circ} = 1$ ), while DGP 2 has two time-varying state variables and no excluded variable (i.e.,  $d_z = 0$ ,  $d_{x^*} = 2$  and  $d_{x^\circ} = 1$ ), but  $\Delta \mathbb{E}(x_{t+1} | x_t)$  is non-linear in  $x_t$ .

**Remark 7** (Stationary models). *If the model is stationary, then  $\Delta_t = \bar{\Delta}$  and  $\delta_{0,t} = \bar{\delta}_0$  for all  $t$ . Due to such restrictions on the parameters, under Assumptions 1 to 6, eq. (18) becomes*

$$y_{T-1} = x'_{T-1} \bar{\Delta} + \beta \Delta \bar{q}_{T-1}^{K'} \gamma^K, \text{ and} \quad (28a)$$

$$y_t = \left( x'_t + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_{1,t}^{\tau'} \right) \bar{\Delta} + \left( \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_t^{\tau'} \right) \bar{\delta}_0 + \beta^{T-t} \Delta \bar{q}_t^{K'} \gamma^K \quad (28b)$$

for  $t = 1, \dots, T-2$ . The (over-)identification of  $\bar{\Delta}$ ,  $\bar{\delta}_0$  and  $\gamma^K$  can therefore be clearly analyzed using eq. (28).

If  $T \geq 3$ , then stationary models may not even need the normalization  $\delta_{0,1} = c$  made in Assumption 5.  $\bar{\Delta}$  and  $\gamma^K$  are identified from eq. (28a) if the variables in  $x_{T-1}$  and  $\Delta \bar{q}_{T-1}^K$  are not perfectly linearly correlated, and then  $\bar{\delta}_0$  is identified from eq. (28b) if the variables in  $\sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_t^{\tau'}$  are not perfectly linearly correlated. The intuition is that the inter-temporal variation in dynamic models provides extra identification source than static models, even if the dynamic models are stationary.

**Remark 8** (Identification with unknown discount factor). *Unknown  $\beta$  can be analyzed under the same framework, and the identification condition is only slightly stronger. We focus on the  $T = 3$  case and the strengthening of Corollary 2 as an illustration. Under condition (i) in Corollary 2 (i.e., the second moment matrix of  $(x'_2, \Delta \bar{q}_2^{K'})'$  has full rank),  $(\Delta'_2, \beta \gamma^{K'})'$  are identified through*

<sup>20</sup>  $\rho_1$ ,  $\ell(\cdot)$  and  $\rho_2$  are allowed to depend on  $t$ , due to the nonstationary feature of the model, but it is unnecessary to make this dependence explicit here.

<sup>21</sup> It is notable that (a) and (b) do not necessarily restrict  $d_z$ , the number of excluded variables. If  $z_t$  enters  $\Delta \mathbb{E}(x_{t+1} | x_t, z_t)$  linearly (i.e.,  $\ell(\cdot)$  are all linear functions in all arguments), then they require that  $d_z \geq d_x$ . The other extreme is that the variation of  $z_t$  and the non-linearity in  $\ell(\cdot)$  combined are such that the variables in  $(x'_t, \ell(z_t))'$  are mutually linearly independent, then one excluded variable suffices. Many cases in between these two extremes are also possible.

eq. (18a) and can then be regarded as known. The unique solution of eq. (18b), at the same time, is equivalent to the unique solution of

$$\begin{bmatrix} \mathbb{E}(x_1 y_1) \\ \mathbb{E}(\Delta \bar{x}_1^2 y_1) \\ \mathbb{E}(\Delta \bar{x}_{1,1}^2 y_1) \\ \mathbb{E}(\Delta \bar{q}_1^K y_1) \end{bmatrix} = \begin{bmatrix} \mathbb{E}(x_1 x_1') & \mathbb{E}(x_1 \Delta \bar{x}_1^{2'}) & \mathbb{E}(x_1 \Delta \bar{x}_{1,1}^{2'}) & \mathbb{E}(x_1 \Delta \bar{q}_1^{K'}) \\ \mathbb{E}(\Delta \bar{x}_1^2 x_1') & \mathbb{E}(\Delta \bar{x}_1^2 \Delta \bar{x}_1^{2'}) & \mathbb{E}(\Delta \bar{x}_1^2 \Delta \bar{x}_{1,1}^{2'}) & \mathbb{E}(\Delta \bar{x}_1^2 \Delta \bar{q}_1^{K'}) \\ \mathbb{E}(\Delta \bar{x}_{1,1}^2 x_1') & \mathbb{E}(\Delta \bar{x}_{1,1}^2 \Delta \bar{x}_1^{2'}) & \mathbb{E}(\Delta \bar{x}_{1,1}^2 \Delta \bar{x}_{1,1}^{2'}) & \mathbb{E}(\Delta \bar{x}_{1,1}^2 \Delta \bar{q}_1^{K'}) \\ \mathbb{E}(\Delta \bar{q}_1^K x_1') & \mathbb{E}(\Delta \bar{q}_1^K \Delta \bar{x}_1^{2'}) & \mathbb{E}(\Delta \bar{q}_1^K \Delta \bar{x}_{1,1}^{2'}) & \mathbb{E}(\Delta \bar{q}_1^K \Delta \bar{q}_1^{K'}) \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \beta \delta_{0,2} \\ \beta \Delta_2 \\ \beta(\beta \gamma^K) \end{bmatrix}, \quad (29)$$

where the last  $d_x + K$  subvector of the unknown vector is proportional to the known parameters  $(\Delta_2', \beta \gamma^{K'})'$  by a factor of  $\beta$ , and the only unknown parameter now is the first  $2d_x$  subvector  $(\Delta_1', \beta \delta_{0,2}')'$ , which can be identified together with  $\beta$  if the second moment matrix of  $(x_1', \Delta \bar{x}_1^{2'}, \Delta \bar{x}_{1,1}^{2'}, \Delta \bar{q}_1^{K'})'$  has full rank. This is only slightly stronger than condition (ii) in Corollary 2 (i.e., the second moment matrix of  $(x_1', \Delta \bar{x}_1^{2'})'$  has full rank).

### 3.3 Estimation

Recall that the available data are a panel of state-variable-and-choice pairs  $D_{i,t} \equiv (a_{i,t}, s'_{i,t})' = (a_{i,t}, x'_{i,t}, z'_{i,t})'$  for  $N$  agents and  $T$  periods ( $i \in \{1, \dots, N\}$  and  $t \in \mathcal{T}_{data} = \{1, \dots, T\}$ ), and  $D_i \equiv (D'_{i,1}, \dots, D'_{i,T})'$ .

We use  $p$  to collectively denote all CCPs, use  $\Delta \bar{\eta}$  to collectively denote all  $\Delta \bar{\eta}_t^\tau$  ( $t = 1, \dots, T-2$ ,  $t < \tau \leq T-1$ ), use  $\Delta \bar{x}$  to collectively denote all  $\Delta \bar{x}_t^\tau$  and  $\Delta \bar{x}_t^\tau$  ( $t = 1, \dots, T-2$ ,  $t < \tau \leq T-1$ ), and use  $\Delta \bar{q}^K$  to collectively denote all  $\Delta \bar{q}_t^K$  ( $t = 1, \dots, T-1$ ). In light of the identification analysis in Section 3.2, the parameters  $\delta$  and  $\gamma^K$  can be estimated via minimum distance (MD) using the following moment functions:

$$m_{T-1}(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K) \equiv -v_{T-1}(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K) X_{T,T-1}, \quad \text{and} \quad (30a)$$

$$m_t(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K) \equiv -v_t(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K) X_{T,t}, \quad (30b)$$

where  $v_{T-1}(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K) \equiv y_{T-1} - x'_{T-1} \Delta_{T-1} - \beta \Delta \bar{q}_{T-1}^{K'} \gamma^K$ ,  $v_t(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K) \equiv y_t - x'_t \Delta_t - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_t^{\tau'} \delta_{0,\tau} - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_{1,t}^{\tau'} \Delta_\tau - \beta^{T-t} \Delta \bar{q}_t^{K'} \gamma^K$  for  $t = 1, \dots, T-2$ , and  $X_{T,t}$  for  $t = 1, \dots, T-1$  are defined in eq. (21). Let  $m(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K)$  be the stack of the moment functions  $m_t(D, \delta, \gamma^K, p, \Delta \bar{\eta}, \Delta \bar{x}, \Delta \bar{q}^K)$  for  $t = T-1, \dots, 1$  (in that order), which is a  $((T-1)^2 d_x + (T-1)K) \times 1$  vector-valued moment function that equals zeros under the true values of  $\delta$  and  $\gamma^K$ . In this moment function, the nuisance parameters  $p$ ,  $\Delta \bar{\eta}$ ,  $\Delta \bar{x}$  and  $\Delta \bar{q}^K$ , which are all conditional mean difference functions, need to be estimated in preliminary steps. Recall that  $\Delta \bar{\eta}$ ,  $\Delta \bar{x}$  and  $\Delta \bar{q}^K$  all take the common form of  $\Delta \mathbb{E}(h_\tau | x_t, z_t)$  ( $t = 1, \dots, T-2$ ,  $t < \tau \leq T-1$ ) in eq. (7), where the generic notation  $h_\tau$  denotes a random variable that is either directly observable or identified after knowing the CCP functions  $p_\tau$ . The specific expressions of  $h_\tau$  for  $\Delta \bar{\eta}_t^\tau$ ,  $\Delta \bar{x}_t^\tau$  and  $\Delta \bar{q}_t^K$  are known and are given in eq. (14), eqs. (13a) to (13c), and eq. (19), respectively.

To summarize, the estimation proceeds in the following three steps.

- (I) For each  $t = 1, \dots, T$ , use the data  $\{D_{i,t}\}_{i=1}^N$  to estimate the CCP function  $p_t$  in eq. (1) and obtain the estimated CCP value  $\hat{p}_{i,t}$  for each agent  $i \in \{1, \dots, N\}$ .<sup>22</sup>
- (II) Obtain  $\hat{h}_{i,\tau}$  by substituting unknown  $p_{i,\tau}$  with  $\hat{p}_{i,\tau}$  whenever applicable. Then, for each  $t = 1, \dots, T-2$  and  $t < \tau \leq T-1$ , use the data  $\{\hat{h}_{i,\tau}, D_{i,t}\}_{i=1}^N$  to estimate the corresponding conditional mean difference functions (i.e., the forward-looking terms) in  $\widehat{\Delta\eta}$ ,  $\widehat{\Delta\bar{x}}$  and  $\widehat{\Delta\bar{q}}^K$  and obtain their estimated values for each agent  $i \in \{1, \dots, N\}$ .
- (III) Let  $\bar{m}_N(\delta, \gamma^K) \equiv \frac{1}{N} \sum_{i=1}^N \hat{m}(D_i, \delta, \gamma^K)$ , where  $\hat{m}(D_i, \delta, \gamma^K) \equiv m(D_i, \delta, \gamma^K, \hat{p}_i, \widehat{\Delta\eta}_i, \widehat{\Delta\bar{x}}_i, \widehat{\Delta\bar{q}}_i^K)$ . Then, the MD estimator  $(\hat{\delta}', \hat{\gamma}^{K'})'$  is the unique closed-form solution to:

$$(\hat{\delta}', \hat{\gamma}^{K'})' \equiv \arg \min_{\delta \in \mathbb{R}^{(2T-3)d_x}, \gamma^K \in \mathbb{R}^K} \bar{m}_N(\delta, \gamma^K)' W_N \bar{m}_N(\delta, \gamma^K), \quad (31)$$

where  $W_N$  is a  $((T-1)^2 d_x + (T-1)K) \times ((T-1)^2 d_x + (T-1)K)$  symmetric weighting matrix that converges in probability to a positive definite matrix  $W$  as  $N \rightarrow \infty$ .<sup>23</sup>

**Remark 9** (Advantages of our estimator). *Our three-step CCP-based semiparametric estimator avoids the estimation of the state transition distributions  $f_{s_t|s_{t-1}, a_{t-1}}$ . Estimating the conditional means (given  $d_s$ -dimensional  $s_t$ ) is a problem with smaller dimension and better finite-sample properties than the state transition distributions (which involve joint densities of  $2d_s$ -dimensional vectors), which can be difficult to estimate or simulate nonparametrically, especially when  $s_t$  has a large dimension or contains both continuous and discrete variables (e.g., spouse income and number of young children). The fact that any finite-order Markov process of finite number of variables can be rewritten as a first-order Markov process by expanding the state vector further exacerbates the problem. Even if a parametric specification is used, specifying a conditional mean model is easier than specifying the entire conditional distribution. Various dimension-reduction methods (e.g., lasso) are better developed, better understood and more readily available for conditional means. In addition, our estimator does not involve any simulation or iteration like many other estimators do, which is typically the most time-consuming task, and therefore is much less demanding computationally. (For example, in the simulation experiments in Section 5 below, the HMSS estimator is at least one hundred times more time-consuming than our estimator.)*

**Remark 10** (Estimation step (II)). *The conditional mean difference functions  $\widehat{\Delta\eta}$ ,  $\widehat{\Delta\bar{x}}$  and  $\widehat{\Delta\bar{q}}^K$ , taking the common form of  $\Delta\mathbb{E}(h_\tau | x_t, z_t) = \mathbb{E}(h_\tau | x_t, z_t, a_t = 1) - \mathbb{E}(h_\tau | x_t, z_t, a_t = 0)$ , resemble*

<sup>22</sup>The CCP functions could be estimated parametrically or nonparametrically, and for generality we assume this is done nonparametrically. The same goes for other unknown functions below unless indicated otherwise. Section F.1 describes series estimator, a particular nonparametric estimator that could be used in steps (I) and (II). The choice of the tuning parameters is discussed in Section F.2.

<sup>23</sup>We skip the expression of this closed-form solution for conciseness.

the average treatment effect in the program evaluation and the missing data literature.<sup>24</sup> Many estimators from that literature can therefore be used, and here we briefly discuss two of them.

A “inverse probability weighted” (IPW) estimator is based on a useful identity  $\Delta \mathbb{E}(h_\tau | x_t, z_t) = \mathbb{E}(\phi(h_\tau, a_t, p_t) | x_t, z_t)$ , where

$$\phi(h_\tau, a_t, p_t) \equiv \frac{a_t h_\tau}{p_t} - \frac{(1 - a_t) h_\tau}{1 - p_t}, \quad (32)$$

and the function  $\Phi(x_t, z_t) \equiv \mathbb{E}(\phi(h_\tau, a_t, p_t) | x_t, z_t)$  can be estimated by regressing  $\phi(\hat{h}_{i,\tau}, a_{i,t}, \hat{p}_{i,t})$  on  $x_{i,t}$  and  $z_{i,t}$ .<sup>25</sup> Then, the estimated value of the corresponding function for agent  $i$  equals to  $\hat{\Phi}(x_{i,t}, z_{i,t})$ .

A “conditional mean projection” (CEP) estimator (or “imputation” estimator) estimates the function  $\Phi_1(x_t, z_t) \equiv \mathbb{E}(h_\tau | x_t, z_t, a_t = 1)$  using the subsample  $\{\hat{h}_{i,\tau}, x_{i,t}, z_{i,t}\}_{a_{i,t}=1}$  and estimates the function  $\Phi_0(x_t, z_t) \equiv \mathbb{E}(h_\tau | x_t, z_t, a_t = 0)$  using the subsample  $\{\hat{h}_{i,\tau}, x_{i,t}, z_{i,t}\}_{a_{i,t}=0}$  for  $t = 1, \dots, T - 2$ ,  $t < \tau \leq T - 1$ . Then, the estimated value of the conditional mean difference for agent  $i$  equals to  $\hat{\Phi}_1(x_{i,t}, z_{i,t}) - \hat{\Phi}_0(x_{i,t}, z_{i,t})$ .

Both the IPW and the CEP estimators are shown to be  $\sqrt{N}$ -consistent and asymptotically normal (Hirano, Imbens and Ridder, 2003; Chen, Hong and Tamer, 2005), but the CEP estimator requires weaker regularity conditions. In particular, it does not require the CCPs to be uniformly bounded away from zero and one. Suppose the IPW estimator is used in step (II) and fulfills the regularity conditions in Section 5 of Newey (1994), then Proposition 1 below gives the asymptotic distribution of  $\hat{\delta}$  by characterizing its influence function. Its derivation heavily uses Newey (1994)’s pathwise-derivative-based characterization of the influence function of semiparametric estimators, which is relegated to Appendix B. Meanwhile, Chen, Hong and Tarozzi (2008) show that the IPW and the CEP estimators are asymptotically equivalent (eqs. (7) and (12), as well as the discussion on p. 819), so if the CEP estimator is used in step (II), the asymptotic distribution of  $\hat{\delta}$  will remain

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<sup>24</sup>It is worth pointing out that the goal of step (II) is simply to estimate the conditional means of realized (directly observed or previously identified)  $h_\tau$  given  $x_t, z_t$  and  $a_t$ , not the (unconditional or conditional) means of “potential outcomes” like in the program evaluation literature. As a result, the “ignorability” condition, which serves the purpose of equating the conditional means of realized variables to structural parameters in the potential outcome model, is *not* required to hold. On the other hand, one can conceptualize the realized  $h_\tau$  in a future period  $\tau$  ( $\tau > t$ ) as  $h_\tau = a_t h_\tau^{(1)} + (1 - a_t) h_\tau^{(0)}$ , where  $h_\tau^{(1)}$  and  $h_\tau^{(0)}$  are “potential outcomes” for scenarios  $a_t = 1$  and  $a_t = 0$ , respectively. Then the DDC model studied in this paper (particularly, Assumptions 1 to 3) permits “ignorability” in the sense that Assumptions 1 to 3 do not necessarily imply or exclude  $(h_\tau^{(1)}, h_\tau^{(0)}) \perp a_t | x_t, z_t$ , and vice versa.

<sup>25</sup>Equation (32) holds due to the law of total probability. To see this, note that

$$\begin{aligned} \mathbb{E}(a_t h_\tau | x_t, z_t) &= \mathbb{E}(a_t h_\tau | x_t, z_t, a_t = 1) \Pr(a_t = 1 | x_t, z_t) + \mathbb{E}(a_t h_\tau | x_t, z_t, a_t = 0) \Pr(a_t = 0 | x_t, z_t) \\ &= \mathbb{E}(h_\tau | x_t, z_t, a_t = 1) p_t(x_t, z_t). \end{aligned}$$

Because  $p_t$  is a function of  $x_t, z_t$  only, it can be moved to the left-hand side of the equation and into the conditional expectation, implying  $\mathbb{E}(a_t h_\tau / p_t | x_t, z_t) = \mathbb{E}(h_\tau | x_t, z_t, a_t = 1)$ . Similarly, it can be shown that  $\mathbb{E}((1 - a_t) h_\tau / (1 - p_t) | x_t, z_t) = \mathbb{E}(h_\tau | x_t, z_t, a_t = 0)$ . Again, note that the “ignorability” condition is *not* required for eq. (32) to hold.

the same as in Proposition 1 below.<sup>26</sup>

We should point out that the number of conditional mean difference functions (given  $d_s$ -dimensional  $s_t$ ) estimated in step (II) is at the order of  $T^2$ , although the total impact of their estimation errors on the standard error of  $\hat{\delta}$  is additive in them (by Proposition 1 below). On the other hand, the number of  $2d_s$ -dimensional state transition distributions is  $2(T-1)$ . In consequence, the advantages of our estimator may be limited for large  $T$  and/or small  $d_s$  samples.

To state the proposition, we let  $m_{t,0}(D) \equiv m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K)$  for  $t = 1, \dots, T-1$  and stack them to get  $m_0(D) \equiv (m_{T-1,0}(D), \dots, m_{1,0}(D))'$ . Here and in the rest of this paper, we use the subscript “ $_0$ ”, with a comma in the front and after all the other subscripts (if applicable), to emphasize that all the parameters (potentially infinite-dimensional) that the object involves take the true values. The influence function of  $(\hat{\delta}', \hat{\gamma}^{K'})'$  takes the form

$$\psi(D) \equiv -(L_T' W L_T)^{-1} L_T' W (m_0(D) + \alpha(D)), \quad (33)$$

where

$$\alpha(D) \equiv \alpha_\eta(D) + \alpha_x(D) + \alpha_q(D) + \alpha_{p,direct}(D) + \alpha_{p,indirect}(D),$$

$L_T$  is defined in eq. (23), and  $\alpha_\eta(D)$ ,  $\alpha_x(D)$ ,  $\alpha_q(D)$ ,  $\alpha_{p,direct}(D)$  and  $\alpha_{p,indirect}(D)$  are defined in eq. (B.2), eq. (B.3), eq. (B.4), eq. (B.5) and eq. (B.6) in Appendix B, respectively.

**Proposition 1** (Asymptotic distribution of  $\hat{\delta}$ ). *Under Assumptions 1 to 7 and the regularity conditions in Section 5 of Newey (1994), the three-step CCP-based semiparametric estimator  $\hat{\delta}$  defined in eq. (31) has the asymptotic distribution*

$$\sqrt{N}(\hat{\delta} - \delta) \xrightarrow{d} \mathcal{N}(0, V) \text{ as } N \rightarrow \infty,$$

where  $V \equiv \mathbb{E}(\psi_\delta(D_i)\psi_\delta'(D_i))$ , and  $\psi_\delta(\cdot)$  is the first  $(2T-3)d_x$  coordinates of the influence function  $\psi(D)$  in eq. (33).

Depending on what nonparametric estimators are used in steps (I) and (II), the primitive conditions for Proposition 1 vary. In Section F, we provide the primitive conditions, the choice of the tuning parameters and the proof of Proposition 1 for power series estimators.

We follow Lemma 5.4 of Newey (1994) to give  $\widehat{V}$ , a consistent estimator of the asymptotic variance  $V$ . Define a consistent estimator of  $\tilde{L}_{T,T-1}$  in eq. (22a) as

$$\widehat{\tilde{L}}_{T,T-1} \equiv \begin{bmatrix} \frac{1}{N} \sum_{i=1}^N x_{i,T-1} x'_{i,T-1} & \frac{\beta}{N} \sum_{i=1}^N x_{i,T-1} \widehat{\Delta q}_{i,T-1}^{K'} \\ \frac{\beta}{N} \sum_{i=1}^N \widehat{\Delta q}_{i,T-1}^K x'_{i,T-1} & \frac{\beta^2}{N} \sum_{i=1}^N \widehat{\Delta q}_{i,T-1}^K \widehat{\Delta q}_{i,T-1}^{K'} \end{bmatrix},$$

where  $\widehat{\Delta q}_{i,T-1}^K \equiv \widehat{\Delta} \mathbb{E}(q^K(x_T, z_T) \mid x_{i,T-1}, z_{i,T-1})$ . Define  $\widehat{\tilde{L}}_{T,t}$  similarly for  $t = 1, \dots, T-2$  so that we obtain  $\widehat{L}_T$ . In addition, let  $\hat{\alpha}_\eta(D_i)$ ,  $\hat{\alpha}_x(D_i)$ ,  $\hat{\alpha}_q(D_i)$ ,  $\hat{\alpha}_{p,direct}(D_i)$  and  $\hat{\alpha}_{p,indirect}(D_i)$  be estimators

<sup>26</sup>We will use the CEP estimator in step (II) in our simulation experiments (Section 5).

of  $\alpha_\eta(D_i)$ ,  $\alpha_x(D_i)$ ,  $\alpha_q(D_i)$ ,  $\alpha_{p,direct}(D_i)$  and  $\alpha_{p,indirect}(D_i)$ , respectively, with unknown parameters and functions in the latter be replaced by the corresponding estimates obtained via our estimator. Define

$$\begin{aligned}\hat{\alpha}(D_i) &\equiv \hat{\alpha}_\eta(D_i) + \hat{\alpha}_x(D_i) + \hat{\alpha}_q(D_i) + \hat{\alpha}_{p,direct}(D_i) + \hat{\alpha}_{p,indirect}(D_i), \\ \hat{\psi}(D_i) &\equiv m(D_i, \hat{\delta}, \hat{\gamma}^K, \hat{p}, \widehat{\Delta\eta}, \widehat{\Delta x}, \widehat{\Delta q}) + \hat{\alpha}(D_i), \\ \widehat{\Sigma} &\equiv \frac{1}{N} \sum_{i=1}^N \hat{\psi}(D_i) \hat{\psi}(D_i)', \text{ and} \\ \widehat{V} &\equiv (\widehat{L}'_T W_N \widehat{L}_T)^{-1} \widehat{L}'_T W_N \widehat{\Sigma} W_N \widehat{L}_T (\widehat{L}'_T W_N \widehat{L}_T)^{-1}.\end{aligned}\tag{34}$$

**Proposition 2** (Consistent estimator of asymptotic variance). *Under Assumptions 1 to 7 and the regularity conditions in Section 5 of Newey (1994),  $\widehat{V} \xrightarrow{P} V$  as  $N \rightarrow \infty$ .*

The results in this subsection premise that all the coordinates of  $\delta$  and  $\gamma^K$  are identified. To accommodate time-invariant and/or exogenously evolving variables in  $x_t$  (recall Remark 5), we need to slightly revise the notation to exclude the unidentified (and unnecessary to estimate) coordinates of  $\delta_{0,t}$ , and all the results in this subsection essentially remain unchanged for the other coordinates. We elaborate this in Appendix C.2.

We conclude this section by remarking that our state-transition-free estimator of the flow utility parameters can be useful for counterfactual analysis, too. Certain counterfactual behaviors are identified without restricting state transition (i.e., they are identified for all state transitions), and certain counterfactual scenarios involve new state transition that is different from the one in the data (see Remark 11 below). In these cases, estimating state transitions is unnecessary for counterfactual analysis. Even for the counterfactuals that require state transition estimates as inputs, our state-transition-free estimator of the flow utility parameter could be useful in guiding the choice among different parametric specifications of state transition distributions.<sup>27</sup> In addition, note that counterfactual analysis can also be conducted using the flow utility parameters estimated via our estimator and a separately estimated state transition.

**Remark 11** (Usefulness in counterfactual analysis). *We explain how our estimator and overall approach can be used in analyzing counterfactual behaviors in two counterfactual scenarios. First, Aguirregabiria (2010); Arcidiacono and Miller (2020) establish that when there is a counterfactual temporary shift (i.e., additive change) in the flow utility in period  $t$ , then the counterfactual behaviors (i.e., CCPs) in and prior to period  $t$  is identified.<sup>28</sup> Without loss of generality, we let  $T = 3$  and consider the scenario where the flow utility in period 1 shifts by  $\xi$  if  $a_1 = 1$  and zero otherwise,*

<sup>27</sup>If  $T = T_{end}$ , our Assumption 6 is not binding, so our model nests those with a specific state transition distribution, and a Hausman type test can be conducted about the state transition specification.

<sup>28</sup>The counterfactual behaviors after period  $t$  remain the same as observed, because neither the current flow utility nor the future payoff is affected by this counterfactual scenario.

while the state transition remains the same. That is,

$$u_1^{cntf}(1, x_1) = u_1(1, x_1) + \xi, \quad \text{and} \quad u_t^{cntf}(a_t, x_t) = u_t(a_t, x_t) \quad \text{for } t > 1 \text{ or } a_t = 0. \quad (35)$$

It follows from eq. (18b) that the counterfactual CCP  $p_1^{cntf}(s_1)$  satisfies

$$\ln\left(\frac{p_1^{cntf}(s_1)}{1 - p_1^{cntf}(s_1)}\right) + \beta\Delta\bar{\eta}_1^2 = x_1'\Delta_1^{cntf} + \beta\Delta\bar{x}_1^{2'}\delta_{0,2} + \beta^2\Delta\bar{q}_1^{K'}\gamma^K,$$

where other than  $\Delta_1^{cntf}$  and  $p_1^{cntf}(s_1)$ , the other terms are unaffected by the counterfactual change and have been estimated in the process of estimating the flow utility parameters. For  $\Delta_1^{cntf}$ , only the intercept coordinate shifts by  $\xi$ , and the others remain the same. So, the value of the counterfactual CCP function  $p_1^{cntf}(s_1)$  can be computed for each  $s_1$  value using the above equation.

Another obvious counterfactual scenario which does not need to estimate state transition is when the state transition changes to a known one, different from the observed one in the data. In this scenario, a general approach is to solve the DP problem based on the estimated flow utility parameters and the new known state transition, and therefore any counterfactual can be constructed for long panels.<sup>29</sup> When the change in the state transition is temporary, however, solving the DP may be spared, too. Consider a simple example where  $T = 3$  and the only counterfactual change in the primitives is that only the state transition from period 1 to period 2 becomes a known  $f_1^{cntf}(s_2|s_1, a_1)$ , and we are again interested in  $p_1^{cntf}(s_1)$ . We assume that researchers do not know the state transitions for the periods that are not affected by the counterfactual change. By eq. (18b), we have

$$\begin{aligned} & \ln\left(\frac{p_1^{cntf}(s_1)}{1 - p_1^{cntf}(s_1)}\right) + \beta\Delta\mathbb{E}^{cntf}[p_2(s_2)\ln p_2(s_2) + (1 - p_2(s_2))\ln(1 - p_2(s_2))|s_1] \\ &= x_1'\Delta_1 + \beta\Delta\mathbb{E}^{cntf}(x_2|s_1)\delta_{0,2} + \beta^2\Delta\mathbb{E}^{cntf}(q^K(s_3)'|s_1)\gamma^K, \end{aligned}$$

where again, only the objects with a superscript  $^{cntf}$  differ from their counterparts in the data. Among them,  $\Delta\mathbb{E}^{cntf}[p_2(s_2)\ln p_2(s_2) + (1 - p_2(s_2))\ln(1 - p_2(s_2))|s_1]$  and  $\Delta\mathbb{E}^{cntf}(x_2|s_1)$  can be computed straightforwardly using the known  $f_1^{cntf}(s_2|s_1, a_1)$  and the estimated CCP function  $\hat{p}_2(\cdot)$ . To compute  $\Delta\mathbb{E}^{cntf}(q^K(s_3)'|s_1)$ , Lemma 1 gives

$$\mathbb{E}^{cntf}(q^K(s_3)'|s_1, a_1) = \mathbb{E}^{cntf}[\mathbb{E}(q^K(s_3)'|s_2)|s_1, a_1],$$

where the inside conditional mean  $\mathbb{E}(q^K(s_3)'|s_2)$  has already been estimated from the actual data, and the outside  $\mathbb{E}^{cntf}(\cdot|s_1, a_1)$  can be computed using the known  $f_1^{cntf}(s_2|s_1, a_1)$ . So, the value of the counterfactual CCP function  $p_1^{cntf}(s_1)$  can be computed for each  $s_1$  value using the above equation.

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<sup>29</sup>For short panels, the counterfactuals depend on whether/what restrictions are imposed on the model primitives beyond the data horizon.

In both scenarios, the computation of the counterfactual CCP  $p_1^{cntf}(s_1)$  is made easy by eq. (18b) (due to the Markov property and subsequent derivation) and does not require the estimation of the actual state transition. We will demonstrate these two scenarios using simulation experiments in Section 5 below.

## 4 Sensitivity of Estimation to Assumption 6

The data-terminal-period integrated value function  $\bar{V}_T(s_T)$  may have a different form from that in Assumption 6. In this section, we will first quantify the estimation bias of the flow utility parameter induced by imposing Assumption 6 when it is violated.<sup>30</sup> Then, we will discuss how to utilize the over-identification opportunity in the linear system eq. (18) to reduce the impact of Assumption 6 on estimation.

### 4.1 Bias in Estimation from Imposing Assumption 6

The parametric form  $\bar{V}_T(s_T) = q^K(x_T, z_T)' \gamma^K$  in Assumption 6 can be interpreted as an approximation of  $\bar{V}_T(s_T)$ , especially when one uses common series basis functions, such as power series, as  $q^K(x_T, z_T)$ . Define the approximation error

$$r^K(x_T, z_T) \equiv \bar{V}_T(s_T) - q^K(x_T, z_T)' \gamma^K$$

and its expected difference in period  $t$  ( $t = 1, \dots, T-1$ )

$$\Delta \bar{r}_t^K \equiv \Delta \mathbb{E}(r^K(x_T, z_T) | x_t, z_t).$$

Next lemma gives conditions under which the expected difference of the approximation error diminishes rapidly with  $K$ , the dimension of the approximation of  $\bar{V}_T(s_T)$ .

**Lemma 2** (Approximation error  $\Delta \bar{r}_t^K$  due to approximating  $\bar{V}_T(s_T)$  by power series). *Assume*

- (i)  $q^K(x_T, z_T)$  is a triangular sequence of powers of  $(x_T, z_T)$ ,<sup>31</sup>
- (ii)  $K = (k+1)^{d_s}$ , where  $d_s = d_x + d_z$  is the dimension of  $s_T$ , so that  $q^K(x_T, z_T)$  has powers in all state variables at least up to  $k$ ;
- (iii) the support  $\mathcal{S}$  of  $s_T = (x_T, z_T)$  is a compact subset of  $\mathbb{R}^{d_s}$ ;
- (iv)  $\bar{V}_T(s_T)$  is  $m$  times continuously differentiable.

<sup>30</sup>Keane and Wolpin (1994); Geweke and Keane (2000) provide Monte Carlo evidence that polynomial approximation appears to work well in their setting.

<sup>31</sup>That is, power of each state variable is added to  $q^K(x_T, z_T)$  one by one, and  $q^K(x_T, z_T)$  includes all lower order powers before adding any higher order power. See eq. (22) in Hirano, Imbens and Ridder (2003) for details.

Then there exists  $\gamma^K$  such that the resulting approximation error satisfies

$$\mathbb{E}((\Delta \bar{r}_t^K)^2) = O\left(K^{-\frac{2m}{d_s}}\right), \text{ for } t = 1, \dots, T-1.$$

Let  $\delta_{pseudo}^K$  denote the probability limit of our estimator  $\hat{\delta}$  in eq. (31). Because the  $\bar{m}_N$  in eq. (31) is linear in  $\delta$  and  $\gamma^K$ , we know that  $\delta_{pseudo}^K$  exists, is unique and has an explicit-form solution under the conditions of Theorem 3 (see, for example, Newey and McFadden, 1994).<sup>32</sup> These properties help translate the approximation error of  $\bar{V}_T(s_T)$  to the asymptotic bias  $\delta_{pseudo}^K - \delta$  straightforwardly.

**Theorem 4** (Asymptotic bias bound of  $\hat{\delta}$  due to approximating  $\bar{V}_T(s_T)$  by power series). *Suppose Assumptions 1 to 5 and the conditions (i) to (iv) in Lemma 2 hold. Also suppose that  $q^K(x_T, z_T)$  and  $s_t$  for  $t = 1, \dots, T-1$  all have finite second moments,  $0 < \lambda_{min}(W) \leq \lambda_{max}(W) < \infty$ , and  $0 < \lambda_{min}(L'L) \leq \lambda_{max}(L'L) < \infty$ , where  $\lambda_{min}$  and  $\lambda_{max}$  denote the smallest and the largest eigenvalues, respectively. Let  $\|\cdot\|$  denote the Frobenius norm, then we have*

$$\|\delta_{pseudo}^K - \delta\| = O\left(K^{-\frac{m}{d_s}}\right).$$

## 4.2 Reducing Sensitivity of Estimation to Assumption 6

When the data horizon is long, a typical implementation of the HMSS estimator would simulate to a large truncation period  $T_{tr}$  and ignore the term that involves  $\bar{V}_{T_{tr}}(s_{T_{tr}})$  (recall the last term in eq. (15b) for  $T^* = T_{tr}$ ), because the pre-multiplied factor  $\beta^{T_{tr}-t-1}$  quickly approaches zero as  $T_{tr} - t$  increases. The same idea, combined with the over-identification (recall Theorem 3(ii)), can be leveraged to reduce the sensitivity of our estimator to Assumption 6.

Start by considering eq. (15b) for  $t = 1$ . It contains the entire parameter vector  $\delta$  and is linear in  $\delta$ ; only the last term of it contains  $\bar{V}_T(s_T)$  and the pre-multiplied factor  $\beta^{T-1}$  quickly approaches zero as  $T$  increases. Truncating the last term, eq. (15b) for  $t = 1$  becomes a linear equation

$$y_1 = x_1' \Delta_1 + \sum_{\tau=2}^{T-1} \beta^{\tau-1} \Delta \bar{x}_t^{\tau'} \delta_{0,\tau} + \sum_{\tau=2}^{T-1} \beta^{\tau-1} \Delta \bar{x}_{1,1}^{\tau'} \Delta_\tau,$$

which is capable of identifying the entire vector  $\delta$  if the usual rank condition holds.<sup>33</sup> Similarly, eq. (15b) for  $t = 2$ , when truncating the last term, is capable of identifying the last  $(2T-5)d_x$  coordinates of  $\delta$ ; and the same logic applies to those  $t$  for which  $\beta^{T-t}$  is reasonably small. Therefore, the truncated version of eq. (15b) for the first few periods constitute a linear system that is insensitive to Assumption 6, but at the same time still provides ample over-identification opportunity.

<sup>32</sup>In addition, Proposition 1 continues to hold with  $\delta$  replaced by  $\delta_{pseudo}^K$ .

<sup>33</sup>Precisely, if the upper-left  $((2T-3)d_x) \times ((2T-3)d_x)$  submatrix of  $\tilde{L}_1$  has full rank.

## 5 Simulation Experiments: EV Adoption 2025

We compare the finite sample performance of our estimator, referred to as the CRS estimator in this section, and the HMSS estimators with both nonparametrically and parametrically estimated state transition distributions. We use two DGPs that help substantiate our discussion in Section 3.

### 5.1 Simulation Model Specification and Parameterization

To ground our simulation design in a tangible setting, we interpret the model as reflecting a dealership’s decision of whether to begin offering electric vehicles (EVs) over a three-year horizon (i.e.,  $T_{start} = 1$  and  $T_{end} = 3$ ). Each year, the dealership chooses whether to adopt EVs ( $a_t = 1$ ) or not ( $a_t = 0$ ). The time-varying observed state variables  $x_t^* = (x_{1t}^*, x_{2t}^*)'$  reflect the dealership’s readiness in two dimensions – for example, infrastructure (e.g., service equipment) and personnel training.

In DGP 1, we allow  $x_t^*$  to be shaped by public interest in EVs  $z_t$ :

$$\begin{aligned} x_{1,t+1}^* &= 0.7x_{1t}^* + 0.2x_{2t}^* + a_t H_1^{(1)}(z_t) + \varepsilon_{x,1,t+1}, & H_1^{(1)}(z_t) &= 0.5(z_t - \chi_t) \\ x_{2,t+1}^* &= 0.6x_{2t}^* + a_t H_2^{(1)}(z_t) + \varepsilon_{x,2,t+1}, & H_2^{(1)}(z_t) &= 0.5z_t^2 \\ z_{t+1} &= 0.5z_t + \varepsilon_{z,t+1}, \end{aligned} \quad (36)$$

where  $z_t$  represents public interest in EVs driven by policy incentives, media, consumer sentiment, etc. This formulation means that those dealerships that adopt EV, for which  $a_t = 1$ , are affected by public interest  $z_t$ . The shocks  $\varepsilon_{x,1,t+1}, \varepsilon_{x,2,t+1}, \varepsilon_{z,t+1}$  follow the independent standard normal distribution. The shift parameter  $\chi_t$  is reserved for the second counterfactual scenario below; when generating the simulation samples, we let  $\chi_t = 0$ . The public interest in EVs  $z_t$  is excluded from the flow utility functions:

$$\begin{aligned} u_t(0, x_t) &= \delta_{0,t,0} + \delta_{0,t,1}x_{1t}^* + \delta_{0,t,2}x_{2t}^* = 0 + 0.5x_{1t}^* + 0.4x_{2t}^*, \\ u_t(1, x_t) &= \delta_{1,t,0} + \delta_{1,t,1}x_{1t}^* + \delta_{1,t,2}x_{2t}^* = -0.5 + 0.7x_{1t}^* + 0.5x_{2t}^*, \\ \Rightarrow u_t(1, x_t) - u_t(0, x_t) &= \Delta_{t,0} + \Delta_{t,1}x_{1t}^* + \Delta_{t,2}x_{2t}^* = -0.5 + 0.2x_{1t}^* + 0.1x_{2t}^*, \end{aligned}$$

and it only affects the agent’s decision through future evolution of  $x_t^*$ .<sup>34</sup> In practice, two dealerships with the same current readiness but different exposure to public interest  $z_t$  will face different expectations about future value, which affects their adoption decision. This reflects real-world dynamics where even well-prepared dealers may delay adoption if public demand appears to be softening.

DGP 2 differs from DGP 1 in its law of state transition; in particular, it does not have any excluded variable:

$$x_{1,t+1}^* = 0.7x_{1t}^* + 0.2x_{2t}^* + a_t H_1^{(2)}(x_{1t}^*, x_{2t}^*) + \varepsilon_{x,1,t+1}, \quad H_1^{(2)}(x_{1t}^*, x_{2t}^*) = \sin(x_{1t}^* + 2x_{2t}^*) - \chi_t \quad (37)$$

<sup>34</sup>This setup resonates with the economic forces studied empirically by [Chou and Derdenger \(2025\)](#), where household EV adoption is influenced by both vehicle characteristics and shifting public interest.

$$x_{2,t+1}^* = 0.6x_{2t}^* + a_t H_2^{(2)}(x_{1t}^*, x_{2t}^*) + \varepsilon_{x,2,t+1}, \quad H_2^{(2)}(x_{1t}^*, x_{2t}^*) = \cos(2x_{1t}^* - x_{2t}^*).$$

This design leads to  $\Delta\mathbb{E}(x_{1,t+1}^* | x_t^*) = H_1^{(2)}(x_t^*)$  and  $\Delta\mathbb{E}(x_{2,t+1}^* | x_t^*) = H_2^{(2)}(x_t^*)$ . The nonlinear functions  $H_1^{(2)}$  and  $H_2^{(2)}$  help to avoid the collinearity among  $x_{1t}^*$ ,  $x_{2t}^*$ ,  $\Delta\mathbb{E}(x_{1,t+1}^* | x_t^*)$  and  $\Delta\mathbb{E}(x_{2,t+1}^* | x_t^*)$  in eq. (18).

We carefully design these two DGPs to substantiate our discussion in Section 3. First, both DGPs have a time-invariant observed state variable – the intercept, so the parameter  $\delta_{0,t}^\circ = \delta_{0,t,0}$  is not identified (recall Remark 5). Second, although DGP 1 has an excluded variable (and  $1 = d_z < d_{x^*} = 2$ ) and DGP 2 does not have any ( $s_t = x_t^*$ ), both DGPs satisfy the rank requirement of Corollary 2 that the variables in the vector  $(x_t^*, \Delta\mathbb{E}(x_{t+1}^* | s_t))$  are not perfectly linearly correlated (recall Remark 6).<sup>35</sup> Finally, note that  $\delta_{0,1}$  can be normalized to any constant vector  $c$  and therefore is not identified (recall Assumption 5).

We assume that the researcher has available all the three periods of data, i.e.,  $T = T_{end}$ , but the researcher may or may not know this fact. If they know  $T = T_{end}$  (i.e., long panel), they would simply use the  $q^K(x_T, z_T)$  function in Remark 3; and if they do not know this (i.e., short panel), they would use power series in Lemma 2.<sup>36</sup>

In addition to the follow utility parameters, we also consider the percentage change in the average period 1 CCP  $\mathbb{E}(p_1(s_t))$ , a summary parameter of counterfactual behavior, in two counterfactual scenarios discussed in Remark 11. The first counterfactual scenario is a temporary shift in flow utility in period 1 for choice  $a_1 = 1$  only, while the state transition remains the same – recall eq. (35) and we let  $\xi = -0.5$ . This can be interpreted as the termination of EV purchase rebate. The second counterfactual scenario is a temporary change in the state transition from period 1 to period 2 only (and the researcher knows this new state transition), while the flow utility remains the same – in eq. (36) and eq. (37) we let  $\chi_1^{cntf} = 1$  for  $t = 1$  and  $\chi_t^{cntf} = \chi_t = 0$  for all the other  $t$ . This exercise demonstrates the usefulness of our estimator in counterfactual analysis (recall Remark 11).

## 5.2 Estimators and Simulation Results

Although we let all the flow utility parameters be time-invariant in both DGPs, we assume that the researcher does not know this fact and therefore treats them as time-varying in estimation. The parameters of interest are  $\Delta_t = (\Delta_{t,0}, \Delta_{t,1}, \Delta_{t,2})' = (-0.5, 0.2, 0.1)'$  for  $t = 1, 2, 3$ ,  $\delta_{0,t}^* = (\delta_{0,t,1}, \delta_{0,t,2})' = (0.5, 0.4)'$  for  $t = 2, 3$  and percentage change in  $\mathbb{E}(p_1(s_1))$  in the two counterfactual scenarios, numerically computed as  $-21.7\%$  and  $-19.0\%$ , respectively, for DGP 1, and  $-24.5\%$  and  $-39.2\%$ , respectively, for DGP 2.

<sup>35</sup>It is easy to check that excluding  $z_t$  from  $s_t$  for DGP 1 will lead to failure of Corollary 2.

<sup>36</sup>For both DGPs,  $K$  is chosen such that  $q^K(x_T, z_T)$  includes all the second-order powers, including the cross-products (i.e.,  $k = 2$  and  $K = (2 + 1)^{d_s}$ , using the notation in Lemma 2(ii).

Table 1: If  $T = T_{end} = 3$  Is Known (i.e., Long Panel), with Excluded Variable  $z_t$  (DGP 1)

Parameter	Truth	CRS estimator			HMSS estimator (NP)			HMSS estimator (P)			
		Bias	Std Dev	MSE	Bias	Std Dev	MSE	Bias	Std Dev	MSE	
$t = 1$	$\Delta_{1,0}$	-0.5	-0.002	0.129	0.017	-0.020	0.109	0.012	0.345	0.112	0.132
	$\Delta_{1,1}$	0.2	0.002	0.075	0.006	0.020	0.070	0.005	-0.013	0.048	0.002
	$\Delta_{1,2}$	0.1	0.007	0.085	0.007	0.033	0.083	0.008	-0.006	0.059	0.004
$t = 2$	$\delta_{0,2,2}$	0.5	-0.018	0.269	0.073	-0.186	0.134	0.053	-0.192	0.147	0.058
	$\delta_{0,2,2}$	0.4	0.022	0.163	0.027	-0.028	0.112	0.013	-0.496	0.101	0.256
	$\Delta_{2,0}$	-0.5	0.006	0.097	0.009	-0.011	0.094	0.009	0.192	0.091	0.045
	$\Delta_{2,1}$	0.2	0.002	0.052	0.003	0.017	0.052	0.003	0.002	0.046	0.002
	$\Delta_{2,2}$	0.1	-0.001	0.054	0.003	0.025	0.057	0.004	0.019	0.047	0.003
	$\delta_{0,3,1}$	0.5	-0.029	0.171	0.030	-0.235	0.090	0.063	-0.128	0.110	0.028
$t = 3$	$\delta_{0,3,2}$	0.4	-0.012	0.100	0.010	-0.077	0.078	0.012	-0.388	0.070	0.155
	$\Delta_{3,0}$	-0.5	-0.002	0.077	0.006	-0.005	0.072	0.005	-0.006	0.072	0.005
	$\Delta_{3,1}$	0.2	0.006	0.043	0.002	0.003	0.043	0.002	0.007	0.045	0.002
	$\Delta_{3,2}$	0.1	-0.001	0.044	0.002	0.002	0.038	0.001	0.002	0.040	0.002

<sup>†</sup> There are  $N = 1000$  agents in each Monte Carlo repetition. The results are based on  $R = 500$  Monte Carlo repetitions.

For each DGP, we generate  $R = 1,000$  samples of  $N = 500$  agents. We compare the finite sample performance of our CRS estimator (Section 3.3) and two versions of the HMSS estimator – one with nonparametrically estimated state transition distributions and another with parametrically estimated state transition distributions. For the CRS estimator, the unknown conditional mean functions are estimated nonparametrically using series estimators. For the nonparametric HMSS estimator, the state transition distributions  $f(s_2 | s_1, a_1)$  and  $f(s_3 | s_2, a_2)$  are estimated by kernel density estimators. For the parametric HMSS estimator, we assume that  $s_t$  follows choice specific VAR(1) models and estimate  $f(s_{t+1} | s_t, a_t)$  for  $t = 1, 2$  and  $a_t = 0, 1$  separately. Knowing the state transition distribution, we can draw multiple sequences of states and choices. When  $T = T_{end} = 3$  is known, we draw 60 sequences of state variables and choices up to period 3 when evaluating the moment functions.<sup>37</sup> When  $T_{end}$  is unknown, we draw 60 sequences of state variables and choices up to period 50 periods when evaluating the moment functions. For period  $t = 3$  beyond, we assume  $f(s_{t+1} | s_t, a_t) = f(s_3 | s_2, a_2)$  in order to draw up to period 50. We only estimate the counterfactual CCP when  $T_{end}$  is assumed as unknown to researchers.

For DGP 1, the true values, the simulation biases, the simulation standard deviations and the simulation mean squared errors (MSE) for the CRS and the two versions of the HMSS estimators are reported in Table 1 and Table 2, for  $T = T_{end}$  known case (long panel) and  $T = T_{end}$  unknown case (short panel), respectively. In both tables, the biases of the CRS estimator are smaller than

<sup>37</sup>We compared the estimates resulted from using 60 and 500 sequences in preliminary experiment and did not find noticeable change in parameter estimates. Using 500 sequences takes substantially more memory and hence reduces the number of cores in parallel computing. So, we chose to use 60 draws for the HMSS estimator for all the reported results.

Table 2: If  $T = T_{end}$  Is Unknown (i.e., Short Panel), with Excluded Variable  $z_t$  (DGP 1)

	Parameter	Truth	CRS estimator			HMSS estimator (NP)			HMSS estimator (P)		
			Bias	Std Dev	MSE	Bias	Std Dev	MSE	Bias	Std Dev	MSE
$t = 1$	$\Delta_{1,0}$	-0.5	-0.003	0.136	0.019	-0.019	0.109	0.012	0.334	0.113	0.124
	$\Delta_{1,1}$	0.2	-0.001	0.084	0.007	0.021	0.071	0.005	-0.015	0.048	0.003
	$\Delta_{1,2}$	0.1	0.002	0.094	0.009	0.035	0.084	0.008	-0.009	0.059	0.004
$t = 2$	$\delta_{0,2,1}$	0.5	-0.032	0.289	0.085	-0.034	0.128	0.018	0.002	0.134	0.018
	$\delta_{0,2,2}$	0.4	0.036	0.170	0.030	0.140	0.102	0.030	-0.401	0.100	0.171
	$\Delta_{2,0}$	-0.5	0.017	0.114	0.013	0.159	0.072	0.030	0.224	0.071	0.055
	$\Delta_{2,1}$	0.2	-0.001	0.065	0.004	0.004	0.044	0.002	0.005	0.043	0.002
	$\Delta_{2,2}$	0.1	-0.010	0.069	0.005	0.018	0.046	0.002	0.018	0.044	0.002
$t = 3$	$\delta_{0,3,1}$	0.5	-	-	-	-0.538	0.028	0.290	-0.532	0.025	0.284
	$\delta_{0,3,2}$	0.4	-	-	-	-0.390	0.027	0.153	-0.459	0.021	0.211
	$\Delta_{3,0}$	-0.5	-	-	-	-0.086	0.074	0.013	-0.021	0.073	0.006
	$\Delta_{3,1}$	0.2	-	-	-	-0.001	0.043	0.002	-0.003	0.043	0.002
	$\Delta_{3,2}$	0.1	-	-	-	-0.003	0.039	0.002	-0.006	0.040	0.002
flow shift	$p_1$ change %	-21.7	-0.1	0.8	0.6	-1.3	0.7	2.1	-2.8	0.7	8.1
trans. shift	$p_1$ change %	-19.0	1.8	6.0	39.3	3.4	4.0	27.7	2.9	3.4	20.1

<sup>†</sup> There are  $T = 3$  periods and  $N = 1000$  agents in each Monte Carlo repetition. The results are based on  $R = 500$  Monte Carlo repetitions.

<sup>‡</sup> In this scenario, the parameters in period  $t = 3$  are not identified under Assumptions 1 to 6, so they are not estimated by the CRS estimator. The HMSS estimators require additional assumptions about  $T_{end}$ , the flow utility and the state transition beyond period  $t = 3$  (details are in the paper).

those of the nonparametric HMSS estimator for all the parameters, which are in turn smaller than the parametric HMSS estimator, except for  $\Delta_{3,1}$  in Table 1. The standard deviations of the CRS estimators, on the other hand, are generally larger than those of the nonparametric HMSS estimator, leading to larger MSEs, but the differences are not big. This is not surprising given the asymptotic equivalence between the NFXP/MLE and the HMSS estimators shown by Aguirregabiria and Mira (2002). The parametric HMSS estimator, on the other hand, have much larger biases for many parameters, although its standard deviations could be the smallest, leading to generally the largest MSEs. This is again not surprising given the bias-variance trade-off between parametric and nonparametric modeling of nuisance parameters.<sup>38</sup> The results for DGP 2, reported in Table 3 and Table 4, are very similar to those for DGP 1.

Several remarks are worth mentioning. First, when  $T = T_{end}$  is unknown (Table 2 and Table 4), the parameters in period  $t = 3$  are not identified without assuming stationarity, and therefore no result is reported for the CRS estimator. If one blindly implement the HMSS estimators regardless of the non-identification, however, they will get something (reported for the HMSS estimators). They do not appear to be terrible, possibly because the true simulation DGPs are stationary, but

<sup>38</sup>See Shi (2024) for an averaging estimator that takes the advantages of both to improve estimation.

Table 3: If  $T = T_{end} = 3$  Is Known (i.e., Long Panel), without Excluded Variable  $z_t$  (DGP 2)

Parameter	Truth	CRS estimator			HMSS estimator (NP)			HMSS estimator (P)			
		Bias	Std Dev	MSE	Bias	Std Dev	MSE	Bias	Std Dev	MSE	
$t = 1$	$\Delta_{1,0}$	-0.5	-0.081	0.120	0.021	-0.114	0.098	0.023	0.062	0.070	0.009
	$\Delta_{1,1}$	0.2	0.022	0.086	0.008	-0.026	0.066	0.005	-0.024	0.048	0.003
	$\Delta_{1,2}$	0.1	0.020	0.094	0.009	-0.010	0.076	0.006	0.019	0.054	0.003
$t = 2$	$\delta_{0,2,2}$	0.5	0.034	0.201	0.042	-0.073	0.150	0.028	-0.581	0.126	0.353
	$\delta_{0,2,2}$	0.4	0.041	0.188	0.037	-0.040	0.141	0.021	-0.432	0.130	0.204
	$\Delta_{2,0}$	-0.5	-0.014	0.076	0.006	-0.027	0.075	0.006	0.020	0.070	0.005
	$\Delta_{2,1}$	0.2	0.005	0.049	0.002	-0.015	0.046	0.002	-0.010	0.045	0.002
	$\Delta_{2,2}$	0.1	0.007	0.060	0.004	-0.004	0.055	0.003	0.000	0.053	0.003
$t = 3$	$\delta_{0,3,1}$	0.5	-0.232	0.144	0.075	-0.303	0.092	0.100	-0.582	0.071	0.345
	$\delta_{0,3,2}$	0.4	-0.175	0.129	0.047	-0.231	0.086	0.061	-0.443	0.076	0.202
	$\Delta_{3,0}$	-0.5	-0.004	0.069	0.005	-0.001	0.071	0.005	-0.011	0.066	0.004
	$\Delta_{3,1}$	0.2	-0.001	0.046	0.002	0.003	0.044	0.002	0.001	0.043	0.002
	$\Delta_{3,2}$	0.1	-0.003	0.051	0.003	0.002	0.048	0.002	0.003	0.050	0.003

† There are  $T = 3$  periods and  $N = 1000$  agents in each Monte Carlo repetition. The results are based on  $R = 500$  Monte Carlo repetitions.

we do not suggest reading too much into these results. Second,  $\Delta_t$  tends to be estimated better than  $\delta_{0,t}$  across all settings and all estimators, because as elaborated in Remark 4 to Remark 6, the non-collinearity in  $x_t^*$  itself provides identifying power for the former, while the later requires the non-collinearity in  $(x_t^*, \Delta\mathbb{E}(x_{t+1}^* | s_t))$  (recall Corollary 2). This indicates that the strength of the rank condition given in this paper reflects an intrinsic feature of the model and the data, and is unlikely to be altered by alternative estimators based on the same optimal decision rule (recall eqs. (3) to (6) and eq. (8)).

It is worth emphasizing that the CRS estimator is much faster than both versions of the HMSS estimator. For DGP 1 (resp. DGP 2), it took a five-core Mac Studio 20 (resp. 30) minutes to complete the calculation of the CRS estimates, but about 2.5 (resp. 4) days to compute the nonparametric HMSS estimator, and about 1.9 (resp. 2.5) days to compute the parametric HMSS estimator – roughly two-hundred-time and one-hundred-and-thirty-time differences. Moreover, the HMSS estimator results reported here are based on a commercial optimization software (Knitro, see Byrd, Nocedal and Waltz, 2006, for details), which we used to speed up and enhance the performance of the HMSS estimator, while the step (III) of the CRS estimator has a closed-form solution.

A final note before concluding this section. To generate a simulation sample, we must know the CCP function  $p_t(x_{t,1}^*, x_{t,2}^*, z_t)$  for any value of the continuous vector  $(x_{t,1}^*, x_{t,2}^*, z_t)$  by solving the dynamic programming (DP) problem. This is not a trivial task, and we proceed in three steps. First, we discretize the continuous state space of  $s_t$  into an efficient grid and obtain the corresponding state transition processes using the “EDS” method (Maliar and Maliar, 2015; Gordon, 2021). This allows

Table 4: If  $T = T_{end}$  Is Unknown (i.e., Short Panel), without Excluded Variable  $z_t$  (DGP 2)

	Parameter	Truth	CRS estimator			HMSS estimator (NP)			HMSS estimator (P)		
			Bias	Std Dev	MSE	Bias	Std Dev	MSE	Bias	Std Dev	MSE
$t = 1$	$\Delta_{1,0}$	-0.5	-0.085	0.119	0.021	-0.113	0.098	0.022	0.062	0.069	0.009
	$\Delta_{1,1}$	0.2	0.020	0.086	0.008	-0.027	0.065	0.005	-0.024	0.048	0.003
	$\Delta_{1,2}$	0.1	0.018	0.094	0.009	-0.011	0.076	0.006	0.018	0.054	0.003
$t = 2$	$\delta_{0,2,1}$	0.5	0.032	0.198	0.040	0.034	0.139	0.020	-0.582	0.126	0.355
	$\delta_{0,2,2}$	0.4	0.041	0.187	0.037	0.058	0.135	0.022	-0.437	0.123	0.205
	$\Delta_{2,0}$	-0.5	-0.011	0.075	0.006	0.006	0.066	0.004	0.018	0.069	0.005
	$\Delta_{2,1}$	0.2	0.006	0.049	0.002	-0.013	0.042	0.002	-0.011	0.045	0.002
	$\Delta_{2,2}$	0.1	0.009	0.060	0.004	-0.001	0.052	0.003	-0.001	0.053	0.003
$t = 3$	$\delta_{0,3,1}$	0.5	-	-	-	-0.541	0.022	0.293	-0.579	0.018	0.336
	$\delta_{0,3,2}$	0.4	-	-	-	-0.420	0.024	0.177	-0.440	0.023	0.194
	$\Delta_{3,0}$	-0.5	-	-	-	-0.021	0.073	0.006	-0.014	0.066	0.005
	$\Delta_{3,1}$	0.2	-	-	-	-0.003	0.042	0.002	-0.004	0.042	0.002
	$\Delta_{3,2}$	0.1	-	-	-	-0.001	0.048	0.002	-0.001	0.049	0.002
flow shift	$p_1$ change (%)	-24.5	-2.0	1.3	5.7	-1.7	0.8	3.5	-2.9	0.6	8.7
trans. shift	$p_1$ change (%)	-39.2	0.9	7.6	58.6	3.3	6.5	53.3	40.6	7.9	1712.8

<sup>†</sup> There are  $T = 3$  periods and  $N = 1000$  agents in each Monte Carlo repetition. The results are based on  $R = 500$  Monte Carlo repetitions.

<sup>‡</sup> In this scenario, the parameters in period  $t = 3$  are not identified under Assumptions 1 to 6, so they are not estimated by the CRS estimator. The HMSS estimators require additional assumptions about  $T_{end}$ , the flow utility and the state transition beyond period  $t = 3$  (details are in the paper).

us to draw from a much smaller number of state grid points but still approximate well the stochastic behavior of the state variables. Second, we solve the DP problem backwardly with the state variables on the discrete EDS grid, which was shown to well approximate the solution of the original DP problem (Maliar and Maliar, 2015). This step gives the CCP value on the discrete EDS grid, from which we obtain the CCP function of continuous state variables by polynomial interpolation. Third, we simulate the three periods of the state variables  $s_t$  and the choices  $a_t$  for  $N$  agents and  $R$  Monte Carlo repetitions. The details of these three steps, which might be of independent interest, can be found in Section E. We acknowledge that this “discretization-interpolation” approach to generating simulation samples will inevitably result in bias in final parameter estimation, but because the CRS and the HMSS estimators are based on the same simulation samples, any approximation errors in generating the samples should affect both estimators equally.

## 6 Concluding Remarks

Lines of inquiry we hope to explore in the future include: (i) identifying the expected flow utility function  $u_t(a_t, x_t)$  while relaxing Assumption 5 (linearity) and/or Assumption 6 (data-terminal-period integrated value function); (ii) generalizing the analysis in this paper to permit unobserved

heterogeneity among the agents, as characterized by latent types; (iii) allowing for serially correlated flow utility shocks.

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Online Appendices for  
 “Identification and Estimation of Nonstationary Dynamic Discrete  
 Choice Models”

by Cheng Chou, Geert Ridder and Ruoyao Shi

## A Proofs of the Results in Section 2.2

**Lemma A.1.** *Under Assumptions 1 and 2(i)-(iii),  $\varepsilon_t \perp (s_{t-j}, a_{t-j}) \mid s_t$  for  $t$  and  $j \in \mathbb{N}^+$  such that  $t$  and  $t-j$  all belong to  $\mathcal{T}$ .*

*Proof.* To prove this, we separate the cases  $j = 1$  and  $j \geq 2$ . For  $j = 1$ , first note that Assumption 1 implies that  $a_{t-1}$  is completely determined by  $s_{t-1}$  and  $\varepsilon_{t-1}$  (see Aguirregabiria and Mira, 2010, p. 39, and it can easily be shown), then  $\varepsilon_t \perp a_{t-1}$  by Assumptions 2(ii) and 2(iii). Together with Assumption 2(i) and 2(ii), this implies that  $\varepsilon_t$  is independent of  $(s'_{t-1}, a_{t-1}, s'_t)'$ , or any subvector of it, so

$$f(\varepsilon_t \mid s_{t-1}, a_{t-1}, s_t) = f(\varepsilon_t) = f(\varepsilon_t \mid s_t). \quad (\text{A.1})$$

In consequence, we have

$$\begin{aligned} f(\varepsilon_t, s_{t-1}, a_{t-1} \mid s_t) &= f(s_{t-1}, a_{t-1} \mid s_t) f(\varepsilon_t \mid s_{t-1}, a_{t-1}, s_t) \\ &= f(s_{t-1}, a_{t-1} \mid s_t) f(\varepsilon_t \mid s_t). \end{aligned}$$

For  $j \geq 2$ , we have

$$\begin{aligned} &f(\varepsilon_t, s_{t-j}, a_{t-j} \mid s_t) \\ &= \int f(\varepsilon_t, \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j} \mid s_t) d\Omega_{t-1} da_{t-1} \\ &= f(s_{t-j}, a_{t-j} \mid s_t) \int f(\varepsilon_t, \Omega_{t-1}, a_{t-1} \mid s_{t-j}, a_{t-j}, s_t) d\Omega_{t-1} da_{t-1} \\ &= f(s_{t-j}, a_{t-j} \mid s_t) \int f(\Omega_{t-1}, a_{t-1} \mid s_{t-j}, a_{t-j}, s_t) f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) d\Omega_{t-1} da_{t-1}. \quad (\text{A.2}) \end{aligned}$$

Note that

$$\begin{aligned} &f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) \\ &= \frac{f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) f(s_t \mid \Omega_{t-1}, a_{t-1})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \\ &= \frac{f(\varepsilon_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j}, s_t) f(s_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \\ &= \frac{f(\varepsilon_t, s_t \mid \Omega_{t-1}, a_{t-1}, s_{t-j}, a_{t-j})}{f(s_t \mid \Omega_{t-1}, a_{t-1})} \end{aligned}$$

$$\begin{aligned}
&= \frac{f(\varepsilon_t, s_t | \Omega_{t-1}, a_{t-1})}{f(s_t | \Omega_{t-1}, a_{t-1})} \\
&= f(\varepsilon_t | \Omega_{t-1}, a_{t-1}, s_t) \\
&= f(\varepsilon_t | s_t),
\end{aligned} \tag{A.3}$$

where the second and the fourth equalities hold by Assumption 1, and the last equality holds by Assumptions 1, 2(i)-(iii) and an argument similar to the one used to show eq. (A.1). Plug eq. (A.3) into eq. (A.2), we get

$$\begin{aligned}
&f(\varepsilon_t, s_{t-j}, a_{t-j} | s_t) \\
&= f(s_{t-j}, a_{t-j} | s_t) f(\varepsilon_t | s_t) \int f(\Omega_{t-1}, a_{t-1} | s_{t-j}, a_{t-j}, s_t) d\Omega_{t-1} da_{t-1} \\
&= f(s_{t-j}, a_{t-j} | s_t) f(\varepsilon_t | s_t).
\end{aligned}$$

This completes the proof. ■

**Lemma A.2** (Markovian state variables). *Under Assumption 1,  $\Omega_t$  is a first order Markov process; that is,  $\Omega_{t+1} \perp \Omega_{t-j} | \Omega_t$  for  $t$  and  $j \in \mathbb{N}^+$  such that  $t, t+1$  and  $t-j$  all belong to  $\mathcal{T}$ .*

*Proof.* First note that  $f(\Omega_{t+1} | a_t, \Omega_t, \Omega_{t-j}) = f(\Omega_{t+1} | a_t, \Omega_t)$  by Assumption 1. Also recall that the choice  $a_t$  is completely determined by  $\Omega_t$  under Assumption 1, so  $f(a_t | \Omega_t, \Omega_{t-j}) = f(a_t | \Omega_t)$ . These two equalities, together with the law of total probabilities, lead to

$$\begin{aligned}
f(\Omega_{t+1} | \Omega_t, \Omega_{t-j}) &= \sum_{a_t=0,1} f(\Omega_{t+1} | a_t, \Omega_t, \Omega_{t-j}) f(a_t | \Omega_t, \Omega_{t-j}) \\
&= \sum_{a_t=0,1} f(\Omega_{t+1} | a_t, \Omega_t) f(a_t | \Omega_t) \\
&= f(\Omega_{t+1} | \Omega_t),
\end{aligned}$$

which is the result of this lemma. ■

*Proof of Theorem 1.* First note that by the first order Markovian property of  $\Omega_t$  shown in Lemma A.2, we have  $s_{t+1} \perp s_{t-j} | (s_t, \varepsilon_t)$ , which indicates  $f(s_{t+1}, s_{t-j} | s_t, \varepsilon_t) = f(s_{t+1} | s_t, \varepsilon_t) f(s_{t-j} | s_t, \varepsilon_t)$ . Moreover, Lemma A.1 further simplifies  $f(s_{t-j} | s_t, \varepsilon_t)$  to  $f(s_{t-j} | s_t)$ , implying  $f(s_{t+1}, s_{t-j} | s_t, \varepsilon_t) = f(s_{t+1} | s_t, \varepsilon_t) f(s_{t-j} | s_t)$ . Finally, applying the integral operator  $\int \cdot dF(\varepsilon_t | s_t)$  to both sides of this equality gives

$$\begin{aligned}
\int f(s_{t+1}, s_{t-j} | s_t, \varepsilon_t) dF(\varepsilon_t | s_t) &= \int f(s_{t+1} | s_t, \varepsilon_t) f(s_{t-j} | s_t) dF(\varepsilon_t | s_t) \\
\implies f(s_{t+1}, s_{t-j} | s_t) &= f(s_{t+1} | s_t) f(s_{t-j} | s_t),
\end{aligned}$$

which is the result of this theorem. ■

**Lemma A.3.** Under Assumptions 1 and 2(i)-(iii),  $s_{t+j} \perp\!\!\!\perp s_t \mid s_{t+1}$  and  $\Omega_{t+j} \perp\!\!\!\perp \Omega_t \mid \Omega_{t+1}$  for  $t$  and  $j \in \mathbb{N}^+$  such that  $t, t+1$  and  $t+j$  all belong to  $\mathcal{T}$ .

*Proof.* We consider

$$\begin{aligned}
& f(s_{t+j}, s_t \mid s_{t+1}) \\
&= \int f(s_{t+j}, s_{t+j-1}, \dots, s_{t+2}, s_t \mid s_{t+1}) ds_{t+j-1} \cdots ds_{t+2} \\
&= f(s_t \mid s_{t+1}) \int f(s_{t+j}, s_{t+j-1}, \dots, s_{t+2} \mid s_t, s_{t+1}) ds_{t+j-1} \cdots ds_{t+2} \\
&= f(s_t \mid s_{t+1}) \int f(s_{t+j}, s_{t+j-1}, \dots, s_{t+2} \mid s_{t+1}) ds_{t+j-1} \cdots ds_{t+2} \\
&= f(s_t \mid s_{t+1}) f(s_{t+j} \mid s_{t+1}),
\end{aligned}$$

where the third equality holds by Theorem 1. The result  $f(\Omega_{t+j}, \Omega_t \mid \Omega_{t+1}) = f(\Omega_t \mid \Omega_{t+1}) f(\Omega_{t+j} \mid \Omega_{t+1})$  can be shown by the same argument using Lemma A.2, and this completes the proof of this lemma.  $\blacksquare$

**Lemma A.4.** Under Assumptions 1 and 2(i)-(iii),  $s_{t+j} \perp\!\!\!\perp \varepsilon_t \mid s_{t+1}$  for  $t$  and  $j \in \mathbb{N}^+$  such that  $t, t+1$  and  $t+j$  all belong to  $\mathcal{T}$ .

*Proof.* We consider

$$\begin{aligned}
& f(s_{t+j}, \varepsilon_t \mid s_{t+1}) \\
&= \int f(s_{t+j}, \varepsilon_{t+1}, \varepsilon_t \mid s_{t+1}) d\varepsilon_{t+1} \\
&= \int f(\varepsilon_{t+1} \mid s_{t+1}) f(s_{t+j}, \varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) d\varepsilon_{t+1} \\
&= \int f(\varepsilon_{t+1} \mid s_{t+1}) f(s_{t+j} \mid s_{t+1}, \varepsilon_{t+1}) f(\varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) d\varepsilon_{t+1} \\
&= \int f(s_{t+j}, \varepsilon_{t+1} \mid s_{t+1}) f(\varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) d\varepsilon_{t+1}, \tag{A.4}
\end{aligned}$$

where the third equality holds by  $s_{t+j} \perp\!\!\!\perp \varepsilon_t \mid (s_{t+1}, \varepsilon_{t+1})$  implied by Lemma A.3. Note that in eq. (A.4),

$$\begin{aligned}
f(\varepsilon_t \mid s_{t+1}, \varepsilon_{t+1}) &= \frac{f(\varepsilon_t, s_{t+1}, \varepsilon_{t+1})}{f(s_{t+1}, \varepsilon_{t+1})} \\
&= \frac{f(\varepsilon_t, s_{t+1}) f(\varepsilon_{t+1})}{f(s_{t+1}) f(\varepsilon_{t+1})} = \frac{f(\varepsilon_t, s_{t+1})}{f(s_{t+1})} = f(\varepsilon_t \mid s_{t+1}),
\end{aligned}$$

where the second equality holds by Assumptions 2(i) and 2(iii). As a result, eq. (A.4) becomes

$$\begin{aligned}
f(s_{t+j}, \varepsilon_t \mid s_{t+1}) &= \int f(s_{t+j}, \varepsilon_{t+1} \mid s_{t+1}) f(\varepsilon_t \mid s_{t+1}) d\varepsilon_{t+1} \\
&= f(\varepsilon_t \mid s_{t+1}) \int f(s_{t+j}, \varepsilon_{t+1} \mid s_{t+1}) d\varepsilon_{t+1} \\
&= f(\varepsilon_t \mid s_{t+1}) f(s_{t+j} \mid s_{t+1}),
\end{aligned}$$

which is the result of this lemma.  $\blacksquare$

*Proof of Lemma 1.* First recall that  $a_t$  is completely determined by  $s_t$  and  $\varepsilon_t$  under Assumption 1, then Lemmas A.3 and A.4 imply that  $s_{t+j} \perp (s_t, a_t) \mid s_{t+1}$ . Then, we have

$$\begin{aligned}
& \mathbb{E}(\mathbb{E}(g(s_{t+j}) \mid s_{t+1}) \mid s_t, a_t) \\
&= \int \int g(s_{t+j}) f(s_{t+j} \mid s_{t+1}) f(s_{t+1} \mid s_t, a_t) ds_{t+j} ds_{t+1} \\
&= \int \int g(s_{t+j}) f(s_{t+j} \mid s_{t+1}, s_t, a_t) f(s_{t+1} \mid s_t, a_t) ds_{t+j} ds_{t+1} \\
&= \int \int g(s_{t+j}) f(s_{t+j}, s_{t+1} \mid s_t, a_t) ds_{t+j} ds_{t+1} \\
&= \int g(s_{t+j}) f(s_{t+j} \mid s_t, a_t) ds_{t+j} \\
&= \mathbb{E}(g(s_{t+j}) \mid s_t, a_t).
\end{aligned}$$

■

## B Influence Function $\psi_\delta(D)$ in Proposition 1

In this section, we use Proposition 5 of Newey (1994) to compute the influence function  $\psi(D)$  of  $(\hat{\delta}', \hat{\gamma}^{K'})'$ . The first  $(2T-3)d_x$  coordinates of  $\psi(D)$ , denoted as  $\psi_\delta(D)$ , are the influence function of  $\hat{\delta}$ .

Proposition 1 and Proposition 2 hold by Lemma 5.3 and Lemma 5.4 of Newey (1994) under the regularity conditions in Section 5 of Newey (1994), for which we provide primitive conditions and proof for series estimators in Section F.

### B.1 Basic Terms in the Influence Function

In eq. (33),  $L_T$  is the Jacobian matrix of the moment functions (with respect to  $\delta$  and  $\gamma^K$ );  $\alpha_\eta(D)$ ,  $\alpha_x(D)$  and  $\alpha_q(D)$  are the respective adjustment terms for the estimation of  $\Delta\bar{\eta}$ ,  $\Delta\bar{x}$  and  $\Delta\bar{q}^K$  in step (II) of the estimation strategy in Section 3.3,  $\alpha_{p,direct}(D)$  is the adjustment term for the estimation of the CCPs in step (I) that appears directly in the moment function, and  $\alpha_{p,indirect}(D)$  is the adjustment term for the estimation of the CCPs in step (II) that appears in the “dependent variables” of the nonparametric regressions leading to  $\Delta\bar{\eta}$ ,  $\Delta\bar{x}$  and  $\Delta\bar{q}^K$  in step (II).

Note that by eq. (18), as well as the definition of the moment function  $m$  in eq. (30), we have  $m_0(D) = 0$ , because  $v_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K) = 0$  for  $t = 1, \dots, T-1$ . In other words, the influence function consists of only the adjustment terms.

In the rest of this subsection, we verify that the  $L_T$  matrix define in eq. (23) indeed equals to the Jacobian matrix, denoted by  $M$ . We know that

$$M_{((T-1)^2 d_x + (T-1)K) \times ((2T-3)d_x + K)} \equiv \begin{bmatrix} M_{T-1,\delta} & M_{T-1,\gamma} \\ \vdots & \vdots \\ M_{1,\delta} & M_{1,\gamma} \end{bmatrix},$$

where

$$M_{t,\delta} \equiv \frac{\partial}{\partial \delta'} \mathbb{E}(m_t(D, \delta, \gamma_0^K, p_0, \Delta \bar{\eta}_0, \Delta \bar{x}_0, \Delta \bar{q}_0^K)) \Big|_{\delta=\delta_0},$$

and

$$M_{t,\gamma} \equiv \frac{\partial}{\partial \gamma^K} \mathbb{E}(m_t(D, \delta_0, \gamma^K, p_0, \Delta \bar{\eta}_0, \Delta \bar{x}_0, \Delta \bar{q}_0^K)) \Big|_{\gamma^K=\gamma_0^K}.$$

So, it is easy to see that, assuming interchangeability of differentiation and integral,

$$M = L_T. \quad (\text{B.1})$$

## B.2 Adjustment Terms in the Influence Function

Note that the moment functions  $m$  depend on the CCP functions in step (I) and the nonparametric regression functions in step (II) only through their values at given argument values, so Proposition 5 of Newey (1994) applies. In the rest of this subsection, we use  $\tilde{p}$ ,  $\widetilde{\Delta \bar{\eta}}$ ,  $\widetilde{\Delta \bar{x}}$  and  $\widetilde{\Delta \bar{q}}^K$  to denote given values that these unknown functions take, and their coordinates are denoted in a similar way. These values are taken as real variables, not functions, in the partial derivatives shown in the rest of this subsection.

### B.2.1 Adjustment Term for Estimated $\Delta \bar{\eta}$ Functions

To compute the adjustment term for estimated  $\Delta \bar{\eta}$  functions, we first need to compute

$$M_{t,\Delta \bar{\eta}_{t'}^\tau} \equiv \frac{\partial}{\partial \widetilde{\Delta \bar{\eta}_{t'}^\tau}} m_t(D, \delta_0, \gamma_0^K, p_0, \widetilde{\Delta \bar{\eta}_{t'}^\tau}, \Delta \bar{\eta}_{t',0}^{\tau,c}, \Delta \bar{x}_0, \Delta \bar{q}_0^K) \Big|_{\widetilde{\Delta \bar{\eta}_{t'}^\tau}=\Delta \bar{\eta}_{t',0}^\tau},$$

where  $\Delta \bar{\eta}_{t'}^{\tau,c}$  denotes all the coordinates of  $\Delta \bar{\eta}$  other than  $\Delta \bar{\eta}_{t'}^\tau$ .

Through basic algebra, we get  $M_{t,\Delta \bar{\eta}_{t'}^\tau} = 0_{((2T-2t-1)d_x+K)\times 1}$  for  $t = 1, \dots, T-2$ ,  $t' \neq t$  and  $\tau = t' + 1, \dots, T-1$ ,  $M_{T-1,\Delta \bar{\eta}_{t'}^\tau} = 0_{(d_x+K)\times 1}$  for  $t' = 1, \dots, T-1$  and  $\tau = t' + 1, \dots, T-1$ , and  $M_{t,\Delta \bar{\eta}_t^\tau} = -\beta^{\tau-t} X_{T,t,0}$  for  $t = 1, \dots, T-2$  and  $\tau = t + 1, \dots, T-1$ . As a result,

$$\alpha_\eta(D) = - \begin{bmatrix} 0_{(d_x+K)\times 1} \\ \beta X_{T,T-2,0} \left[ \left( \frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) \eta_{T-1,0} - \Delta \bar{\eta}_{T-2,0}^{T-1} \right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0} \left[ \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) \eta_{\tau,0} - \Delta \bar{\eta}_{t,0}^\tau \right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0} \left[ \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) \eta_{\tau,0} - \Delta \bar{\eta}_{1,0}^\tau \right] \end{bmatrix}. \quad (\text{B.2})$$

### B.2.2 Adjustment Term for Estimated $\Delta \bar{x}$ Functions

Note that  $\Delta \bar{x}$  contains two types of conditional mean functions,  $\Delta \bar{x}_t^\tau$  defined in eq. (13c) and  $\Delta \bar{x}_{1,t}^\tau$  define in eq. (13a). To compute the adjustment term for estimated  $\Delta \bar{x}$  functions, therefore, we first

need to compute

$$M_{t,\Delta\bar{x}_{t'}^\tau} \equiv \frac{\partial}{\partial \widetilde{\Delta\bar{x}_{t'}^\tau}} m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \widetilde{\Delta\bar{x}_{t'}^\tau}, \Delta\bar{x}_{t',0}^{\tau,c}, \Delta\bar{q}_0^K) \Big|_{\widetilde{\Delta\bar{x}_{t'}^\tau} = \Delta\bar{x}_{t',0}^\tau}$$

and

$$M_{t,\Delta\bar{x}_{1,t'}^\tau} \equiv \frac{\partial}{\partial \widetilde{\Delta\bar{x}_{1,t'}^\tau}} m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \widetilde{\Delta\bar{x}_{1,t'}^\tau}, \Delta\bar{x}_{1,t',0}^{\tau,c}, \Delta\bar{q}_0^K) \Big|_{\widetilde{\Delta\bar{x}_{1,t'}^\tau} = \Delta\bar{x}_{1,t',0}^\tau},$$

where  $\Delta\bar{x}_{t'}^{\tau,c}$  denotes all the coordinates of  $\Delta\bar{x}$  other than  $\Delta\bar{x}_{t'}^\tau$ , and  $\Delta\bar{x}_{1,t'}^{\tau,c}$  denotes all the coordinates of  $\Delta\bar{x}$  other than  $\Delta\bar{x}_{1,t'}^\tau$ .

Through basic algebra, we get  $M_{t,\Delta\bar{x}_{t'}^\tau} = 0_{((2T-2t-1)d_x+K) \times d_x}$  for  $t = 1, \dots, T-2$ ,  $t' \neq t$  and  $\tau = t'+1, \dots, T-1$ ,  $M_{T-1,\Delta\bar{x}_{t'}^\tau} = 0_{(d_x+K) \times d_x}$  for  $t' = 1, \dots, T-1$  and  $\tau = t+1, \dots, T-1$ , and  $M_{t,\Delta\bar{x}_t^\tau} = \beta^{\tau-t} X_{T,t,0} \delta'_{0,\tau,0}$  for  $t = 1, \dots, T-2$  and  $\tau = t+1, \dots, T-1$ . In addition,  $M_{t,\Delta\bar{x}_{1,t'}^\tau} = 0_{((2T-2t-1)d_x+K) \times d_x}$  for  $t = 1, \dots, T-2$ ,  $t' \neq t$  and  $\tau = t'+1, \dots, T-1$ ,  $M_{T-1,\Delta\bar{x}_{1,t'}^\tau} = 0_{(d_x+K) \times d_x}$  for  $t' = 1, \dots, T-1$  and  $\tau = t+1, \dots, T-1$ , and  $M_{t,\Delta\bar{x}_{1,t}^\tau} = \beta^{\tau-t} X_{T,t,0} \Delta'_{\tau,0}$  for  $t = 1, \dots, T-2$  and  $\tau = t+1, \dots, T-1$ . As a result,

$$\alpha_x(D) = \begin{bmatrix} 0_{(d_x+K) \times 1} \\ \beta X_{T,T-2,0} \delta'_{0,T-1,0} \left[ \left( \frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) x_{T-1} - \Delta\bar{x}_{T-2,0}^{T-1} \right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0} \delta'_{0,\tau,0} \left[ \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) x_\tau - \Delta\bar{x}_{t,0}^\tau \right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0} \delta'_{0,\tau,0} \left[ \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) x_\tau - \Delta\bar{x}_{1,0}^\tau \right] \end{bmatrix} + \begin{bmatrix} 0_{(d_x+K) \times 1} \\ \beta X_{T,T-2,0} \Delta'_{T-1,0} \left[ \left( \frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) p_{T-1,0} x_{T-1} - \Delta\bar{x}_{1,T-2,0}^{T-1} \right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0} \Delta'_{\tau,0} \left[ \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) p_{\tau,0} x_\tau - \Delta\bar{x}_{1,t,0}^\tau \right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0} \Delta'_{\tau,0} \left[ \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) p_{\tau,0} x_\tau - \Delta\bar{x}_{1,1,0}^\tau \right] \end{bmatrix}. \quad (\text{B.3})$$

Note that the first term in eq. (B.3) adjusts for the estimated  $\widetilde{\Delta\bar{x}_t^\tau}$ , and the second,  $\widetilde{\Delta\bar{x}_{1,t}^\tau}$ .

### B.2.3 Adjustment Term for Estimated $\Delta\bar{q}^K$ Functions

To compute the adjustment term for estimated  $\Delta\bar{q}^K$  functions, we first need to compute

$$M_{t,\Delta\bar{q}_{t'}^K} \equiv \frac{\partial}{\partial \widetilde{\Delta\bar{q}_{t'}^K}} m_t(D, \delta_0, \gamma_0^K, p_0, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \widetilde{\Delta\bar{q}_{t'}^K}, \Delta\bar{q}_{t',0}^{K,c}) \Big|_{\widetilde{\Delta\bar{q}_{t'}^K} = \Delta\bar{q}_{t',0}^K},$$

where  $\Delta\bar{q}_{t'}^{K,c}$  denotes all the coordinates of  $\Delta\bar{q}^K$  other than  $\Delta\bar{q}_{t'}^K$ .

Through basic algebra, we get  $M_{t,\Delta\bar{q}_t^K} = 0_{((2T-2t-1)d_x+K)\times K}$  for  $t = 1, \dots, T-1$  and  $t' \neq t$ , and  $M_{t,\Delta\bar{q}_t^K} = \beta^{T-t} X_{T,t,0} \gamma_0^{K'}$  for  $t = 1, \dots, T-1$ . As a result,

$$\alpha_q(D) = \begin{bmatrix} \beta X_{T,T-1,0} \gamma_0^{K'} \left[ \left( \frac{a_{T-1}}{p_{T-1,0}} - \frac{1-a_{T-1}}{1-p_{T-1,0}} \right) q^K(x_T, z_T) - \Delta\bar{q}_{T-1,0}^K \right] \\ \vdots \\ \beta^{T-t} X_{T,t,0} \gamma_0^{K'} \left[ \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) q^K(x_T, z_T) - \Delta\bar{q}_{t,0}^K \right] \\ \vdots \\ \beta^{T-1} X_{T,1,0} \gamma_0^{K'} \left[ \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) q^K(x_T, z_T) - \Delta\bar{q}_{1,0}^K \right] \end{bmatrix}. \quad (\text{B.4})$$

#### B.2.4 Adjustment Terms for Estimated CCPs

We first compute  $\alpha_{p,direct}(D)$ , which captures the impact of estimated CCPs that appear directly in the moment functions. To do so, we need to compute

$$M_{t,p_{t'}} \equiv \frac{\partial}{\partial \widetilde{p_{t'}}} m_t(D, \delta_0, \gamma_0^K, \widetilde{p_{t'}}, p_{t',0}^c, \Delta\bar{\eta}_0, \Delta\bar{x}_0, \Delta\bar{q}_0^K) \Big|_{\widetilde{p_{t'}}=p_{t',0}},$$

where  $p_{t'}^c$  denotes all the coordinates of  $p$  other than  $p_{t'}$ . Note that the CCPs only appear directly as the first term of the contemporaneous  $y_t$ , in the form of  $\ln\left(\frac{p_t}{1-p_t}\right)$ , so  $M_{t,p_{t'}} = 0_{((2T-2t-1)d_x+K)\times 1}$  for  $t = 1, \dots, T-1$  and  $t' \neq t$ , and  $M_{t,p_t} = X_{T,t,0}/(p_{t,0}(1-p_{t,0}))$  for  $t = 1, \dots, T-1$ . As a result,

$$\alpha_{p,direct}(D) = \begin{bmatrix} \frac{X_{T,T-1,0}}{p_{T-1,0}(1-p_{T-1,0})} (a_{T-1} - p_{T-1,0}) \\ \vdots \\ \frac{X_{T,t,0}}{p_{t,0}(1-p_{t,0})} (a_t - p_{t,0}) \\ \vdots \\ \frac{X_{T,1,0}}{p_{1,0}(1-p_{1,0})} (a_1 - p_{1,0}) \end{bmatrix}. \quad (\text{B.5})$$

We then compute  $\alpha_{p,indirect}(D)$ , which captures the impact of estimated CCPs that appear as the “dependent variables” of the unknown functions  $\Delta\bar{\eta}$  and  $\Delta\bar{x}$ . Due to eq. (32),  $p_\tau$  is part of the numerators of the “dependent variables” of the nonparametric regressions for  $\Delta\bar{\eta}_t^\tau$  and  $\Delta\bar{x}_{1,t}^\tau$ , and  $p_t$  is part of their denominators. For  $\tau > t$  and  $t = 1, \dots, T-2$ , define

$$\begin{aligned} \mathcal{H}_{t,p_\tau}^\tau &\equiv \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) \ln \frac{p_{\tau,0}}{1-p_{\tau,0}}, & \mathcal{H}_{t,p_t}^\tau &\equiv - \left( \frac{a_t}{p_{t,0}^2} + \frac{1-a_t}{(1-p_{t,0})^2} \right) \eta_{\tau,0}, \\ \mathcal{X}_{t,p_\tau}^\tau &\equiv \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) x_\tau, & \text{and } \mathcal{X}_{t,p_t}^\tau &\equiv - \left( \frac{a_t}{p_{t,0}^2} + \frac{1-a_t}{(1-p_{t,0})^2} \right) p_{\tau,0} x_\tau, \end{aligned}$$

where we recall that  $\eta_\tau$  is defined in eq. (14). By chain rule, we combine these with  $M_{t,\Delta\bar{\eta}_t^\tau}$  and

$M_{t,\Delta\bar{x}_{1,t}^\tau}$  for  $t = 1, \dots, T-2$  and  $\tau = t+1, \dots, T-1$  to get

$$\begin{aligned}
\alpha_{p,indirect}(D) = & - \left[ \begin{array}{c} 0_{(d_x+K) \times 1} \\ \beta \ln \frac{p_{T-1,0}}{1-p_{T-1,0}} \mathbb{E} \left( X_{T,T-2,0} \left( \frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) | x_{T-1}, z_{T-1} \right) (a_{T-1} - p_{T-1,0}) \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \ln \frac{p_{\tau,0}}{1-p_{\tau,0}} \mathbb{E} \left( X_{T,t,0} \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) | x_\tau, z_\tau \right) (a_\tau - p_{\tau,0}) \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} \ln \frac{p_{\tau,0}}{1-p_{\tau,0}} \mathbb{E} \left( X_{T,1,0} \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) | x_\tau, z_\tau \right) (a_\tau - p_{\tau,0}) \end{array} \right] \\
& + \left[ \begin{array}{c} 0_{(d_x+K) \times 1} \\ \beta X_{T,T-2,0} \left( \frac{\mathbb{E}(\eta_{T-1,0} a_{T-2} | x_{T-2}, z_{T-2})}{p_{T-2,0}^2} + \frac{\mathbb{E}(\eta_{T-1,0} (1-a_{T-2}) | x_{T-2}, z_{T-2})}{(1-p_{T-2,0})^2} \right) (a_{T-2} - p_{T-2,0}) \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0} \left( \frac{\mathbb{E}(\eta_{\tau,0} a_t | x_t, z_t)}{p_{t,0}^2} + \frac{\mathbb{E}(\eta_{\tau,0} (1-a_t) | x_t, z_t)}{(1-p_{t,0})^2} \right) (a_t - p_{t,0}) \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0} \Delta'_{\tau,0} \left( \frac{\mathbb{E}(\eta_{\tau,0} a_1 | x_1, z_1)}{p_{1,0}^2} + \frac{\mathbb{E}(\eta_{\tau,0} (1-a_1) | x_1, z_1)}{(1-p_{1,0})^2} \right) (a_1 - p_{1,0}) \end{array} \right] \\
& + \left[ \begin{array}{c} 0_{(d_x+K) \times 1} \\ \beta \Delta'_{T-1,0} \mathbb{E} \left( X_{T,T-2,0} \left( \frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) | x_{T-1}, z_{T-1} \right) x_{T-1} (a_{T-1} - p_{T-1,0}) \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta'_{\tau,0} \mathbb{E} \left( X_{T,t,0} \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) | x_\tau, z_\tau \right) x_\tau (a_\tau - p_{\tau,0}) \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} \Delta'_{\tau,0} \mathbb{E} \left( X_{T,1,0} \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) | x_\tau, z_\tau \right) x_\tau (a_\tau - p_{\tau,0}) \end{array} \right] \\
& - \left[ \begin{array}{c} 0_{(d_x+K) \times 1} \\ \beta X_{T,T-2,0} \Delta'_{T-1,0} \left( \frac{\mathbb{E}(p_{T-1,0} x_{T-1} a_{T-2} | x_{T-2}, z_{T-2})}{p_{T-2,0}^2} + \frac{\mathbb{E}(p_{T-1,0} x_{T-1} (1-a_{T-2}) | x_{T-2}, z_{T-2})}{(1-p_{T-2,0})^2} \right) (a_{T-2} - p_{T-2,0}) \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0} \Delta'_{\tau,0} \left( \frac{\mathbb{E}(p_{\tau,0} x_\tau a_t | x_t, z_t)}{p_{t,0}^2} + \frac{\mathbb{E}(p_{\tau,0} x_\tau (1-a_t) | x_t, z_t)}{(1-p_{t,0})^2} \right) (a_t - p_{t,0}) \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0} \Delta'_{\tau,0} \left( \frac{\mathbb{E}(p_{\tau,0} x_\tau a_1 | x_1, z_1)}{p_{1,0}^2} + \frac{\mathbb{E}(p_{\tau,0} x_\tau (1-a_1) | x_1, z_1)}{(1-p_{1,0})^2} \right) (a_1 - p_{1,0}) \end{array} \right].
\end{aligned} \tag{B.6}$$

Note that the first and the second terms in eq. (B.6) adjust for the estimated  $\hat{p}_\tau$  and  $\hat{p}_t$ , respectively, in  $\widehat{\Delta\bar{\eta}}_t^\tau$ , and the third and the fourth terms adjust for the estimated  $\hat{p}_\tau$  and  $\hat{p}_t$ , respectively, in  $\widehat{\Delta\bar{x}}_{1,t}^\tau$ .

## C Time-invariant and Exogenously Evolving Variables in $x_t$

In this section, we modify the theorems in Section 3.2 and Section 3.3 to accommodate the possibility that  $x_t$  may contain time-invariant and exogenously evolving variables.

Following the notation in Remark 5, let  $x_t^*$  be the  $d_{x^*} \times 1$  ( $0 \leq d_{x^*} \leq d_x$ ) subvector of time-varying and endogenously evolving state variables in  $x_t$ , and let  $\delta_{0,t}^*$  denote the corresponding subvector  $\delta_{0,t}$  (for  $t = 2, \dots, T-1$ ) that can be identified. Our goal is to give identification condition and estimation procedure for  $\delta^* \equiv (\Delta'_1, \delta_{0,2}^{*'}, \Delta'_2, \dots, \delta_{0,T-1}^{*'}, \Delta'_{T-1})'$  and  $\gamma^K$ . All the theorems in this section have an apostrophe to emphasize their connection to those in the main text.

## C.1 Identification

Equation (18) becomes a system of linear equations of  $\Delta_t$  ( $t = 1, \dots, T-1$ ),  $\delta_{0,t}^*$  ( $t = 2, \dots, T-1$ ) and  $\gamma^K$ :

$$\begin{aligned} y_{T-1} &= x'_{T-1} \Delta_{T-1} + \beta \Delta \bar{q}'_{T-1} \gamma^K, \text{ and} \\ y_t &= x'_t \Delta_t + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_t^{*\tau'} \delta_{0,\tau}^* + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta \bar{x}_{1,t}^{\tau'} \Delta_\tau \\ &\quad + \beta^{T-t} \Delta \bar{q}_t^{K'} \gamma^K \end{aligned}$$

for  $t = 1, \dots, T-2$ . Based on these equations, we define

$$\begin{aligned} \underbrace{X_{T,T-1}^*}_{(d_x+K) \times 1} &\equiv X_{T,T-1} \text{ and} \\ \underbrace{X_{T,t}^*}_{((T-t)d_x+(T-t-1)d_{x^*}+K) \times 1} &\equiv (x'_t, \beta \Delta \bar{x}_t^{*t+1'}, \beta \Delta \bar{x}_{1,t}^{t+1'}, \dots, \beta^{T-1-t} \Delta \bar{x}_t^{*T-1'}, \beta^{T-1-t} \Delta \bar{x}_{1,t}^{T-1'}, \beta^{T-t} \Delta \bar{q}_t^{K'})' \end{aligned}$$

for  $T \geq 2$  and  $t = 1, 2, \dots, T-2$ , and let

$$\begin{aligned} \underbrace{\tilde{L}_{T,t}^*}_{((T-t)d_x+(T-t-1)d_{x^*}+K) \times ((T-t)d_x+(T-t-1)d_{x^*}+K)} &\equiv \mathbb{E}(X_{T,t}^* X_{T,t}^{*'}) \text{ and} \\ \underbrace{L_{T,t}^*}_{((T-t)d_x+(T-t-1)d_{x^*}+K) \times ((T-1)d_x+(T-2)d_{x^*}+K)} &\equiv \begin{bmatrix} 0_{((T-t)d_x+(T-t-1)d_{x^*}+K) \times ((t-1)d_x+(t-1)d_{x^*})} & \tilde{L}_{T,t}^* \end{bmatrix} \end{aligned}$$

for  $T \geq 2$  and  $t = 1, \dots, T-1$ . Like in the main text, we stack  $\tilde{L}_{T,t}^*$  for  $t = T-1, \dots, 1$  (in that order) to get the  $(T(T-1)d_x/2 + (T-1)(T-2)d_{x^*}/2 + (T-1)K) \times ((T-1)d_x + (T-2)d_{x^*} + K)$  block-triangular Jacobian matrix  $L_T^*$ :

$$L_T^* = \begin{bmatrix} 0_{(d_x+K) \times (d_x+d_{x^*})} & 0_{(d_x+K) \times ((T-4)d_x+(T-4)d_{x^*})} & 0_{(d_x+K) \times (d_x+d_{x^*})} & \mathbb{E}(X_{T,T-1}^* X_{T,T-1}^{*'}) \\ 0_{(2d_x+d_{x^*}+K) \times (d_x+d_{x^*})} & 0_{(2d_x+d_{x^*}+K) \times ((T-4)d_x+(T-4)d_{x^*})} & & \mathbb{E}(X_{T,T-2}^* X_{T,T-2}^{*'}) \\ \vdots & & \ddots & \\ 0_{((T-2)d_x+(T-3)d_{x^*}+K) \times (d_x+d_{x^*})} & & & \mathbb{E}(X_{T,2}^* X_{T,2}^{*'}) \\ & & & \mathbb{E}(X_{T,1}^* X_{T,1}^{*'}) \end{bmatrix}.$$

**Theorem 3'** (Identification and over-identification of  $\delta^*$  and  $\gamma^K$  when  $x_t$  contains time-invariant or exogenously evolving variables). *Under Assumptions 1 to 7, (i) the parameters  $\delta^*$  and  $\gamma^K$  are identified if and only if the  $L_T^*$  matrix defined above has full column rank (i.e.,  $(T-1)d_x + (T-2)d_{x^*} + K$ ); (ii) if more than one matrix that consists of  $(T-1)d_x + (T-2)d_{x^*} + K$  distinct rows from  $L_T^*$  has full rank, then the parameters  $\delta^*$  and  $\gamma^K$  are over-identified.*

**Corollary 1'**. *Under Assumptions 1 to 7,  $\delta^*$  and  $\gamma^K$  are identified if the second moment matrix of  $(x'_1, \Delta\bar{x}_1^{*2'}, \Delta\bar{x}_{1,1}^{2'}, \dots, \Delta\bar{x}_1^{*\tau'}, \Delta\bar{x}_{1,1}^{\tau'}, \dots, \Delta\bar{x}_1^{*T-1'}, \Delta\bar{x}_{1,1}^{T-1'}, \Delta\bar{q}_1^{K'})'$  has full rank.*

**Corollary 2'**. *Under Assumptions 1 to 7,  $\delta^*$  and  $\gamma^K$  are identified if: (i) the second moment matrix of  $(x'_{T-1}, \Delta\bar{q}_{T-1}^{K'})'$  has full rank (i.e.,  $d_x + K$ ); and (ii) the second moment matrix of  $(x'_t, \Delta\bar{x}_t^{*t+1'})'$  has full rank (i.e.,  $d_x + d_{x^*}$ ) for all  $t = 1, \dots, T-2$ .*

## C.2 Estimation

We use  $\Delta\bar{x}^*$  to collectively denote all  $\Delta\bar{x}_{1,t}^{\tau'}$  and  $\Delta\bar{x}_t^{*\tau}$  ( $t = 1, \dots, T-2$ ,  $t < \tau \leq T-1$ ). Redefine the moment functions as:

$$\begin{aligned} m_{T-1}^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K) &\equiv -v_{T-1}^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K) X_{T,T-1}^*, \text{ and} \\ m_t^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K) &\equiv -v_t^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K) X_{T,t}^*, \end{aligned}$$

where  $v_{T-1}^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K) \equiv y_{T-1} - x'_{T-1} \Delta_{T-1} - \beta \Delta\bar{q}_{T-1}^{K'} \gamma^K$  and  $v_t^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K) \equiv y_t - x'_t \Delta_t - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta\bar{x}_t^{*\tau'} \delta_{0,\tau}^* - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \Delta\bar{x}_{1,t}^{\tau'} \Delta_\tau - \beta^{T-t} \Delta\bar{q}_t^{K'} \gamma^K$  for  $t = 1, \dots, T-2$ . Let  $m^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K)$  be the stack of the moment functions  $m_t^*(D, \delta^*, \gamma^K, p, \Delta\bar{\eta}, \Delta\bar{x}^*, \Delta\bar{q}^K)$  for  $t = T-1, \dots, 1$  (in that order).  $\delta^*$  and  $\gamma^K$  can still be estimated through steps (I) to (III) in Section 3.3, provided that we replace  $\Delta\bar{x}_t^{\tau'}$  with  $\Delta\bar{x}_t^{*\tau}$  in the collection of “dependent variables”  $\Delta\bar{x}$  in step (II) (now denoted as  $\Delta\bar{x}^*$ ) and let

$$(\hat{\delta}^{*t}, \hat{\gamma}^{K'})' \equiv \arg \min_{\delta^* \in \mathbb{R}^{(T-1)d_x + (T-2)d_{x^*}}, \gamma^K \in \mathbb{R}^K} \bar{m}_N(\delta^*, \gamma^K)' W_N^* \bar{m}_N(\delta^*, \gamma^K), \quad (\text{C.1})$$

where  $\bar{m}_N^*(\delta^*, \gamma^K) \equiv \frac{1}{N} \sum_{i=1}^N \hat{m}^*(D_i, \delta^*, \gamma^K)$ ,  $\hat{m}^*(D, \delta^*, \gamma^K) \equiv m^*(D, \delta^*, \gamma^K, \hat{p}, \widehat{\Delta\bar{\eta}}, \widehat{\Delta\bar{x}^*}, \widehat{\Delta\bar{q}^K})$  and  $W_N^*$  is a symmetric weighting matrix of conformable dimensions that converges in probability to a positive definite matrix  $W^*$  as  $N \rightarrow \infty$ . Note that the “regressors” of the nonparametric regressions in step (II) are still  $s_t$ , including the time-invariant and exogenously evolving variables in  $x_t$ .

**Proposition 1'** (Asymptotic distribution of  $\hat{\delta}^*$  when  $x_t$  contains time-invariant or exogenously evolving variables). *Under Assumptions 1 to 7 and the regularity conditions in Section 5 of Newey (1994), the three-step CCP-based semiparametric estimator  $\hat{\delta}^*$  defined in eq. (C.1) has the asymptotic distribution*

$$\sqrt{N} (\hat{\delta}^* - \delta^*) \xrightarrow{d} \mathcal{N}(0, V^*),$$

where  $V^* \equiv \mathbb{E}(\psi_{\delta^*}(D_i)\psi'_{\delta^*}(D_i))$ , and  $\psi_{\delta^*}(\cdot)$  is the first  $(T-1)d_x + (T-2)d_{x^*}$  coordinates of the following influence function  $\psi^*(\cdot)$ :

$$\psi^*(D) \equiv -(L_T^*W^*L_T^*)^{-1}L_T^*W^*\alpha^*(D),$$

in which

$$\begin{aligned} \alpha^*(D) &\equiv \alpha_\eta(D) + \alpha_{x^*}(D) + \alpha_q(D) + \alpha_{p,direct}(D) + \alpha_{p,indirect}(D), \\ \alpha_{x^*}(D) &\equiv \begin{bmatrix} 0_{(d_x+K) \times 1} \\ \beta X_{T,T-2,0}^* \delta_{0,T-1,0}^{*'} \left[ \left( \frac{a_{T-2}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) x_{T-1}^* - \Delta \bar{x}_{T-2,0}^{*T-1} \right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0}^* \delta_{0,\tau,0}^{*'} \left[ \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) x_\tau^* - \Delta \bar{x}_{t,0}^{*\tau} \right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0}^* \delta_{0,\tau,0}^{*'} \left[ \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) x_\tau^* - \Delta \bar{x}_{1,0}^{*\tau} \right] \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{(d_x+K) \times 1} \\ \beta X_{T,T-1,0}^* \Delta'_{T-2,0} \left[ \left( \frac{a_{T-1}}{p_{T-2,0}} - \frac{1-a_{T-2}}{1-p_{T-2,0}} \right) p_{T-1,0} x_{T-1} - \Delta \bar{x}_{1,T-2,0}^{T-1} \right] \\ \vdots \\ \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} X_{T,t,0}^* \Delta'_{\tau,0} \left[ \left( \frac{a_t}{p_{t,0}} - \frac{1-a_t}{1-p_{t,0}} \right) p_{\tau,0} x_\tau - \Delta \bar{x}_{1,t,0}^\tau \right] \\ \vdots \\ \sum_{\tau=2}^{T-1} \beta^{\tau-1} X_{T,1,0}^* \Delta'_{\tau,0} \left[ \left( \frac{a_1}{p_{1,0}} - \frac{1-a_1}{1-p_{1,0}} \right) p_{\tau,0} x_\tau - \Delta \bar{x}_{1,1,0}^\tau \right] \end{bmatrix}, \end{aligned}$$

and  $\alpha_\eta(D)$ ,  $\alpha_q(D)$ ,  $\alpha_{p,direct}(D)$  and  $\alpha_{p,indirect}(D)$  are the same as those defined above in eq. (B.2), eq. (B.4), eq. (B.5) and eq. (B.6), respectively, except for the only difference that  $X_{T,t,0}$  is replaced by  $X_{T,t,0}^*$  for  $t = 1, \dots, T-1$  in every occurrence. Note that again, the influence function consists of only the “adjustment terms” since  $m_0^*(D) \equiv m_0(D, \delta_0^*, \gamma_0^K, p_0, \Delta \bar{\eta}_0, \Delta \bar{x}_0^*, \Delta \bar{q}_0^K) = 0$ .

A consistent estimator of  $V^*$  is provided in eq. (34), with relevant objects replaced by their “\*” counterparts whenever applicable. Again, its consistency can be justified by a proposition that is essentially the same as Proposition 2, which we omit here for conciseness.

## D Proofs of the Results in Section 4.1

*Proof of Lemma 2.* By the definition of  $\Delta \bar{r}_t^K$  and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}((\Delta \bar{r}_t^K)^2) &= \mathbb{E}([\Delta \mathbb{E}(r^K(x_T, z_T) | x_t, z_t)]^2) \\ &= \mathbb{E}([\mathbb{E}(r^K(x_T, z_T) | x_t, z_t, a_t = 1) - \mathbb{E}(r^K(x_T, z_T) | x_t, z_t, a_t = 0)]^2) \\ &\leq 2\mathbb{E}([\mathbb{E}(r^K(x_T, z_T) | x_t, z_t, a_t = 1)]^2) \\ &\quad + 2\mathbb{E}([\mathbb{E}(r^K(x_T, z_T) | x_t, z_t, a_t = 0)]^2). \end{aligned} \tag{D.1}$$

Under the conditions (i) to (iv) of this lemma and by Theorem 8 of [Lorentz \(1966, p. 90\)](#), we have

$$\sup_{s \in \mathcal{S}} |r^K(s)| \leq C_1 K^{-\frac{m}{d_s}},$$

for some constant  $C_1$ . By this uniform bound of  $|r^K(s)|$  and the Jensen's inequality, we have

$$\begin{aligned} [\mathbb{E}(r^K(x_T, z_T) | x_t, z_t, a_t = a)]^2 &\leq \mathbb{E}(|r^K(x_T, z_T)|^2 | x_t, z_t, a_t = a) \\ &\leq C_1^2 K^{-\frac{2m}{d_s}} \text{ for } a = 0, 1. \end{aligned} \quad (\text{D.2})$$

Combining eq. (D.1) and eq. (D.2), we get

$$\mathbb{E}((\Delta \bar{r}_t^K)^2) \leq 4C_1^2 K^{-\frac{2m}{d_s}},$$

and the result of the lemma follows. ■

*Proof of Theorem 4.* It is easy to see that the probability limit of the linear MD estimators  $(\hat{\delta}, \hat{\gamma}^K)'$  is  $(\delta_{pseudo}^{K'}, \gamma_{pseudo}^{K'})' = (L'WL)^{-1} L'WR$ . Recalling the definitions of  $R$  and  $\Delta \bar{r}_t^K$  ( $t = 1, \dots, T-1$ ), we get

$$\begin{aligned} \left\| \begin{bmatrix} \delta_{pseudo}^K - \delta \\ \gamma_{pseudo}^K - \gamma^K \end{bmatrix} \right\| &\leq \lambda_{max} \left( (L'WL)^{-1} \right) \sqrt{\lambda_{max}(LL')} \lambda_{max}(W) \left\| \begin{bmatrix} \mathbb{E}(X_{T,T-1} \Delta \bar{r}_{T-1}^K) \\ \vdots \\ \mathbb{E}(X_{T,1} \Delta \bar{r}_1^K) \end{bmatrix} \right\| \\ &= \frac{\sqrt{\lambda_{max}(LL')} \lambda_{max}(W)}{\lambda_{min}(L'WL)} \sqrt{\sum_{t=1}^{T-1} \|\mathbb{E}(X_{T,t} \Delta \bar{r}_t^K)\|^2} \\ &\leq C_2 \sqrt{\sum_{t=1}^{T-1} \|\mathbb{E}(X_{T,t} \Delta \bar{r}_t^K)\|^2} \\ &\leq C_3 K^{-\frac{m}{d_s}} \sqrt{\sum_{t=1}^{T-1} \|\sqrt{\mathbb{E}(X_{T,t}^2)}\|} \\ &\leq C_4 K^{-\frac{m}{d_s}}, \end{aligned}$$

where  $C_2$  is some constant. In the above, the first inequality holds by the Cauchy-Schwarz inequality and the properties of eigenvalues, the first equality holds by the property of eigenvalues and the definition of the Frobenius norm, the second inequality holds by the conditions on the eigenvalues of  $L'L$  and  $W$ , the third inequality holds by the Cauchy-Schwarz inequality and Lemma 2, and the last inequality holds by finite second moments of  $q^K(x_T, z_T)$  and  $x_t$  for  $t = 1, \dots, T-1$ . Then, the result of the proposition follows. ■

## E Generating Simulation Sample

To generate a simulation sample from the model specified in Section 5.1 is not a trivial task, because all the state variables in our parameterization are continuous. In this appendix, we provide

details on the three steps we took to solve the dynamic programming (DP) problem to generate a simulation sample. Section E.1 describes how to discretize the choice-specific VAR(1) state transition processes, so the DP problem with continuous state variables can be approximated by a DP problem with discrete states. Section E.2 explains how to solve the DP problem with discrete states backwardly. Section E.3 describes how to draw a simulation sample forwardly.

## E.1 Efficient Discretization of States and State Transition Distributions

In general, randomly drawing state variables  $s_t$  from their stationary distribution is not efficient, especially when  $d_s$  is large, because most of the draws will end up in regions with low probabilities. In addition, the VAR(1) processes of  $s_t$  in our model depend on whether  $a_t = 0$  or  $a_t = 1$  is chosen.

The original DP problem in Section 5.1 involves a  $3 \times 1$  vector of continuous state variables  $s_{i,t}$ , whose transition is governed by two choice-specific VAR(1) processes.<sup>39</sup> Let  $f(s_{t+1} | a_t = a)$  for  $a = 0, 1$  denote the density function of the stationary distribution of  $s_{t+1}$  when  $a_t = a$ . The choice  $a_t = 1$  is a “reset” choice such that the distribution of the next-period state variables  $s_{t+1}$  does not depend on the current state variables  $s_t$ , that is,  $f(s_{t+1} | s_t, a_t = 1) = f(s_{t+1} | a_t = 1)$ . It is tempting to separately discretize the two choice-specific VAR(1) processes, but this will result in complicated choice-specific discretized state transition matrices.

To make the discretized state transition matrices simple, we impose the restriction that the stationary distributions of the state variables to be the same across both choices:  $f(s_{t+1} | a_t = 1) = f(s_{t+1} | a_t = 0)$ . We first discretize the VAR(1) process when  $a_t = 0$  by using the “EDS” method, proposed by Maliar and Maliar (2015) and specialized for a VAR(1) process by Gordon (2021) to obtain a discrete grid of  $s_t$  values that consists of only 1000 points but approximates the stochastic behavior of the original VAR(1) process well.<sup>40</sup> Let  $\vec{s} = (s^1, \dots, s^{1000})'$  denote the 1000 “EDS” grid points, and each point  $s^j$  ( $j = 1, \dots, 1000$ ) is to a vector of  $(x_1, x_2, z)'$  values. The “EDS” method also gives us the discretized state transition matrix  $f^{dis}(s_{t+1} = s^j | s_t = s^{j'}, a_t = 0)$  and the stationary probability mass function  $f^{dis}(s_{t+1} = s^j | a_t = 0)$ , for  $\forall j, j' = 1, \dots, 1000$ . This probability mass function equals to each row of the state transition matrix  $f^{dis}(s_{t+1} = s^j | s_t = s^{j'}, a_t = 1)$  when  $a_t = 1$ , because  $f(s_{t+1} | s_t, a_t = 1) = f(s_{t+1} | a_t = 0)$ .

## E.2 Solving the Model Backwardly with the Discrete States

Having obtained the discrete states and discrete state transition matrices, we can solve the discretized DP problem backwardly.

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<sup>39</sup>Our method can handle larger  $d_s$ , but we chose  $d_s = 3$  because simulating the sequences of state variables and choices to implement the HMSS estimator puts a rapidly increasing strain on memory as  $d_s$  increases.

<sup>40</sup>The number of points is up to the researchers’ choice.

### E.2.1 $t = 3$ (Decision Terminal period)

We start from the terminal period. First, the choice-specific expected payoff in period  $t = 3$  is

$$v_3(a, s^j) = u_3(a, x^j) = \delta_{a,3,0} + \delta_{a,3,1}x_1^j + \delta_{a,3,2}x_2^j, \text{ for } j = 1, \dots, 1000, \quad (\text{E.1})$$

where  $x^j = (x_1^j, x_2^j)'$  is the first  $2 \times 1$  subvector of  $s^j$ , and we get a vector  $\vec{v}_3(a) \equiv (v_3(a, s^1), \dots, v_3(a, s^{1000}))'$  using this formula. Then, the CCPs in period  $t$  is

$$p_t(s^j) = \frac{\exp v_t(1, s^j)}{\exp v_t(0, s^j) + \exp v_t(1, s^j)}, \text{ for } j = 1, \dots, 1000. \quad (\text{E.2})$$

Let  $t = 3$  in eq. (E.2) and we get a vector  $\vec{p}_3 \equiv (p_3(s^1), \dots, p_3(s^{1000}))'$ . Lastly, the expected optimal payoff in period  $t$  (i.e., the value of the integrated value function) is (see [Arcidiacono and Miller, 2011](#), for example)

$$\bar{V}_t(s^j) = v_t(0, s^j) - \ln(1 - p_t(s^j)), \text{ for } j = 1, \dots, 1000. \quad (\text{E.3})$$

Let  $t = 3$  in eq. (E.3) and we get a vector  $\vec{\bar{V}}_3 \equiv (\bar{V}_3(s^1), \dots, \bar{V}_3(s^{1000}))'$ .

### E.2.2 $t = 2$

The choice-specific expected payoff in non-terminal period  $t$  is

$$v_t(a, s^j) = u_t(a, x^j) + \beta \mathbb{E}(\bar{V}_{t+1}(s^j) \mid s_t = s^j, a_t = a), \quad (\text{E.4})$$

where  $u_t(a, x^j)$  is computed in a similar way as  $u_3(a, x^j)$  in eq. (E.1). For  $t = 2$ , the second term in eq. (E.4) can be numerically computed using the vector  $\vec{\bar{V}}_3$  obtained in Section E.2.1 and the choice-specific discrete state transition matrices  $f^{dis}(s_{t+1} \mid s_t, a_t)$  obtained in Section E.1. Let  $\vec{v}_2(a) \equiv (v_2(a, s^1), \dots, v_2(a, s^{1000}))'$ . Let  $t = 2$  in eq. (E.2) eq. (E.3) and plug in  $v_2(a, s^j)$  for  $j = 1, \dots, 1000$ , then we get a vector  $\vec{p}_2 = (p_2(s^1), \dots, p_2(s^{1000}))'$  and another vector  $\vec{\bar{V}}_2 \equiv (\bar{V}_2(s^1), \dots, \bar{V}_2(s^{1000}))' = \vec{v}_2(0) - \ln(1 - \vec{p}_2)$ .

### E.2.3 $t = 1$

Let  $t = 1$  in eq. (E.4), in which again the second term can be numerically computed using the vector  $\vec{\bar{V}}_2$  obtained in Section E.2.2 and the choice-specific discrete state transition matrices  $f^{dis}(s_{t+1} \mid s_t, a_t)$  obtained in Section E.1. We let  $\vec{v}_1(a) \equiv (v_1(a, s^1), \dots, v_1(a, s^{1000}))'$ . Let  $t = 1$  in eq. (E.2) and we get  $\vec{p}_1 = (p_1(s^1), \dots, p_1(s^{1000}))'$ .  $\vec{\bar{V}}_1$  is unnecessary since  $t = 1$  is the sample initial period.

## E.3 Simulate A Sample Forwardly with the Original State Space

Section E.2 gives the solution of the DP problem on the discrete ‘‘EDS’’ grid points  $s^1, \dots, s^{1000}$  – the CCP vectors  $\vec{p}_1, \vec{p}_2, \vec{p}_3$ . Before simulating a sample, however, we need to interpolate CCPs

for any continuous value of  $s$  using  $\tilde{p}_t$  as observations. In particular, for each  $t = 1, 2, 3$ , we regress  $\ln(p_t(s^j)/(1 - p_t(s^j)))$  on a power series of  $x_1^j$ ,  $x_2^j$  and  $z^j$  ( $j = 1, \dots, 1000$ ) to get the approximated CCP function  $\tilde{p}_t(x_1, x_2, z)$ .

Then we simulate a sample forwardly. For period  $t = 1$ , we first generate  $N = 250$  random draws of  $s_1 = (x_{1,1}, x_{1,2}, z_1)'$  from the stationary distribution of  $s_1$ , which is the same across the choices 0 and 1. We then evaluate the approximated CCP function  $\tilde{p}_1(x_{1,1}, x_{1,2}, z_1)$  at the  $s_1$  draws and randomly generate a choice  $a_1$  for each draw. For period  $t = 2$ , we first generate random draws of  $s_2$  from choice-specific VAR(1) processes based on the  $s_1$  and  $a_1$  values. We then evaluate the approximated CCP function  $\tilde{p}_2(x_{2,1}, x_{2,2}, z_2)$  at the  $s_2$  draws and randomly generate a choice  $a_2$  for each draw. For period  $t = 3$ , repeat what we did for  $t = 2$  with all time indices increased by one.

## F Primitive Conditions for Proposition 1

The regularity conditions in Section 5 of [Newey \(1994\)](#), alluded to in Proposition 1, may be fulfilled under various primitive conditions for different estimators used in steps (I) and (II) of our estimation procedure. In this section, we elaborate the primitive conditions under which Proposition 1 holds for series estimators and provide the proof.

### F.1 Series Estimators

Series estimator could be used to nonparametrically estimator the CCP functions and the conditional mean difference functions in steps (I) and (II).

We follow [Hirano, Imbens and Ridder \(2003\)](#) to use the Series Logit Estimator (SLE) for the CCPs in step (I) as they are conditional probabilities. For any positive integer  $J$ , let  $R^J(s) \equiv (r_{1J}(s), r_{2J}(s), \dots, r_{JJ}(s))'$  be a  $J$ -vector of functions, and we focus on the orthonormal power series described in Appendix A of [Hirano, Imbens and Ridder \(2003\)](#) in particular. Define  $\zeta(J) \equiv \sup_{s \in \mathcal{S}} \|R^J(s)\|$  where  $\|\cdot\|$  denotes the Frobenius norm, then we have  $\zeta(J) \leq CJ$  ([Hirano, Imbens and Ridder, 2003](#), p. 1177). If a function  $g$  is  $\nu_g$  times continuously differentiable, then by Theorem 8 of [Lorentz \(1966, p. 90\)](#), there exist a  $\pi_J$  such that on the compact set  $\mathcal{S}$ ,

$$\sup_{s \in \mathcal{S}} \|g(s) - R^J(s)' \pi_J\| \leq CJ^{-\frac{\nu_g}{d_s}}. \quad (\text{F.1})$$

We denote the logistic cumulative distribution function by  $L(u) = \exp(u)/(1 + \exp(u))$ , then the SLE for the  $t$ -period CCP function  $p_t(s)$  is defined by  $\hat{p}_{t,J}(s) \equiv L(R^J(s)' \hat{\pi}_{t,J})$  with

$$\hat{\pi}_{t,J} \equiv \arg \max_{\pi} \sum_{i=1}^N (a_{i,t} \cdot \ln L(R^J(s_{i,t})' \pi) + (1 - a_{i,t}) \cdot \ln(1 - L(R^J(s_{i,t})' \pi))).$$

Let  $p_{i,t} \equiv p_t(s_{i,t})$  and  $\hat{p}_{i,t,J} \equiv \hat{p}_{t,J}(s_{i,t})$ . For a typical conditional mean difference function  $\Delta \mathbb{E}(h_{\tau} | s_t = s)$  in step (II), we use  $L$  orthonormal power series, so the series IPW estimator (see

Remark 10) is  $\Delta \mathbb{E}(\widehat{h_\tau} | s_t = s) \equiv R^L(s)' \hat{\omega}_{h,\tau,L}$  with

$$\hat{\omega}_{h,\tau,L} \equiv \left( \sum_{i=1}^N R^L(s_{i,t}) R^L(s_{i,t})' \right)^{-1} \sum_{i=1}^N R^L(s_{i,t}) \left( \frac{a_{i,t} \hat{h}_{i,\tau,J}}{\hat{p}_{i,t,J}} - \frac{(1-a_{i,t}) \hat{h}_{i,\tau,J}}{1-\hat{p}_{i,t,J}} \right),$$

where recall that in step (II),  $\hat{h}_{i,\tau,J}$  is the counterpart of  $h_{i,\tau}$  with  $p_{i,\tau}$  replaced by  $\hat{p}_{i,\tau,J}$ . To facilitate our discussion in the next subsection, we look at a specific example:  $h_{T-1} = x_{T-1} p_{T-1}$ . Define

$$\hat{\omega}_L \equiv \hat{\Xi}_{T-2,L}^{-1} \frac{1}{N} \sum_{i=1}^N R^L(s_{i,T-2}) x_{i,T-1} \left( \frac{a_{i,T-2} \hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{(1-a_{i,T-2}) \hat{p}_{i,T-1,J}}{1-\hat{p}_{i,T-2,J}} \right), \quad (\text{F.2})$$

$$\widehat{\Delta \bar{x}}_{1,T-2}^{T-1}(s) \equiv R^L(s)' \hat{\omega}_L, \quad (\text{F.3})$$

where

$$\hat{\Xi}_{T-2,L} \equiv \frac{1}{N} \sum_{i=1}^N R^L(s_{i,T-2}) R^L(s_{i,T-2})'. \quad (\text{F.4})$$

## F.2 Regularity and Asymptotic Linearity

The key to Proposition 1 is to characterize the regularity conditions under which  $\|\sqrt{N}((\hat{\delta}', \hat{\gamma}^{K'})' - (\delta_0', \gamma_0^{K'})') - \frac{1}{\sqrt{N}} \sum_{i=1}^N \psi(D_i)\| = o_p(1)$ . Because step (III) has a unique closed-form solution, the key becomes to characterize regularity conditions under which its “numerator part” has an asymptotically linear expansion as follows

$$\begin{aligned} & \left\| \sqrt{N} \bar{m}_N(\delta_0, \gamma_0^K) - \frac{1}{\sqrt{N}} \sum_{i=1}^N (m_0(D_i) + \alpha(D_i)) \right\| \\ &= \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N m(D_i, \delta_0, \gamma_0^K, \hat{p}_i, \widehat{\Delta \bar{\eta}}_i, \widehat{\Delta \bar{x}}_i, \widehat{\Delta \bar{q}}_i^K) - m(D_i, \delta_0, \gamma_0^K, p_0, \Delta \bar{\eta}_{i,0}, \Delta \bar{x}_{i,0}, \Delta \bar{q}_{i,0}^K) - \alpha(D_i) \right\| \\ &= o_p(1). \end{aligned} \quad (\text{F.5})$$

To show eq. (F.5), it is helpful to make a few observations. First, because the adjustment terms are based on linearization of  $m$  with respect to  $p$ ,  $\Delta \bar{\eta}$ ,  $\Delta \bar{x}$  and  $\Delta \bar{q}^K$ , the estimation error in them can be investigated one at a time, while holding the others at the true value (Newey, 1994, p. 1357). Second, both the direct impact of  $\hat{p}$  and their indirect impact through  $\Delta \bar{\eta}$ ,  $\Delta \bar{x}$  and  $\Delta \bar{q}^K$  need to be accounted for. These two observations enable us to focus on a typical component of the adjustment terms with the understanding that detailing all the components leads to substantial notational expense without bringing any new insight. In light of the definitions in eq. (13), eq. (14) and eq. (19), it suffices to characterize the primitive conditions under which the adjustment terms related to  $\hat{p}$  and  $\widehat{\Delta \bar{x}}_{1,t}^\tau$  (whose dependent variable contains  $\hat{p}_\tau$ ) are given by the formula in Proposition 1; that is,

$$\left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( m(D_i, \delta_0, \gamma_0^K, \hat{p}_i, \Delta \bar{\eta}_{i,0}, \widehat{\Delta \bar{x}}_{i,1,t}^\tau, \Delta \bar{x}_{i,0}, \Delta \bar{q}_{i,0}^K) - \alpha_{x,1}(D_i) - \alpha_{p,1,indirect}(D_i) - \alpha_{p,direct}(D_i) \right) \right\| = o_p(1), \quad (\text{F.6})$$

where we let  $\alpha_{x,1}(D)$  denote the second term in eq. (B.3),  $\alpha_{p,1,indirect}(D)$  denote the sum of the last two terms in eq. (B.6), and  $\alpha_{p,direct}(D)$  is defined in eq. (B.5). (Recall that  $\alpha_{x,1}(D)$  and  $\alpha_{p,1,indirect}(D)$  correspond to  $\widehat{\Delta\bar{x}}_{1,t}^\tau$ , while the other terms in eq. (B.3) and eq. (B.6) correspond to  $\widehat{\Delta\bar{x}}_t^\tau$ .) By the definition of the moment functions  $m$  in eq. (30) and of  $X_{T,t}$  in eq. (21), we further focus on  $\tau = T - 1$  and  $t = T - 2$  in eq. (F.6), and it suffices to show that the asymptotic expansion of

$$-\frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \ln \left( \frac{\hat{p}_{i,T-2}}{1 - \hat{p}_{i,T-2}} \right) + \beta \Delta \bar{\eta}_{i,T-2,0}^{T-1} - x'_{i,T-2} \Delta_{T-2,0} - \beta \Delta \bar{x}_{i,T-2,0}^{T-1'} - \beta \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1'} \Delta_{T-1,0} - \beta^2 \Delta \bar{q}_{i,T-2,0}^{K'} \gamma_0^K \right) \cdot \beta \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} = W_{1,N} + W_{2,N}$$

is characterized by the influence function in eq. (F.6). Recall the equality eq. (18b), so in the above,

$$\begin{aligned} W_{1,N} &\equiv -\frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \ln \left( \frac{\hat{p}_{i,T-2}}{1 - \hat{p}_{i,T-2}} \right) - \ln \left( \frac{p_{i,T-2,0}}{1 - p_{i,T-2,0}} \right) \right) \cdot \beta \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1}, \\ &= -\frac{1}{\sqrt{N}} \sum_{i=1}^N \beta \left( \ln \left( \frac{\hat{p}_{i,T-2}}{1 - \hat{p}_{i,T-2}} \right) - \ln \left( \frac{p_{i,T-2,0}}{1 - p_{i,T-2,0}} \right) \right) \left( \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} - \Delta\bar{x}_{i,1,T-2}^{T-1} \right) \end{aligned} \quad (\text{F.7})$$

$$-\frac{1}{\sqrt{N}} \sum_{i=1}^N \beta \Delta \bar{x}_{i,1,T-2}^{T-1} \left( \ln \left( \frac{\hat{p}_{i,T-2}}{1 - \hat{p}_{i,T-2}} \right) - \ln \left( \frac{p_{i,T-2,0}}{1 - p_{i,T-2,0}} \right) \right), \quad (\text{F.8})$$

$$\begin{aligned} W_{2,N} &\equiv \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta \Delta'_{T-1,0} \left( \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} - \Delta\bar{x}_{i,1,T-2}^{T-1} \right) \cdot \left( \beta \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} \right) \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta^2 \Delta'_{T-1,0} \left( \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} - \Delta\bar{x}_{i,1,T-2}^{T-1} \right) \cdot \left( \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} - \Delta\bar{x}_{i,1,T-2}^{T-1} \right) \end{aligned} \quad (\text{F.9})$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta^2 \Delta'_{T-1,0} \left( \widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} - \widetilde{\Delta\bar{x}}_{i,1,T-2}^{T-1} \right) \cdot \Delta\bar{x}_{i,1,T-2}^{T-1} \quad (\text{F.10})$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta^2 \Delta'_{T-1,0} \left( \widetilde{\Delta\bar{x}}_{i,1,T-2}^{T-1} - \Delta\bar{x}_{i,1,T-2}^{T-1} \right) \cdot \Delta\bar{x}_{i,1,T-2}^{T-1}, \quad (\text{F.11})$$

where  $\widehat{\Delta\bar{x}}_{i,1,T-2}^{T-1} = \widehat{\Delta\bar{x}}_{1,T-2}^{T-1}(s_{i,T-2})$  is defined in eq. (F.3) and

$$\tilde{\omega}_L \equiv \hat{\Xi}_{T-2,L}^{-1} \frac{1}{N} \sum_{i=1}^N R^L(s_{i,T-2}) x_{i,T-1} \left( \frac{a_{i,T-2} p_{i,T-1}}{p_{i,T-2}} - \frac{(1 - a_{i,T-2}) p_{i,T-1}}{1 - p_{i,T-2}} \right), \quad (\text{F.12})$$

$$\widetilde{\Delta\bar{x}}_{i,1,T-2}^{T-1} \equiv \widetilde{\Delta\bar{x}}_{1,T-2}^{T-1}(s_{i,T-2}) = R^L(s_{i,T-2})' \tilde{\omega}_L.$$

The goal of this subsection, therefore, becomes to characterize the primitive conditions for and to prove

$$\left\| W_{1,N} + W_{2,N} - \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \alpha_{x,1}(D_i) + \alpha_{p,1,indirect}(D_i) + \alpha_{p,direct}(D_i) \right) \right\| = o_p(1). \quad (\text{F.13})$$

Before proceeding, it helps to first recall and to clarify some notational convention that we stick to throughout this section. Recall that we use the subscript “<sub>0</sub>” after all the other subscripts

to emphasize the true value of a parameter (which is potentially infinite-dimensional); similarly, we use the subscripts “ $,J$ ” and “ $,L$ ” to denote the series estimators with  $J$  and  $L$  basis functions, respectively. For the CCP function  $p_t(s_t)$  and the conditional mean function  $\Delta\bar{x}_{1,T-2}^{T-1}(s)$  (and their estimators), the argument is sometimes omitted when not confusing. When the argument is omitted, we use the subscript “ $,i$ ,” in front of all the other subscript to denote the value of the function evaluated at agent  $i$ ’s covariates. For example, the following notation is used interchangeably:  $\hat{p}_{i,t} = \hat{p}_{i,t,J} = \hat{p}_{t,J}(s_{i,t})$ ,  $\Delta\bar{x}_{i,1,T-2}^{T-1} = \Delta\bar{x}_{1,T-2}^{T-1}(s_{i,T-2})$  and  $\widehat{\Delta\bar{x}}_{1,T-2}^{T-1}(u) - \widetilde{\Delta\bar{x}}_{1,T-2}^{T-1}(u) = R^L(u)'(\hat{\omega}_L - \tilde{\omega}_L)$ . Let  $C$  be a generic positive constant that may take different value at each appearance.

**Assumption 8** (Second moment of  $x_t$ ). *Suppose  $\mathbb{E}(x_t^2) < \infty$ .*

**Assumption 9** (Distribution of  $s_t$ ). *Suppose: (i) the support of  $s_t$  is a compact set  $\mathcal{S}$ ; (ii) the density of  $s_t$  is bounded, and bounded away from zero, on  $\mathcal{S}$ .<sup>41</sup>*

**Assumption 10** (CCP). *Suppose: (i)  $p_{t,0}(s)$  is a  $\nu_p$  times continuously differentiable function with  $\nu_p \geq 13d_s$ ; (ii)  $p_{t,0}(s)$  is bounded away from 0 and 1, i.e.  $0 < \underline{p} \leq p_{t,0}(s) \leq \bar{p} < 1$ .*

**Assumption 11** ( $\Delta\bar{x}_{1,T-2}^{T-1}(s)$  smoothness). *Suppose  $\Delta\bar{x}_{1,T-2}^{T-1}(s)$  is a  $\nu_x$  times continuously differentiable function with  $\nu_x \geq 5d_s$ .*

**Assumption 12** (Conditional mean smoothness). *Suppose: (i) each coordinate of  $\mathbb{E}\left(\frac{\Delta\bar{x}_{1,T-2}^{T-1} a_{T-2}}{p_{T-2}} \mid s_{T-1} = s\right)$  and  $\mathbb{E}\left(\frac{\Delta\bar{x}_{1,T-2}^{T-1}(1-a_{T-2})}{1-p_{T-2}} \mid s_{T-1} = s\right)$  is a  $\nu_{\mu_1}$  times continuously differentiable function of  $s$  with  $\nu_{\mu_1} \geq 1$ ; (ii) each coordinate of  $\mathbb{E}(x_{T-1} p_{T-1} a_{T-2} \mid s_{T-2} = s) \Delta\bar{x}_{1,T-2}^{T-1}(s) / p_{T-2}^2(s)$  and  $\mathbb{E}(x_{T-1} p_{T-1} (1-a_{T-2}) \mid s_{T-2} = s) \cdot \Delta\bar{x}_{1,T-2}^{T-1}(s) / (1-p_{T-2}(s))^2$  is a  $\nu_{\mu_2}$  times continuously differentiable function of  $s$  with  $\nu_{\mu_2} \geq 1$ .*

**Assumption 13** (Series estimators). *Suppose: (i) the SLE of  $p_{t,0}$  uses the orthonormal power series described in Appendix A of [Hirano, Imbens and Ridder \(2003\)](#) with  $J = N^{c_p}$  for some  $1/(N^{d_s} - 2) < c_p < \frac{1}{11}$ ; (ii) the series estimator of  $\Delta\bar{x}_{1,T-2}^{T-1}$  uses the same power series, but with  $L = N^{c_x}$  for some  $1/(2(N^{d_s} - 1)) < c_x < \frac{1}{8}$ .*

**Theorem 5** (Regularity and asymptotic linearity). *Under Assumptions 8 to 13, eq. (F.13) holds.*

The proof of Theorem 5 proceeds in three parts. Section F.2.1 gives the asymptotic properties of the SLE of the CCP functions that are used later. Section F.2.2 states and proves two lemmas that are used later. Section F.2.3 use these results to prove Theorem 5.

### F.2.1 Asymptotic Properties of the Series Logit Estimator of the CCP

This subsection is largely a reiteration of Appendix A of [Hirano, Imbens and Ridder \(2003\)](#).

<sup>41</sup>Same as in [Hirano, Imbens and Ridder \(2003\)](#), the results in this section can accommodate discrete  $s_t$  at notational expense, and this modification is omitted here for simplicity.

Recall eq. (F.1), then under Assumptions 9 to 10, there exists a  $\pi_{t,J}$  such that

$$\sup_{s \in \mathcal{S}} \left| \ln \left( \frac{p_{t,0}(s)}{1 - p_{t,0}(s)} \right) - R^J(s)' \pi_{t,J} \right| < C J^{-\frac{\nu_p}{d_s}}. \quad (\text{F.14})$$

Define the pseudo-true parameter

$$\pi_{t,J}^* \equiv \arg \max_{\pi} \mathbb{E}(p_{t,0}(S) \ln L(R^J(S)'\pi + (1 - p_{t,0}(S)) \ln(1 - L(R^J(S)'\pi))) \quad (\text{F.15})$$

and the pseudo-true CCP function  $p_{t,J}^*(s) \equiv L(R^J(s)'\pi_{t,J}^*)$ .

We have the following two lemmas. Note that Assumptions 9 to 10 fulfill the conditions (i)-(iv) in Lemma F.1, and Assumption 13 fulfills the condition (v) in Lemma F.2.

**Lemma F.1** (Approximation of the CCP, Lemma 1 of Hirano, Imbens and Ridder (2003)). *Suppose that:*

- (i) the support  $\mathcal{S}$  of  $s_t$  is a compact subset of  $\mathbb{R}^{d_s}$ ;
- (ii) the CCP function  $p_{t,0}(s)$  is  $\nu_p$  times continuously differentiable, with  $\nu_p/d_s \geq 4$ ;
- (iii) the CCP function  $p_{t,0}(s)$  is bounded away from zero and one on  $\mathcal{S}$ ;
- (iv) the density of  $s_t$  is bounded away from zero on  $\mathcal{S}$ .

Then for  $\pi_{t,J}$  in eq. (F.14) and  $\pi_{t,J}^*$  in eq. (F.15), we have

$$\|\pi_{t,J} - \pi_{t,J}^*\| = O(J^{-\frac{\nu_p}{2d_s}}),$$

and

$$\sup_{s \in \mathcal{S}} |p_{t,0}(s) - p_{t,J}^*(s)| = O(\zeta(J) J^{-\frac{\nu_p}{2d_s}}).$$

**Lemma F.2** (Convergence of  $\hat{\pi}_{t,J} - \pi_{t,J}^*$ , Lemma 2 of Hirano, Imbens and Ridder (2003)). *Suppose that the same four conditions as in Lemma F.1 hold. In addition, suppose that:*

- (v)  $J$  is a sequence of values of  $J$  satisfying  $J \rightarrow \infty$  and  $\zeta(J)^4/N \rightarrow 0$ . Then

$$\|\hat{\pi}_{t,J} - \pi_{t,J}^*\| = O_p \left( \sqrt{\frac{J}{N}} \right).$$

By Lemmas F.1 to F.2, we immediately get

$$\sup_{s \in \mathcal{S}} |\hat{p}_{t,J} - p_{t,0}| = O_p \left( \zeta(J) J^{-\frac{\nu_p}{2d_s}} \right) + O_p \left( \zeta(J) \sqrt{\frac{J}{N}} \right). \quad (\text{F.16})$$

## F.2.2 Preparatory Lemmas

**Lemma F.3.** *Recall that  $\Omega_t = (s_t', \varepsilon_t')$ . Consider an arbitrary  $\Omega_{T-1}$ -measurable, potentially vector-valued function  $g_1(\cdot)$  and an arbitrary  $\Omega_{T-2}$ -measurable, potentially vector-valued function  $g_2(\cdot)$ . Suppose that each coordinate of  $\mathbb{E}(g_1(\Omega_{T-1})g_2(\Omega_{T-2}) | s_{T-1})$  is  $\nu_1$  times continuously differentiable, each coordinate of  $\mathbb{E}(g_1(\Omega_{T-1})g_2(\Omega_{T-2}) | s_{T-2})$  is  $\nu_2$  times continuously differentiable, and  $p_{T-2,0}(s)$*

and  $p_{T-1,0}$  are both  $\nu_p$  times continuously differentiable. Let  $\nu_{s_1} \equiv \min\{\nu_1, \nu_p\}$  and  $\nu_{s_2} \equiv \min\{\nu_2, \nu_p\}$ . Under Assumptions 8 to 13, we have

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) (\hat{p}_{i,T-2,J} - p_{i,T-2,0}) \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E}(g_1(\Omega_{T-1}) g_2(\Omega_{T-2}) | s_{i,T-2}) (a_{i,T-2} - p_{T-2,0}(s_{i,T-2})) \\ & \quad + O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J) J^{-\frac{\nu_p}{2d_s}}) + O_p(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}}) + O_p(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{s_2}}{d_s}}), \quad (\text{F.17}) \end{aligned}$$

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) (\hat{p}_{i,T-1,J} - p_{i,T-1,0}) \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E}(g_1(\Omega_{T-1}) g_2(\Omega_{T-2}) | s_{i,T-1}) (a_{i,T-1} - p_{T-1,0}(s_{i,T-1})) \\ & \quad + O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J) J^{-\frac{\nu_p}{2d_s}}) + O_p(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}}) + O_p(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{s_1}}{d_s}}). \quad (\text{F.18}) \end{aligned}$$

*Proof.* Equation (F.17) and eq. (F.18) are essentially the same, with the roles of  $T-2$  and  $T-1$  exchanged, so it suffices to show eq. (F.17).

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) (\hat{p}_{i,T-2,J} - p_{i,T-2,0}) \\ &= \frac{1}{\sqrt{N}} \sum_{i=1}^N (g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) (\hat{p}_{i,T-2,J} - p_{i,T-2,0}) \\ & \quad - \int_{\mathcal{S}} \mathbb{E}(g_1(\Omega_{T-1}) g_2(\Omega_{T-2}) | u) (\hat{p}_{T-2,J}(u) - p_{T-2,0}(u)) dF_{T-2,0}(u)) \end{aligned} \quad (\text{F.19})$$

$$\begin{aligned} & + \sqrt{N} \int_{\mathcal{S}} \mathbb{E}(g_1(\Omega_{T-1}) g_2(\Omega_{T-2}) | u) (\hat{p}_{T-2,J}(u) - p_{T-2,0}(u)) dF_{T-2,0}(u) \\ & - \frac{1}{\sqrt{N}} \sum_{i=1}^N \tilde{\zeta}_{T-2,J}(s_{i,T-2}) \frac{a_{i,T-2} - p_{T-2,J}^*(s_{i,T-2})}{\sqrt{p_{T-2,J}^*(s_{i,T-2})(1 - p_{T-2,J}^*(s_{i,T-2}))}} \end{aligned} \quad (\text{F.20})$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^N (\tilde{\zeta}_{T-2,J}(s_{i,T-2}) - \zeta_{T-2,J}(s_{i,T-2})) \frac{a_{i,T-2} - p_{T-2,J}^*(s_{i,T-2})}{\sqrt{p_{T-2,J}^*(s_{i,T-2})(1 - p_{T-2,J}^*(s_{i,T-2}))}} \quad (\text{F.21})$$

$$\begin{aligned} & + \frac{1}{\sqrt{N}} \sum_{i=1}^N (\zeta_{T-2,J}(s_{i,T-2}) \frac{a_{i,T-2} - p_{T-2,J}^*(s_{i,T-2})}{\sqrt{p_{T-2,J}^*(s_{i,T-2})(1 - p_{T-2,J}^*(s_{i,T-2}))}} \\ & - \zeta_{T-2,0}(s_{i,T-2}) \frac{a_{i,T-2} - p_{T-2,0}(s_{i,T-2})}{\sqrt{p_{T-2,0}(s_{i,T-2})(1 - p_{T-2,0}(s_{i,T-2}))}}) \end{aligned} \quad (\text{F.22})$$

$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E}(g_1(s_{T-1}) | s_{i,T-2}) g_2(s_{i,T-2}) (a_{i,T-2} - p_{T-2,0}(s_{i,T-2})). \quad (\text{F.23})$$

In this expression,  $F_{T-2,0}$  is the population distribution of  $s_{T-2}$  and

$$\begin{aligned} & \tilde{\zeta}_{T-2,J}(s_{T-2}) \\ & \equiv \int_{\mathcal{S}} \mathbb{E}(g_1(\Omega_{T-1}) g_2(\Omega_{T-2}) | u) L^{(1)}(R^J(u)' \tilde{\pi}_{T-2,J}) R^J(u)' dF_{T-2,0}(u) \tilde{\Sigma}_{T-2,J}^{-1} \sqrt{L^{(1)}(R^J(s_{T-2})' \pi_{T-2,J}^*)} R^J(s_{T-2}), \end{aligned}$$

$$\begin{aligned}
& \varsigma_{T-2,J}(s_{T-2}) \\
& \equiv \int_{\mathcal{S}} \mathbb{E}(g_1(\Omega_{T-1})g_2(\Omega_{T-2}) | u)L^{(1)}(R^J(u)'\pi_{T-2,J}^*)R^J(u)'dF_{T-2,0}(u)\Sigma_{T-2,J}^{-1}\sqrt{L^{(1)}(R^J(s_{T-2})'\pi_{T-2,J}^*)}R^J(s_{T-2}), \\
& \quad \varsigma_{T-2,0}(s_{T-2}) \\
& \equiv \mathbb{E}(g_1(\Omega_{T-1})g_2(\Omega_{T-2}) | s_{T-2})\sqrt{p_{T-2,0}(s_{T-2})(1-p_{T-2,0}(s_{T-2}))},
\end{aligned}$$

where the superscribed  $L^{(1)}(u) = L(u) \cdot (1 - L(u))$  indicates the first order derivative of the function  $L(u)$  and

$$\begin{aligned}
\Sigma_{T-2,J} & \equiv \mathbb{E}\left(R^J(S_{T-2})R^J(S_{T-2})'L^{(1)}(R^J(S_{T-2})'\pi_{T-2,J}^*)\right), \\
\tilde{\Sigma}_{T-2,J} & \equiv \frac{1}{N} \sum_{i=1}^N R^J(s_{i,T-2})R^J(s_{i,T-2})'L^{(1)}(R^J(s_{i,T-2})'\tilde{\pi}_{T-2,J}),
\end{aligned}$$

and  $\tilde{\pi}_{T-2,J}$  is between  $\hat{\pi}_{T-2,J}$  and  $\pi_{T-2,J}^*$ . By the definition of  $\varsigma_{T-2,0}(s)$ , we know that each coordinate of  $\varsigma_{T-2,0}(s)$  is  $\nu_{\varsigma_2}$  times continuously differentiable. Following the proof of Theorem 1 of [Hirano, Imbens and Ridder \(2003\)](#), details are available from the authors), we can show the following bounds using eq. (F.1) and Lemmas F.1 to F.2.

$$\begin{aligned}
\|\text{eq. (F.19)}\| & \leq O_p\left(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)J^{-\frac{\nu_p}{2d_s}}\right) + O_p\left(\frac{\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J}{\sqrt{N}}\right), \\
\|\text{eq. (F.20)}\| & \leq O_p\left(\sqrt{N}\|g_1\| \cdot \|g_2\| \cdot \zeta(J)J^{-\frac{\nu_p}{2d_s}}\right), \\
\|\text{eq. (F.21)}\| & \leq O_p\left(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}}\right), \\
\|\text{eq. (F.22)}\| & \leq O_p\left(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\varsigma_2}}{d_s}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)J^{-\frac{\nu_p}{2d_s}}\right).
\end{aligned}$$

Combine these orders with eq. (F.23), recall that for the polynomials  $\zeta(J) \leq CJ$ , and eliminate the dominated ones when  $\zeta(J) \rightarrow \infty$ , we get

$$\begin{aligned}
& \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(s_{i,T-1})g_2(s_{i,T-2})\left(\hat{p}_{i,T-2,J} - p_{i,T-2,0}\right) \\
& = \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E}(g_1(s_{T-1}) | s_{i,T-2})g_2(s_{i,T-2})\left(a_{i,T-2} - p_{T-2,0}(s_{i,T-2})\right) \\
& \quad + O_p\left(\sqrt{N}\|g_1\| \cdot \|g_2\| \cdot \zeta(J)J^{-\frac{\nu_p}{2d_s}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\varsigma_2}}{d_s}}\right),
\end{aligned}$$

which is eq. (F.17). This completes the proof of Lemma F.3. ■

**Lemma F.4.** Consider an arbitrary  $\Omega_{T-1}$ -measurable, potentially vector-valued function  $g_1(\cdot)$  and an arbitrary  $\Omega_{T-2}$ -measurable, potentially vector-valued function  $g_2(\cdot)$ . Suppose that each coordinate of  $\mathbb{E}\left(\frac{g_1(\Omega_{T-1})g_2(\Omega_{T-2})}{p_{T-2,0}(s_{T-2})} \mid s_{T-1}\right)$  and  $\mathbb{E}\left(\frac{g_1(\Omega_{T-1})g_2(\Omega_{T-2})}{1-p_{T-2,0}(s_{T-2})} \mid s_{T-1}\right)$  is  $\nu_1$  times continuously differentiable, each coordinate of  $\mathbb{E}\left(\frac{g_1(\Omega_{T-1})g_2(\Omega_{T-2})p_{T-1,0}(s_{T-1})}{p_{T-2,0}^2(s_{T-2})} \mid s_{T-2}\right)$  and  $\mathbb{E}\left(\frac{g_1(\Omega_{T-1})g_2(\Omega_{T-2})p_{T-1,0}(s_{T-1})}{(1-p_{T-2,0}(s_{T-2}))^2} \mid s_{T-2}\right)$  is

$\nu_2$  times continuously differentiable, and  $p_{T-2,0}(s)$  and  $p_{T-1,0}$  are both  $\nu_p$  times continuously differentiable. Let  $\nu_{\varsigma_1} \equiv \min\{\nu_1, \nu_p\}$  and  $\nu_{\varsigma_2} \equiv \min\{\nu_2, \nu_p\}$ . Under Assumptions 8 to 13, we have

$$\begin{aligned}
& \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) \left( \frac{\hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{p_{i,T-1,0}}{p_{i,T-2,0}} \right) \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left( \frac{g_1(\Omega_{T-1}) g_2(\Omega_{T-2})}{p_{T-2,0}(s_{T-2})} \middle| s_{i,T-1} \right) (a_{i,T-1} - p_{T-1,0}(s_{i,T-1})) \\
&\quad - \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{\mathbb{E}(g_1(\Omega_{T-1}) p_{T-1,0}(s_{T-1}) g_2(\Omega_{T-2}) | s_{i,T-2})}{p_{i,T-2,0}^2} (a_{i,T-2} - p_{T-2,0}(s_{i,T-2})) \\
&\quad + O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J^{-\frac{\nu_p}{d_s}}) + O_p\left(\frac{\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J}{\sqrt{N}}\right) + O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J) J^{-\frac{\nu_p}{2d_s}}) \\
&\quad + O_p\left(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\varsigma_1}}{d_s}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\varsigma_2}}{d_s}}\right), \tag{F.24}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) \left( \frac{\hat{p}_{i,T-1,J}}{1 - \hat{p}_{i,T-2,J}} - \frac{p_{i,T-1,0}}{1 - p_{i,T-2,0}} \right) \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left( \frac{g_1(\Omega_{T-1}) g_2(\Omega_{T-2})}{1 - p_{T-2,0}(s_{T-2})} \middle| s_{i,T-1} \right) (a_{i,T-1} - p_{T-1,0}(s_{i,T-1})) \\
&\quad + \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{\mathbb{E}(g_1(\Omega_{T-1}) p_{T-1,0}(s_{T-1}) g_2(\Omega_{T-2}) | s_{i,T-2})}{(1 - p_{i,T-2,0})^2} (a_{i,T-2} - p_{T-2,0}(s_{i,T-2})) \\
&\quad + O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J^{-\frac{\nu_p}{d_s}}) + O_p\left(\frac{\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J}{\sqrt{N}}\right) + O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J) J^{-\frac{\nu_p}{2d_s}}) \\
&\quad + O_p\left(\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\varsigma_1}}{d_s}}\right) + O_p\left(\|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\varsigma_2}}{d_s}}\right). \tag{F.25}
\end{aligned}$$

*Proof.* Equation (F.24) and eq. (F.25) can be shown by exactly the same argument, so it suffices to prove eq. (F.24).

$$\begin{aligned}
& \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) \left( \frac{a_{i,T-2} \hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{a_{i,T-2} p_{i,T-1,0}}{p_{i,T-2,0}} \right) \\
&= - \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) \frac{(\hat{p}_{i,T-1,J} - p_{i,T-1,0})(\hat{p}_{i,T-2,J} - p_{i,T-2,0})}{p_{i,T-2,0}^2} \\
&\quad + \frac{1}{\sqrt{N}} \sum_{i=1}^N g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) \frac{\hat{p}_{i,T-1,J}(\hat{p}_{i,T-2,J} - p_{i,T-2,0})^2}{p_{i,T-2,0}^2 \hat{p}_{i,T-2,J}} \tag{F.26}
\end{aligned}$$

$$\begin{aligned}
& \quad + \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2})}{p_{i,T-2,0}} (\hat{p}_{i,T-1,J} - p_{i,T-1,0}) \tag{F.27}
\end{aligned}$$

$$\begin{aligned}
& \quad - \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{g_1(\Omega_{i,T-1}) g_2(\Omega_{i,T-2}) p_{i,T-1,0}}{p_{i,T-2,0}^2} (\hat{p}_{i,T-2,J} - p_{i,T-2,0}). \tag{F.28}
\end{aligned}$$

By eq. (F.16), we have

$$\|\text{eq. (F.26)}\| = O_p(\sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J^{-\frac{\nu_p}{d_s}}) + O_p\left(\frac{\|g_1\| \cdot \|g_2\| \cdot \zeta(J)^2 J}{\sqrt{N}}\right). \tag{F.29}$$

Apply Lemma F.3 to eq. (F.27) with  $g_1(\Omega_{T-1}) = g_1(\Omega_{T-1})$  and  $g_2(\Omega_{T-2}) = g_2(\Omega_{T-2})/p_{T-2,0}(s_{T-2})$ , we get

$$\begin{aligned} \text{eq. (F.27)} &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \mathbb{E} \left( \frac{g_1(\Omega_{T-1})g_2(\Omega_{T-2})}{p_{T-2,0}(s_{T-2})} \middle| s_{i,T-1} \right) (a_{i,T-1} - p_{T-1,0}(s_{i,T-1})) \\ &\quad + O_p \left( \sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J) J^{-\frac{\nu_p}{2d_s}} \right) + O_p \left( \|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}} \right) + O_p \left( \|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\zeta_1}}{d_s}} \right). \end{aligned} \quad (\text{F.30})$$

Apply Lemma F.3 to eq. (F.28) with  $g_1(\Omega_{T-1}) = g_1(\Omega_{T-1})p_{T-1,0}(s_{T-1})$  and  $g_2(\Omega_{T-2}) = g_2(\Omega_{T-2})/p_{T-2,0}^2(s_{T-2})$ , we get

$$\begin{aligned} \text{eq. (F.28)} &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \frac{\mathbb{E}(g_1(\Omega_{T-1})p_{T-1,0}(s_{T-1})g_2(\Omega_{T-2}) | s_{i,T-2})}{p_{i,T-2,0}^2} (a_{i,T-2} - p_{T-2,0}(s_{i,T-2})) \\ &\quad + O_p \left( \sqrt{N} \|g_1\| \cdot \|g_2\| \cdot \zeta(J) J^{-\frac{\nu_p}{2d_s}} \right) + O_p \left( \|g_1\| \cdot \|g_2\| \cdot \zeta(J)^5 \sqrt{\frac{J}{N}} \right) + O_p \left( \|g_1\| \cdot \|g_2\| \cdot J^{-\frac{\nu_{\zeta_2}}{d_s}} \right). \end{aligned} \quad (\text{F.31})$$

Equation (F.24) follows by combining eq. (F.29), eq. (F.30) and eq. (F.31). This completes the proof of Lemma F.4.  $\blacksquare$

### F.2.3 Proof of Theorem 5

We consider eq. (F.8) first.

$$\begin{aligned} \text{eq. (F.8)} &= -\frac{1}{\sqrt{N}} \sum_{i=1}^N \beta \Delta \bar{x}_{i,1,T-2}^{T-1} \left( \ln \left( \frac{\hat{p}_{T-2,J}(s_{i,T-2})}{1 - \hat{p}_{T-2,J}(s_{i,T-2})} \right) - \ln \left( \frac{p_{T-2,0}(s_{i,T-2})}{1 - p_{T-2,0}(s_{i,T-2})} \right) \right) \\ &\quad - \frac{\hat{p}_{T-2,J}(s_{i,T-2}) - p_{T-2,0}(s_{i,T-2})}{p_{T-2,0}(s_{i,T-2})(1 - p_{T-2,0}(s_{i,T-2}))} \end{aligned} \quad (\text{F.32})$$

$$- \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta \Delta \bar{x}_{i,1,T-2}^{T-1} \frac{\hat{p}_{T-2,J}(s_{i,T-2}) - p_{T-2,0}(s_{i,T-2})}{p_{T-2,0}(s_{i,T-2})(1 - p_{T-2,0}(s_{i,T-2}))}. \quad (\text{F.33})$$

By Assumptions 10 to 13, eq. (F.16) and the mean value theorem, we can show that

$$\|\text{eq. (F.32)}\| \leq O_p \left( \sqrt{N} \zeta(J)^2 J^{-\frac{\nu_p}{d_s}} \right) + O_p \left( \frac{\zeta(J)^2 J}{\sqrt{N}} \right) = o_p(1).$$

As for eq. (F.33), use Lemma F.3 with  $g_1(s) = 1$  and  $g_2(s) = \beta \Delta \bar{x}_{1,T-2}^{T-1}(s) / (p_{T-2,0}(s)(1 - p_{T-2,0}(s)))$  in eq. (F.17), we get

$$\begin{aligned} \text{eq. (F.33)} &= -\frac{1}{\sqrt{N}} \sum_{i=1}^N \beta \Delta \bar{x}_{i,1,T-2}^{T-1} \frac{a_{i,T-2} - p_{T-2,0}(s_{i,T-2})}{p_{T-2,0}(s_{i,T-2})(1 - p_{T-2,0}(s_{i,T-2}))} \\ &\quad + O_p \left( \sqrt{N} \zeta(J) J^{-\frac{\nu_p}{2d_s}} \right) + O_p \left( \zeta(J)^5 \sqrt{\frac{J}{N}} \right) + O_p \left( J^{-\frac{\nu_{\zeta}}{d_s}} \right). \end{aligned} \quad (\text{F.34})$$

Note that the first term in eq. (F.34) is the rescaled sum of the adjustment term  $\alpha_{p,direct}(D)$  defined in eq. (B.5) and the other three terms are all  $o_p(1)$  under Assumptions 10 to 13, so we have shown that

$$\text{eq. (F.8)} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \alpha_{p,direct}(D_i) + o_p(1). \quad (\text{F.35})$$

We consider eq. (F.11) next.

$$\begin{aligned} \text{eq. (F.11)} &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta^2 \Delta'_{T-1,0} \left( \Delta \bar{x}_{i,1,T-2}^{T-1} \left( \widetilde{\Delta \bar{x}}_{i,1,T-2}^{T-1} - \Delta \bar{x}_{i,1,T-2}^{T-1} \right) \right. \\ &\quad \left. - \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) \left( \widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(u) - \Delta \bar{x}_{1,T-2}^{T-1}(u) \right) dF_{T-2,0}(u) \right) \end{aligned} \quad (\text{F.36})$$

$$\begin{aligned} &+ \beta^2 \sqrt{N} \Delta'_{T-1,0} \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) \left( \widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(u) - \Delta \bar{x}_{1,T-2,0}^{T-1}(u) \right) dF_{T-2,0}(u) \\ &- \frac{\beta^2 \Delta'_{T-1,0}}{\sqrt{N}} \sum_{i=1}^N \tilde{\vartheta}_{T-2,L}(s_{i,T-2}) \\ &\cdot \left( \frac{a_{i,T-2} p_{i,T-1} x_{i,T-1}}{p_{i,T-2}} - \frac{(1-a_{i,T-2}) p_{i,T-1} x_{i,T-1}}{1-p_{i,T-2}} - \Delta \bar{x}_{1,T-2,0}^{T-1}(s_{i,T-2}) \right) \end{aligned} \quad (\text{F.37})$$

$$\begin{aligned} &+ \frac{\beta^2 \Delta'_{T-1,0}}{\sqrt{N}} \sum_{i=1}^N \left( \tilde{\vartheta}_{T-2,L}(s_{i,T-2}) - \vartheta_{T-2,L}(s_{i,T-2}) \right) \\ &\cdot \left( \frac{a_{i,T-2} p_{i,T-1} x_{i,T-1}}{p_{i,T-2}} - \frac{(1-a_{i,T-2}) p_{i,T-1} x_{i,T-1}}{1-p_{i,T-2}} - \Delta \bar{x}_{1,T-2,0}^{T-1}(s_{i,T-2}) \right) \end{aligned} \quad (\text{F.38})$$

$$\begin{aligned} &+ \frac{\beta^2 \Delta'_{T-1,0}}{\sqrt{N}} \sum_{i=1}^N \left( \vartheta_{T-2,L}(s_{i,T-2}) - \vartheta_{T-2,0}(s_{i,T-2}) \right) \\ &\cdot \left( \frac{a_{i,T-2} p_{i,T-1} x_{i,T-1}}{p_{i,T-2}} - \frac{(1-a_{i,T-2}) p_{i,T-1} x_{i,T-1}}{1-p_{i,T-2}} - \Delta \bar{x}_{1,T-2,0}^{T-1}(s_{i,T-2}) \right) \end{aligned} \quad (\text{F.39})$$

$$\begin{aligned} &+ \frac{\beta^2 \Delta'_{T-1,0}}{\sqrt{N}} \sum_{i=1}^N \Delta \bar{x}_{1,T-2}^{T-1}(s_{i,T-2}) \\ &\cdot \left( \frac{a_{i,T-2} p_{i,T-1} x_{i,T-1}}{p_{i,T-2}} - \frac{(1-a_{i,T-2}) p_{i,T-1} x_{i,T-1}}{1-p_{i,T-2}} - \Delta \bar{x}_{1,T-2}^{T-1}(s_{i,T-2}) \right) \end{aligned} \quad (\text{F.40})$$

In this expression,

$$\begin{aligned} \tilde{\vartheta}_{T-2,L}(s) &\equiv \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) R^L(u)' dF_{T-2,0}(u) \hat{\Xi}_{T-2,L}^{-1} R^L(s), \\ \vartheta_{T-2,L}(s) &\equiv \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) R^L(u)' dF_{T-2,0}(u) \Xi_{T-2,L}^{-1} R^L(s), \\ \vartheta_{T-2,0}(s) &\equiv \Delta \bar{x}_{1,T-2}^{T-1}(s). \end{aligned}$$

where  $\hat{\Xi}_{T-2,L}$  is defined in eq. (F.4) and

$$\Xi_{T-2,L} \equiv \mathbb{E} \left( R^L(s_{T-2}) R^L(s_{T-2})' \right).$$

Define

$$\Delta \bar{x}_{1,T-2,L}^{T-1*}(s) \equiv R^L(s)' \hat{\Xi}_{T-2,L}^{-1} \frac{1}{N} \sum_{i=1}^N R^L(s_{i,T-2}) \Delta \bar{x}_{i,1,T-2}^{T-1},$$

which is a series estimator of  $\Delta \bar{x}_{1,T-2}^{T-1}(s)$  without “regression error”. In addition, recall the definitions of  $\widetilde{\Delta \bar{x}}_{i,1,T-2}^{T-1}$  and  $\Delta \bar{x}_{i,1,T-2}^{T-1}$ , we observe that the difference between  $\widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(s)$  and  $\Delta \bar{x}_{1,T-2}^{T-1*}(s)$  capture the impact of the “regression error”. Then by the proof of Theorem 1 of [Newey \(1997\)](#), we have

$$\begin{aligned} \sup_{s \in \mathcal{S}} \|\Delta \bar{x}_{1,T-2,L}^{T-1*}(s) - \Delta \bar{x}_{1,T-2,0}^{T-1}(s)\| &= O_p\left(\zeta(L)L^{-\frac{\nu_x}{d_s}}\right), \\ \sup_{s \in \mathcal{S}} \|\widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(s) - \Delta \bar{x}_{1,T-2,L}^{T-1*}(s)\| &= O_p\left(\zeta(L)\sqrt{\frac{L}{N}}\right), \end{aligned}$$

which implies

$$\sup_{s \in \mathcal{S}} \|\widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(s) - \Delta \bar{x}_{1,T-2}^{T-1}(s)\| = O_p\left(\zeta(L)L^{-\frac{\nu_x}{d_s}}\right) + O_p\left(\zeta(L)\sqrt{\frac{L}{N}}\right). \quad (\text{F.41})$$

By eq. (F.41) and Assumptions 10 to 13, we can show the following bounds using a similar argument as in the proof of Theorem 1 of [Hirano, Imbens and Ridder \(2003\)](#).

$$\begin{aligned} \|\text{eq. (F.36)}\| &\leq O_p\left(\zeta(L)L^{-\frac{\nu_x}{d_s}}\right) + O_p\left(\zeta(L)\sqrt{\frac{L}{N}}\right) = o_p(1), \\ \|\text{eq. (F.37)}\| &\leq O_p\left(\sqrt{N}\zeta(L)L^{-\frac{\nu_x}{d_s}}\right) = o_p(1), \\ \|\text{eq. (F.38)}\| &\leq O_p\left(\frac{\zeta(L)^4}{\sqrt{N}}\right) = o_p(1), \\ \|\text{eq. (F.39)}\| &\leq O_p\left(L^{-\frac{\nu_x}{d_s}}\right) = o_p(1). \end{aligned}$$

Note that eq. (F.40) is the rescaled sum of the adjustment term  $\alpha_{x,1}(D)$  in eq. (F.6), so we have shown that

$$\text{eq. (F.11)} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \alpha_{x,1}(D_i) + o_p(1). \quad (\text{F.42})$$

In order to consider the asymptotic expansion of eq. (F.7), eq. (F.9) and eq. (F.10), recall the definitions of  $\hat{\omega}_L$  and  $\tilde{\omega}_L$  in eq. (F.2) and eq. (F.12), then by Lemma F.4 and under Assumptions 10 to 13, we have

$$\|\hat{\omega}_L - \tilde{\omega}_L\| = O_p\left(\frac{\zeta(L)}{\sqrt{N}}\right), \quad (\text{F.43})$$

which implies that

$$\sup_{s \in \mathcal{S}} \|\widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(s) - \widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(s)\| = O_p\left(\frac{\zeta(L)^2}{\sqrt{N}}\right) = o_p(1). \quad (\text{F.44})$$

Now we consider eq. (F.10).

$$\begin{aligned} \text{eq. (F.10)} &= \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta^2 \Delta'_{T-1,0} \left( \Delta \bar{x}_{i,1,T-2}^{T-1} \left( \widetilde{\Delta \bar{x}}_{i,1,T-2}^{T-1} - \widetilde{\Delta \bar{x}}_{i,1,T-2}^{T-1} \right) \right. \\ &\quad \left. - \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) \left( \widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(u) - \widetilde{\Delta \bar{x}}_{1,T-2}^{T-1}(u) \right) dF_{T-2,0}(u) \right) \end{aligned} \quad (\text{F.45})$$

$$\begin{aligned}
& + \beta^2 \Delta'_{T-1,0} \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) R^L(u)' dF_{T-2,0}(u) \sqrt{N} (\hat{\omega}_L - \tilde{\omega}_L) \\
& - \beta^2 \Delta'_{T-1,0} \Psi_{T-2,L} \Xi_{T-2,L}^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^N R^L(s_{i,T-2}) x_{i,T-1} \\
& \cdot \left( \frac{a_{i,T-2} \hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{(1-a_{i,T-2}) \hat{p}_{i,T-1,J}}{1-\hat{p}_{i,T-2,J}} - \frac{a_{i,T-2} p_{i,T-1,0}}{p_{i,T-2,0}} + \frac{(1-a_{i,T-2}) p_{i,T-1,0}}{1-p_{i,T-2,0}} \right) \quad (\text{F.46})
\end{aligned}$$

$$\begin{aligned}
& + \beta^2 \Delta'_{T-1,0} \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \varphi_{T-2,L}(s_{i,T-2}) - \varphi_{T-2,0}(s_{i,T-2}) \right) x_{i,T-1} \\
& \cdot \left( \frac{a_{i,T-2} \hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{(1-a_{i,T-2}) \hat{p}_{i,T-1,J}}{1-\hat{p}_{i,T-2,J}} - \frac{a_{i,T-2} p_{i,T-1,0}}{p_{i,T-2,0}} + \frac{(1-a_{i,T-2}) p_{i,T-1,0}}{1-p_{i,T-2,0}} \right) \quad (\text{F.47})
\end{aligned}$$

$$\begin{aligned}
& + \beta^2 \Delta'_{T-1,0} \frac{1}{\sqrt{N}} \sum_{i=1}^N \Delta \bar{x}_{1,T-2}^{T-1}(s_{i,T-2}) x_{i,T-1} \\
& \cdot \left( \frac{a_{i,T-2} \hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{(1-a_{i,T-2}) \hat{p}_{i,T-1,J}}{1-\hat{p}_{i,T-2,J}} - \frac{a_{i,T-2} p_{i,T-1,0}}{p_{i,T-2,0}} + \frac{(1-a_{i,T-2}) p_{i,T-1,0}}{1-p_{i,T-2,0}} \right). \quad (\text{F.48})
\end{aligned}$$

In this expression,

$$\begin{aligned}
\varphi_{T-2,L}(s) & \equiv \Psi_{T-2,L} \Xi_{T-2,L}^{-1} R^L(s), \\
\varphi_{T-2,0}(s) & \equiv \Delta \bar{x}_{1,T-2}^{T-1}(s), \\
\Psi_{T-2,L} & \equiv \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) R^L(u)' dF_{T-2,0}(u).
\end{aligned}$$

By eq. (F.43) and Assumptions 10 to 13, we have

$$\begin{aligned}
\|\text{eq. (F.45)}\| & = \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N \beta^2 \Delta'_{T-1,0} \left( \Delta \bar{x}_{1,T-2}^{T-1}(s_{i,T-2}) R^L(s_{i,T-2})' - \int_{\mathcal{S}} \Delta \bar{x}_{1,T-2}^{T-1}(u) R^L(u)' dF_{T-2,0}(u) \right) (\hat{\omega}_L - \tilde{\omega}_L) \right\| \\
& = O_p \left( \zeta(L) \cdot \frac{\zeta(L)}{\sqrt{N}} \right) = O_p \left( \frac{\zeta(L)^2}{\sqrt{N}} \right) = o_p(1).
\end{aligned}$$

By Assumptions 10 to 13, Lemma F.4 and applying an argument similar to the proof of Theorem 1 of Hirano, Imbens and Ridder (2003), we get

$$\begin{aligned}
\|\text{eq. (F.46)}\| & = \left\| \beta^2 \Delta'_{T-1,0} \Psi_L \left( \hat{\Xi}_{T-2,L}^{-1} - \Xi_{T-2,L}^{-1} \right) \frac{1}{\sqrt{N}} \sum_{i=1}^N R^L(s_{i,T-2}) x_{i,T-1} \right. \\
& \quad \cdot \left. \left( \frac{a_{i,T-2} \hat{p}_{i,T-1,J}}{\hat{p}_{i,T-2,J}} - \frac{(1-a_{i,T-2}) \hat{p}_{i,T-1,J}}{1-\hat{p}_{i,T-2,J}} - \frac{a_{i,T-2} p_{i,T-1,0}}{p_{i,T-2,0}} + \frac{(1-a_{i,T-2}) p_{i,T-1,0}}{1-p_{i,T-2,0}} \right) \right\| \\
& = O_p \left( \|\Psi_L\| \cdot \|\hat{\Xi}_{T-2,L}^{-1} - \Xi_{T-2,L}^{-1}\| \cdot \zeta(L) \right) = O_p \left( \frac{\zeta(L)^4}{\sqrt{N}} \right) = o_p(1).
\end{aligned}$$

Note that  $\varphi_{T-2,L}(s)$  is the least squares projection of  $\varphi_{T-2,0}(s) = \Delta \bar{x}_{1,T-2}^{T-1}(s)$  on  $R^L(s)$ , so by eq. (F.1) we have

$$\sup_{s \in \mathcal{S}} \|\varphi_{T-2,L}(s) - \varphi_{T-2,0}(s)\| = O_p \left( L^{-\frac{\nu_x}{d_s}} \right),$$

which, under Assumptions 10 to 13, implies that

$$\|\text{eq. (F.47)}\| = O_p \left( \sup_{s \in \mathcal{S}} \|\varphi_{T-2,L}(s) - \varphi_{T-2,0}(s)\| \right) = O_p \left( L^{-\frac{\nu_x}{d_s}} \right) = o_p(1).$$

By applying eq. (F.24) in Lemma F.4 with  $g_1(\Omega_{T-1}) = x_{T-1}$  and  $g_2(\Omega_{T-2}) = \Delta \bar{x}_{1,T-2}^{T-1}(s_{T-2})a_{T-2}(\Omega_{T-2})$  and applying eq. (F.25) in Lemma F.4 with  $g_1(\Omega_{T-1}) = x_{T-1}$  and  $g_2(\Omega_{T-2}) = \Delta \bar{x}_{1,T-2}^{T-1}(s_{T-2})(1 - a_{T-2}(\Omega_{T-2}))$ , we get

$$\begin{aligned}
\text{eq. (F.48)} &= \beta^2 \Delta'_{T-1,0} \frac{1}{\sqrt{N}} \sum_{i=1}^N \left( \mathbb{E} \left( \frac{\Delta \bar{x}_{1,T-2}^{T-1} a_{T-2}}{p_{T-2,0}} - \frac{\Delta \bar{x}_{1,T-2}^{T-1} (1 - a_{T-2})}{1 - p_{T-2,0}} \middle| s_{i,T-1} \right) x_{i,T-1} (a_{i,T-1} - p_{i,T-1,0}) \right. \\
&\quad \left. - \Delta \bar{x}_{i,1,T-2}^{T-1} \left( \frac{\mathbb{E}(x_{T-1} p_{T-1} a_{T-2} | s_{i,T-2})}{p_{i,T-2,0}^2} + \frac{\mathbb{E}(x_{T-1} p_{T-1} (1 - a_{T-2}) | s_{i,T-2})}{(1 - p_{i,T-2,0})^2} \right) (a_{i,T-2} - p_{i,T-2,0}) \right) \\
&\quad + O_p \left( \sqrt{N} \zeta(J)^2 J^{-\frac{\nu_p}{d_s}} \right) + O_p \left( \frac{\zeta(J)^2 J}{\sqrt{N}} \right) + O_p \left( \sqrt{N} \zeta(J) J^{-\frac{\nu_p}{2d_s}} \right) + O_p \left( \zeta(J)^5 \sqrt{\frac{J}{N}} \right) \\
&\quad + O_p \left( J^{-\frac{\nu_{\mu_1}}{d_s}} \right) + O_p \left( J^{-\frac{\nu_{\mu_2}}{d_s}} \right). \tag{F.49}
\end{aligned}$$

As a result of these expansions of eq. (F.45) to eq. (F.48), we get that under Assumptions 10 to 13, all the other terms in eq. (F.10) are  $o_p(1)$  except the first term in eq. (F.49), which is the rescaled sum of the adjustment term  $\alpha_{p,1,indirect}(D)$  in eq. (F.6), so we have shown that

$$\text{eq. (F.10)} = \frac{1}{\sqrt{N}} \sum_{i=1}^N \alpha_{p,1,indirect}(D_i) + o_p(1). \tag{F.50}$$

We consider eq. (F.9) next. By eq. (F.41) and eq. (F.44), we get

$$\sup_{s \in \mathcal{S}} \|\widehat{\Delta \bar{x}}_{1,T-2}^{T-1}(s) - \Delta \bar{x}_{1,T-2}^{T-1}(s)\| = O_p \left( \zeta(L) L^{-\frac{\nu_x}{d_s}} \right) + O_p \left( \zeta(L) \sqrt{\frac{L}{N}} \right) + O_p \left( \frac{\zeta(L)^2}{\sqrt{N}} \right), \tag{F.51}$$

so under Assumptions 10 to 13,

$$\|\text{eq. (F.9)}\| = O_p \left( \sqrt{N} \zeta(L)^2 L^{-\frac{2\nu_x}{d_s}} \right) + O_p \left( \frac{\zeta(L)^2 L}{\sqrt{N}} \right) + O_p \left( \frac{\zeta(L)^4}{\sqrt{N}} \right) = o_p(1). \tag{F.52}$$

We consider eq. (F.7) next. Note that by the mean value theorem,

$$\ln \left( \frac{\hat{p}_{i,T-2}}{1 - \hat{p}_{i,T-2}} \right) - \ln \left( \frac{p_{i,T-2}}{1 - p_{i,T-2}} \right) = \frac{\hat{p}_{i,T-2} - p_{i,T-2}}{\tilde{p}_{i,T-2} (1 - \tilde{p}_{i,T-2})}$$

for some  $\tilde{p}_{i,T-2}$  between  $\hat{p}_{i,T-2}$  and  $p_{i,T-2}$ . Then by Assumptions 10 to 13, eq. (F.16) and eq. (F.51), we get

$$\begin{aligned}
\|\text{eq. (F.7)}\| &= O_p \left( \sqrt{N} \zeta(J) J^{-\frac{\nu_p}{2d_s}} \zeta(L) L^{-\frac{\nu_x}{d_s}} \right) + O_p \left( \zeta(J) J^{-\frac{\nu_p}{2d_s}} \zeta(L) \sqrt{L} \right) + O_p \left( \zeta(J) J^{-\frac{\nu_p}{2d_s}} \zeta(L)^2 \right) \\
&\quad + O_p \left( \zeta(J) \sqrt{J} \zeta(L) L^{-\frac{\nu_x}{d_s}} \right) + O_p \left( \zeta(J) \sqrt{\frac{JL}{N}} \right) + O_p \left( \zeta(J) \sqrt{\frac{J}{N}} \zeta(L)^2 \right) = o_p(1). \tag{F.53}
\end{aligned}$$

Finally, combine eq. (F.35), eq. (F.42), eq. (F.50), eq. (F.52) and eq. (F.53), we get eq. (F.13). This completes the proof of Theorem 5.

## G Nonstationary Dynamic Multinomial Choice Models

This section generalizes the main results in the main text to nonstationary dynamic multinomial choice models. All the assumptions and theorems in this section have double apostrophes to emphasize their connection to those in the main text.

### G.1 Model Setup

The model setup is the same as in Section 2.1 except the following differences. Each agent makes a multinomial discrete choice  $a_t \in \mathcal{A} \equiv \{0, 1, \dots, A\}$  in each period  $t \in \mathcal{T}$ . For each  $a_t \in \mathcal{A}$ , let  $\varepsilon_{a_t t}$  be unobserved (to researchers) scalar flow utility shocks. Let  $\varepsilon_t \equiv (\varepsilon_{0t}, \varepsilon_{1t}, \dots, \varepsilon_{At})'$  denote the vector of unobserved state variables and let  $\Omega_t \equiv (s'_t, \varepsilon'_t)'$  denote all state variables. For each  $a \in \mathcal{A} \setminus \{0\}$  and  $\tau > t$ , we use  $\mathbb{E}_a(h_\tau | s_t)$  to denote the difference in the conditional means of any future random variable  $h_\tau$  given  $s_t$  and between  $a_t = a$  and  $a_t = 0$ , i.e.,

$$\mathbb{E}_a(h_\tau | s_t) \equiv \mathbb{E}(h_\tau | s_t, a_t = a) - \mathbb{E}(h_\tau | s_t, a_t = 0).$$

The CCP function in period  $t$  is a  $(A+1)$ -valued function, defined as  $p_t(s) \equiv (p_{0,t}(s), p_{1,t}(s), \dots, p_{A,t}(s))$ , where for each  $a \in \mathcal{A}$ ,

$$p_{a,t}(s) \equiv \Pr.(a_t = a | s_t = s). \tag{G.1}$$

In the rest of this appendix, we will suppress the argument of the CCP functions and use  $p_t$  (resp.  $p_{a,t}$  for each  $a \in \mathcal{A}$ ) to denote  $p_t(s_t)$  (resp.  $p_{a,t}(s_t)$  for each  $a \in \mathcal{A}$ ) for notational conciseness, whenever it is not confusing.

We maintain Assumptions 1, 2(i)-(iii) and 3, and we modify Assumption 2(iv) as follows.

**Assumption 2''(iv)** (Flow utility shocks). *For each  $t \in \mathcal{T}$ , assume  $\varepsilon_{at}$  are mutually independent among all  $a \in \mathcal{A}$ .*

Under Assumptions 1, 2(i)-(iii), Theorem 1 and Lemma 1 still hold as is. Theorem 2 only needs minor notational modification as follows.

**Theorem 2''** (Telescoping). *Under Assumptions 1, 2(i)-(iii), 2''(iv) and 3, for  $t, T^* \in \mathcal{T}$   $t < T^*$  and each  $a \in \mathcal{A} \setminus \{0\}$ ,*

$$\begin{aligned} \psi_{0t}(p_t) - \psi_{at}(p_t) &= u_t(a, x_t) - u_t(0, x_t) + \sum_{\tau=t+1}^{T^*-1} \beta^{\tau-t} \mathbb{E}_a(U_\tau^o(s_\tau) | s_t) \\ &\quad + \beta^{T^*-t} \mathbb{E}_a(\bar{V}_{T^*}(s_{T^*}) | s_t). \end{aligned} \tag{G.2}$$

### G.2 Identification

We further maintain Assumption 6 and modify Assumptions 4 and 5 as follows.

**Assumption 4''** (Logit). For each  $t \in \mathcal{T}$  and each  $a \in \mathcal{A}$ , assume  $\varepsilon_{at}$  follows a type I extreme value distribution.

**Assumption 5''** (Linear expected flow utility). Let  $x_t$  denote a  $d_x \times 1$  subvector of  $s_t$ , with  $d_x \leq d_s$ . For each  $t \in \mathcal{T}$  and each  $a \in \mathcal{A}$ , assume  $u_t(a, x_t) = x_t' \delta_{a,t}$  for some time-varying and choice-specific  $\delta_{a,t}$ . We normalize  $\delta_{0,1}$  to an arbitrary  $d_x \times 1$  vector of constants, denoted by  $c$ .

Under Assumption 4'',  $\psi_{0t}(p_t) - \psi_{at}(p_t) = \ln(p_{a,t}/p_{0,t})$  in eq. (G.2). Under Assumptions 1, 2(i)-(iii), 2''(iv), 3, 4'' and 5'', the optimal flow utility function equals to

$$U_t^o(s_t) = \sum_{a=0}^A p_{a,t} x_t' \delta_{a,t} + \gamma - \sum_{a=0}^A p_{a,t} \ln(p_{a,t}).$$

Based on eq. (G.2), we replace  $T^*$  by  $T$ , reparameterize  $\Delta_{a,t} \equiv \delta_{a,t} - \delta_{0,t}$  for  $t = 1, \dots, T$  and  $a \in \mathcal{A} \setminus \{0\}$ , plug in the above expressions of  $\psi_{0t}(p_t) - \psi_{at}(p_t)$  and  $U_t^o(s_t)$ , and assume Assumption 6 holds, then we obtain a system of equations similar to eq. (18), linear in  $\Delta_{a,t}$ ,  $\delta_{0,t}$  and  $\gamma^K$ :

$$y_{a,T-1} = x_{T-1}' \Delta_{a,T-1} + \beta \mathbb{E}_a(q^K(s_T) | s_{T-1})' \gamma^K, \text{ for } a \in \mathcal{A} \setminus \{0\}, \text{ and} \quad (\text{G.3a})$$

$$y_{a,t} = x_t' \Delta_{a,t} + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \mathbb{E}_a(x_\tau | s_t)' \delta_{0,\tau} + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \sum_{\bar{a}=1}^A \mathbb{E}_a(p_{\bar{a},\tau} x_\tau | s_t)' \Delta_{\bar{a},\tau} + \beta^{T-t} \mathbb{E}_a(q^K(s_T) | s_t)' \gamma^K, \text{ for } a \in \mathcal{A} \setminus \{0\}, \text{ and } t = 1, \dots, T-2, \quad (\text{G.3b})$$

where

$$\begin{aligned} y_{a,T-1} &\equiv \ln\left(\frac{p_{a,T-1}}{p_{0,T-1}}\right), \\ y_{a,t} &\equiv \ln\left(\frac{p_{a,t}}{p_{0,t}}\right) + \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \mathbb{E}_a(\eta_\tau | s_t), \\ \eta_\tau &\equiv \sum_{\bar{a}=0}^A p_{\bar{a},\tau} \ln(p_{\bar{a},\tau}), \end{aligned}$$

for  $a \in \mathcal{A} \setminus \{0\}$ ,  $t = 1, \dots, T-2$  and  $\tau > t$ .

The identification of the parameters  $\theta \equiv (\Delta'_{1,1}, \dots, \Delta'_{A,1}, \delta'_{0,2}, \Delta'_{1,2}, \dots, \Delta'_{A,2}, \dots, \delta'_{0,T-1}, \Delta'_{1,T-1}, \dots, \Delta'_{A,T-1}, \gamma^{K'})'$  can be analyzed based on eq. (G.3). For this purpose, we define the following notation.  $p_{1:A,t} \equiv (p_{1,t}, \dots, p_{A,t})'$  for  $t \in \mathcal{T}$ . For  $T \geq 2$  and each  $a \in \mathcal{A} \setminus \{0\}$ , let

$$\underbrace{X_{a,T,T-1}}_{K \times 1} \equiv \beta \mathbb{E}_a(q^K(s_T) | s_{T-1}), \quad \underbrace{X_{a,T,t}}_{((A+1)(T-t-1)d_x + K) \times 1} \equiv \begin{bmatrix} \beta \mathbb{E}_a(x_{t+1} | s_t) \\ \beta \mathbb{E}_a(p_{1:A,t+1} \otimes x_{t+1} | s_t) \\ \vdots \\ \beta^{T-1-t} \mathbb{E}_a(x_{T-1} | s_t) \\ \beta^{T-1-t} \mathbb{E}_a(p_{1:A,T-1} \otimes x_{T-1} | s_t) \\ \beta^{T-t} \mathbb{E}_a(q^K(s_T) | s_t) \end{bmatrix}$$

for  $t = 1, 2, \dots, T - 2$ . With “0” denoting matrices with conformable dimensions, let

$$\underbrace{\tilde{L}_{T,t}}_{(((A+1)(T-t)-A)d_x+K)A \times (((A+1)(T-t)-1)d_x+K)} \equiv \begin{bmatrix} \mathbb{E}(x_t x_t') & 0 & \cdots & 0 & \mathbb{E}(x_t X'_{1,T,t}) \\ \mathbb{E}(X_{1,T,t} x_t') & 0 & \cdots & 0 & \mathbb{E}(X_{1,T,t} X'_{1,T,t}) \\ 0 & \mathbb{E}(x_t x_t') & \cdots & 0 & \mathbb{E}(x_t X'_{2,T,t}) \\ 0 & \mathbb{E}(X_{2,T,t} x_t') & \cdots & 0 & \mathbb{E}(X_{2,T,t} X'_{2,T,t}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \mathbb{E}(x_t x_t') & \mathbb{E}(x_t X'_{A,T,t}) \\ 0 & \cdots & 0 & \mathbb{E}(X_{A,T,t} x_t') & \mathbb{E}(X_{A,T,t} X'_{A,T,t}) \end{bmatrix},$$

and

$$\underbrace{L_{T,t}}_{(((A+1)(T-t)-A)d_x+K)A \times (((T-1)A+(T-2))d_x+K)} \equiv \begin{bmatrix} 0_{(((A+1)(T-t)-A)d_x+K)A \times (A+1)(t-1)d_x} & \tilde{L}_{T,t} \end{bmatrix}$$

for  $T \geq 2$  and  $t = 1, \dots, T - 1$ . Then the identification and over-identification conditions of the parameters, given in the next theorem, depends on the  $\left(\frac{T(T-1)(A+1)d_x}{2} - A(T-1)d_x + (T-1)K\right)A \times (((T-1)A+(T-2))d_x+K)$  Jacobian matrix  $L_T$  that stacks  $L_{T,t}$  for  $t = T-1, \dots, 1$  (in that order). It is obvious that  $L_T$  has a block-triangular structure.<sup>42</sup> We state, without proof, the following (over-)identification theorem, which is a straightforward generalization of Theorem 3.

**Theorem 3”** (Identification and over-identification of  $\theta$ ). *Under Assumptions 1, 2(i)-(iii), 2”(iv), 3, 4”, 5” and 6-7, (i)  $\theta$  is identified if and only if the  $L_T$  matrix defined in this section has full column rank; (ii) if more than one matrix that consists of  $((T-1)A+(T-2))d_x+K$  distinct rows from  $L_T$  defined in this section has full rank, then  $\theta$  is over-identified.*

### G.3 Estimation

The estimation procedure is almost identical to that introduced in Section 3.3, save for notational modification to accommodate more than two choices. The available data are a panel of state-variable-and-choice pairs  $D_{i,t} \equiv (a_{i,t}, s'_{i,t})'$  for  $N$  agents and  $T$  periods ( $i \in \{1, \dots, N\}$  and  $t \in \mathcal{T}_{data} \equiv \{1, \dots, T\}$ ). Let  $D_i \equiv (D'_{i,1}, \dots, D'_{i,T})'$ . We use  $p$  to collectively denote all CCPs, use  $\Delta\bar{\eta}$  to collectively denote all  $\mathbb{E}_a(\eta_\tau | s_t)$  ( $a \in \mathcal{A} \setminus \{0\}$ ,  $t = 1, \dots, T-2$ ,  $t < \tau \leq T-1$ ), use  $\Delta\bar{x}$  to collectively denote  $\mathbb{E}_a(x_\tau | s_t)$  and  $\mathbb{E}_a(p_{\bar{a},\tau} x_\tau | s_t)$  ( $a, \bar{a} \in \mathcal{A} \setminus \{0\}$ ,  $t = 1, \dots, T-2$ ,  $t < \tau \leq T-1$ ), and use  $\Delta\bar{q}^K$  to collectively denote  $\mathbb{E}_a(q^K(s_T) | s_t)$  ( $a \in \mathcal{A} \setminus \{0\}$ ,  $t = 1, \dots, T-1$ ). In light of Theorem 3”, the

<sup>42</sup>We recycled some symbols (e.g.,  $\tilde{L}_{T,t}$ ,  $L_{T,t}$  and  $L_T$ ) from the main text, but they are not confusing given the context of this section.

parameters  $\theta$  can be estimated via minimum distance (MD) using the following moment functions:

$$m_{T-1}(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv -v_{T-1}(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \cdot \begin{bmatrix} x_{T-1} \\ X_{1,T,T-1} \\ \vdots \\ x_{T-1} \\ X_{A,T,T-1} \end{bmatrix}, \text{ and}$$

$$m_t(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv -v_t(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \cdot \begin{bmatrix} x_t \\ X_{1,T,t} \\ \vdots \\ x_t \\ X_{A,T,t} \end{bmatrix},$$

where  $a \in \mathcal{A} \setminus \{0\}$  and  $t = 1, \dots, T-2$ ,  $v_{T-1}(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K)$  is the stack of  $v_{a,T-1}(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv y_{a,T-1} - x'_{T-1} \Delta_{a,T-1} - \beta \mathbb{E}_a(q^K(s_T) | s_{T-1})' \gamma^K$  in the order of  $a = 1, \dots, A$ , and  $v_t(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K)$  is the stack of  $v_{a,t}(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K) \equiv y_t - x'_t \Delta_{a,t} - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \mathbb{E}_a(x_\tau | s_t)' \delta_{0,\tau} - \sum_{\tau=t+1}^{T-1} \beta^{\tau-t} \sum_{\bar{a}=1}^A \mathbb{E}_a(p_{\bar{a},\tau}(s_\tau) x_\tau | s_t)' \Delta_{\bar{a},\tau} - \beta^{T-t} \mathbb{E}_a(q^K(s_T) | s_t)' \gamma^K$  in the order of  $a = 1, \dots, A$  for  $t = 1, \dots, T-2$ . Let  $m(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K)$  be the stack of the moment functions  $m_t(D, \theta, p, \Delta\bar{\eta}, \Delta\bar{x}, \Delta\bar{q}^K)$  for  $t = T-1, \dots, 1$  (in that order), which is a  $\left(\frac{T(T-1)(A+1)d_x}{2} - A(T-1)d_x + (T-1)K\right) A \times 1$ -vector-valued moment function that equals zeros under the true values of  $\theta$ .

The estimation proceeds in the following three steps.

- (I) For each  $a \in \mathcal{A}$  and each  $t = 1, \dots, T$ , use the data  $\{D_{i,t}\}_{i=1}^N$  to estimate the CCP functions  $p_{a,t}$  in eq. (G.1) and obtain the estimated CCP values  $\hat{p}_{i,a,t}$  for each agent  $i \in \{1, \dots, N\}$ .
- (II) Obtain  $\hat{h}_{i,\tau}$  by substituting unknown  $p_{i,\tau}$  with  $\hat{p}_{i,\tau}$  whenever applicable. Then, for each  $t = 1, \dots, T-2$  and  $t < \tau \leq T-1$ , use the data  $\{\hat{h}_{i,\tau}, D_{i,t}\}_{i=1}^N$  to estimate the corresponding conditional mean difference functions in  $\widehat{\Delta\bar{\eta}}, \widehat{\Delta\bar{x}}$  and  $\widehat{\Delta\bar{q}^K}$  and obtain their estimated values for each agent  $i \in \{1, \dots, N\}$ .
- (III) Let  $\bar{m}_N(\theta) \equiv \frac{1}{N} \sum_{i=1}^N \hat{m}(D_i, \theta)$ , where  $\hat{m}(D_i, \theta) \equiv m(D_i, \theta, \hat{p}_i, \widehat{\Delta\bar{\eta}}_i, \widehat{\Delta\bar{x}}_i, \widehat{\Delta\bar{q}^K}_i)$ . Then,  $\hat{\theta}$  is the MD estimator as follows:

$$\hat{\theta} \equiv \arg \min_{\theta \in \mathbb{R}^{((T-1)A+(T-2))d_x+K}} \bar{m}_N(\theta)' W_N \bar{m}_N(\theta), \quad (\text{G.4})$$

where  $W_N$  is a  $\left(\frac{T(T-1)(A+1)d_x}{2} - A(T-1)d_x + (T-1)K\right) A \times \left(\frac{T(T-1)(A+1)d_x}{2} - A(T-1)d_x + (T-1)K\right) A$  symmetric weighting matrix that converges in probability to a positive definite matrix  $W$  as  $N \rightarrow \infty$ .

In eq. (G.4), the first  $((T-1)A+(T-2))d_x$  coordinates of  $\hat{\theta}$  are the flow utility parameter estimators  $(\hat{\Delta}'_{1,1}, \dots, \hat{\Delta}'_{A,1}, \hat{\delta}'_{0,2}, \hat{\Delta}'_{1,2}, \dots, \hat{\Delta}'_{A,2}, \dots, \hat{\delta}'_{0,T-1}, \hat{\Delta}'_{1,T-1}, \dots, \hat{\Delta}'_{A,T-1})'$ , and the last  $K$  coordinates of  $\hat{\theta}$  are  $\hat{\gamma}^K$ .