An Averaging GMM Estimator Robust to Misspecification[*]

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Abstract

This paper studies the averaging GMM estimator that combines a conservative GMM estimator based on valid moment conditions and an aggressive GMM estimator based on both valid and possibly misspecified moment conditions, where the weight is the sample analog of an infeasible optimal weight. We establish asymptotic theory on uniform approximation of the upper and lower bounds of the finite-sample truncated risk difference between any two estimators, which is used to compare the averaging GMM estimator and the conservative GMM estimator. Under some sufficient conditions, we show that the asymptotic lower bound of the truncated risk difference between the averaging estimator and the conservative estimator is strictly less than zero, while the asymptotic upper bound is zero uniformly over any degree of misspecification. Extending seminal results on the James-Stein estimator, this uniform dominance is established in non-Gaussian semiparametric nonlinear models. The simulation results support our theoretical findings.

Keywords: Asymptotic Risk, Finite-Sample Risk, Generalized Shrinkage Estimator, GMM, Misspecification, Model Averaging, Non-Standard Estimator, Uniform Approximation

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1 Introduction

We are interested in estimating some finite dimensional parameter $\theta_F \in \mathbb{R}^{d_\theta}$ which is uniquely identified by the moment restrictions

$$\mathbb{E}_F[g_1(W, \theta_F)] = 0_{r_1 \times 1} \tag{1.1}$$

for some known vector functions $g_1(\cdot) : \mathcal{W} \times \Theta \to \mathbb{R}^{r_1}$, where Θ is a compact subset of $\mathbb{R}^{d_\theta}$, W is a random vector with support $\mathcal{W}$ and joint distribution F, and $\mathbb{E}_F[\cdot]$ denotes the expectation operator under F. Suppose we have i.i.d. data $\{W_i\}_{i=1}^n$, where W_i has distribution F for any $i = 1, \ldots, n$.[1] Let $\overline{g}_1(\theta) = n^{-1} \sum_{i=1}^n g_1(W_i, \theta)$. One efficient GMM estimator for θ_F is

$$\widehat{\theta}_1 = \underset{\theta \in \Theta}{\arg\min}\, \overline{g}_1(\theta)'(\overline{\Omega}_1)^{-1}\overline{g}_1(\theta), \tag{1.2}$$

where $\overline{\Omega}_1 = n^{-1}\sum_{i=1}^n g_1(W_i, \widetilde{\theta}_1)g_1(W_i, \widetilde{\theta}_1)' - \overline{g}_1(\widetilde{\theta}_1)\overline{g}_1(\widetilde{\theta}_1)'$ is the efficient weighting matrix with some preliminary consistent estimator $\widetilde{\theta}_1$.[2] In a linear instrumental variable (IV) example, $Y = X'\theta_F + U$ where the IV $Z_1 \in \mathbb{R}^{r_1}$ satisfies $\mathbb{E}_F[Z_1 U] = 0_{r_1 \times 1}$. The moments in (1.1) hold with $g_1(W, \theta) = Z_1(Y - X'\theta)$ and θ_F is uniquely identified if $\mathbb{E}_F[Z_1 X']$ has full column rank. Under certain regularity conditions, it is well-known that $\widehat{\theta}_1$ is consistent and achieves the lowest asymptotic variance among GMM estimators based on the moments in (1.1), see Hansen (1982).

If one has additional moments

$$\mathbb{E}_F[g^*(W, \theta_F)] = 0_{r^* \times 1} \tag{1.3}$$

for some known function $g^*(\cdot) : \mathcal{W} \times \Theta \to \mathbb{R}^{r^*}$, imposing them together with (1.1) can further reduce the asymptotic variance of the GMM estimator. However, if these additional moments are misspecified in the sense that $\mathbb{E}_F[g^*(W, \theta_F)] \neq 0_{r^* \times 1}$, imposing (1.3) may result in inconsistent estimation. The choice of moment conditions is routinely faced by empirical researchers. Take the linear IV model for example. One typically starts with a large number of candidate IVs but only has confidence that a small number of them are valid, denoted by Z_1. The rest of them, denoted by Z^*, are valid only under certain economic hypothesis that yet to be tested. In this example, $g^*(W, \theta) = Z^*(Y - X'\theta)$. In contrast to the conservative estimator $\widehat{\theta}_1$, an aggressive estimator $\widehat{\theta}_2$ always imposes (1.3) regardless of its validity. Let $g_2(W_i, \theta) = (g_1(W_i, \theta)', g^*(W_i, \theta)')'$ for any

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[1]The main theory of the paper can be easily extended to time series models with dependent data, as long as the preliminary results in Lemma A.1 hold.

[2]For example, $\widetilde{\theta}_1$ could be the GMM estimator similar to $\widehat{\theta}_1$ but with an identity weighting matrix, see (A.5) in the Appendix.

Figure 1. Finite Sample $(n = 500)$ MSEs of the Pre-test and the Averaging GMM Estimators
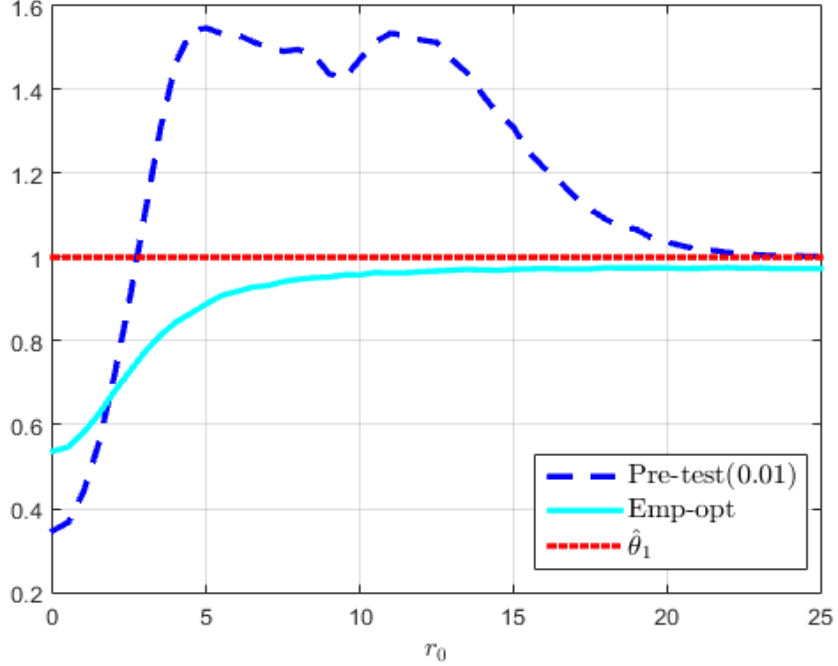
Note: "Pre-test(0.01)" refers to the pre-test GMM estimator based on the J test statistic $n\bar{g}_2(\widehat{\theta}_2)'(\overline{\Omega}_2)^{-1}\bar{g}_2(\widehat{\theta}_2)$ with a norminal size 0.01. "Emp-opt" refers to the averaging GMM estimator defined in equation (4.7) of the paper. In this simulation, we set $\delta_F = r_0 c n^{-1/2}$ where c is a r^* by 1 real vector. At each r_0, we consider 127 different values for c and report the largest finite sample MSEs of the estimators. Details of the simulation design for Figure 1 is provided in Section 7.

$i = 1, \ldots, n$, and $\bar{g}_2(\theta) = n^{-1}\sum_{i=1}^{n} g_2(W_i, \theta)$. The aggressive estimator $\widehat{\theta}_2$ takes the form

$$\widehat{\theta}_2 = \arg\min_{\theta \in \Theta} \bar{g}_2(\theta)'(\overline{\Omega}_2)^{-1}\bar{g}_2(\theta), \tag{1.4}$$

where $\overline{\Omega}_2$ is constructed in the same way as $\overline{\Omega}_1$ except that $g_1(W_i, \theta)$ is replaced by $g_2(W_i, \theta)$.[3]

Because imposing $\mathbb{E}_F[g^*(W, \theta_F)] = 0_{r^* \times 1}$ is a double-edged sword, a data-dependent decision usually is made to choose between $\widehat{\theta}_1$ and $\widehat{\theta}_2$. To study such a decision and the subsequent estimator, let

$$\delta_F = \mathbb{E}_F[g^*(W, \theta_F)] \in \mathbb{R}^{r^*}. \tag{1.5}$$

The pre-testing approach tests the null hypothesis $H_0 : \delta_F = 0_{r^* \times 1}$ and constructs an estimator

$$\widehat{\theta}_{pre} = 1\{T_n > c_\alpha\}\widehat{\theta}_1 + 1\{T_n \le c_\alpha\}\widehat{\theta}_2 \tag{1.6}$$

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[3]See the first line of equations (C.13) in the Supplemental Appendix for the definition of $\overline{\Omega}_2$. In particular, $\overline{\Omega}_2$ is constructed using $\widetilde{\theta}_1$, the preliminary consistent estimator based on the valid moment conditions in (1.1) only.

for some test statistic T_n with the critical value c_α at the significance level α. One popular test is the J-test, see Hansen (1982), and c_α is the $1-\alpha$ quantile of the chi-squared distribution with degree of freedom $r_2 - d_\theta$ where $r_2 = r_1 + r^*$. Because the power of this test against the fixed alternative is 1, $\widehat{\theta}_{pre}$ equals $\widehat{\theta}_1$ with probability 1 asymptotically ($n \to \infty$) for those fixed misspecified model where $\delta_F \neq 0_{r^* \times 1}$. Thus, it seems that $\widehat{\theta}_{pre}$ is immune to moment misspecification. However, we care about the finite-sample mean squared error (MSE) of $\widehat{\theta}_{pre}$ in practice and this standard pointwise asymptotic analysis (δ_F is fixed and $n \to \infty$) provides a poor approximation to the former.[4] To see the comparison between $\widehat{\theta}_{pre}$ and $\widehat{\theta}_1$, the dashed line in Figure 1 plots the maximum finite-sample ($n = 500$) MSE of $\widehat{\theta}_{pre}$ while the MSE of $\widehat{\theta}_1$ is normalized to be 1. For some values of δ_F, the MSE of $\widehat{\theta}_{pre}$ is larger than that of $\widehat{\theta}_1$, sometimes by 50%. Note that the pre-test estimator exhibits multiple peaks because the simulation design allows for multiple potentially misspecified moments and the horizontal axis only shows the main component r_0 which determines the norm of the vector that measures the degree of misspecification.

The goal of this paper is twofold. First, we propose a data-dependent averaging of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ that takes the form

$$\widehat{\theta}_{eo} = (1 - \widetilde{\omega}_{eo})\widehat{\theta}_1 + \widetilde{\omega}_{eo}\widehat{\theta}_2 \tag{1.7}$$

where $\widetilde{\omega}_{eo} \in [0, 1]$ is a data-dependent weight specified in (4.7) below. The subscript in $\widetilde{\omega}_{eo}$ is short for empirical optimal because this weight is an empirical analog of an infeasible optimal weight ω_F^* defined in (4.3) below. We plot the finite-sample MSE of this averaging estimator as the solid line in Figure 1. This averaging estimator is robust to misspecification in the sense that the solid line is below 1 for all values of δ_F, in contrast to the bump in the dashed line that represents the pre-test estimator. Second, we develop a *uniform* asymptotic theory to justify the finite-sample robustness of this averaging estimator. We show that this averaging estimator dominates the conservative estimator uniformly over a large class of models with different degrees of misspecification. The uniform dominance is established under the truncated weighted loss function which is defined in (3.11) below.[5] Truncation at a large number is necessary for the asymptotic analysis. The standard asymptotic theory is pointwise and fails to reveal the fragile nature of the pre-test estimator. A stronger uniform notion of robustness is crucial for this model. Furthermore, we quantify the upper and lower bounds of the asymptotic risk differences between the averaging estimator and the conservative estimator.[6]

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[4]The poor approximation of the pointwise asymptotics to the finite sample properties of the pre-test estimator has been noted in Shibata (1986), Pötscher (1991), Kabaila (1995, 1998) and Leeb and Pötscher (2005, 2008), among others.

[5]Truncation at a large number is needed for the asymptotic analysis of the risk of general estimator without imposing stringent conditions such as uniform integrability.

[6]The lower and upper bounds of asymptotic risk difference are defined in (3.12) below.

The rest of the paper is organized as follows. Section 2 discusses the literature related to our paper. Section 3 defines the parameter space over which the uniform result is established and defines uniform dominance. Section 4 introduces the averaging weight. Section 5 provides an analytical representation of the bounds of the asymptotic risk differences and applies it to show that the averaging GMM estimator uniformly dominates the conservative estimator. Different from the global uniform dominance in Section 5, Section 6 studies local uniform dominance over a shrinking parameter space. Section 7 investigates the finite sample performance of our averaging estimator using Monte Carlo simulations. Section 8 concludes. Proof of the main results of the paper is given in the Appendix. Analysis of the pre-test estimator, extra simulation studies and proofs of some auxiliary results are included in the Supplemental Appendix of the paper.

Notation. For any real matrix A, we use $||A||$ to denote the Frobenius norm of A. If A is a real square matrix, we use $tr(A)$ denote the trace of A. If A is a real symmetric matrix, $\rho_{\min}(A)$ and $\rho_{\max}(A)$ to denote the smallest and largest eigenvalues of A, respectively. For any positive integers d_1 and d_2, I_{d_1} and $0_{d_1 \times d_2}$ stand for the $d_1 \times d_1$ identity matrix and $d_1 \times d_2$ zero matrix, respectively. Let $\text{vec}(\cdot)$ denotes vectorization of a matrix and $\text{vech}(\cdot)$ denotes the half vectorization of a symmetric matrix. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_\infty = \mathbb{R} \cup \{\pm\infty\}$ and $\mathbb{R}_{+,\infty} = \mathbb{R}_+ \cup \{+\infty\}$. For any positive integers d and any set $\mathbb{S}$, $\mathbb{S}^d$ denotes the Cartesian product of d many sets: $\mathbb{S}_1 \times \cdots \times \mathbb{S}_d$ with $\mathbb{S}_j = \mathbb{S}$ for $j = 1, \ldots, d$. For any set $\mathbb{S}$, $int(\mathbb{S})$ denotes the interior of $\mathbb{S}$. We use $\{n\}$ to denote the set of natural numbers and $\{p_n\} = \{p_n : n \geq 1\}$ denote a subsequence of $\{n\}$. For any (possibly random) positive sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$, $a_n = O_p(b_n)$ means that $\lim_{c \to \infty} \lim_{n \to \infty} \Pr(a_n/b_n > c) = 0$; $a_n = o_p(b_n)$ means that for all $\varepsilon > 0$, $\lim_{n \to \infty} \Pr(a_n/b_n > \varepsilon) = 0$. Let "$\to_p$" and "$\to_D$" stand for convergence in probability and convergence in distribution, respectively. The notation $a \equiv b$ means a is defined as b.

2   Related Literature

In this section, we discuss the related literature. Our uniform dominance result is related to the Stein's phenomenon (Stein, 1956) in parametric models. The James-Stein (JS) estimator (James and Stein, 1961) is shown to dominate the maximum likelihood estimator in exact normal sampling. Hansen (2016) considers local asymptotic analysis of the JS type averaging estimator in general parametric models and substantially extends its application in econometrics. Many other frequentist model averaging estimators are studied in the literature, including Buckland, Burnham, and Augustin (1997), Hjort and Claeskens (2003, 2006), Hansen (2007, 2015, 2017), Claeskens and Carroll (2007), Hansen and Racine (2012), Cheng and Hansen (2014) and Lu and Su (2015), DiTraglia

(2016) to name only a few. Our paper is closely related to DiTraglia (2016) and Hansen (2017). Hansen (2017) proposes an averaging estimator that combines the ordinary least squares (OLS) estimator and the two-stage-least-squares (2SLS) estimator in linear IV models. Under some sufficient conditions, he shows that the averaging estimator has smaller asymptotic risk than the OLS estimator under any *given* sequence of $n^{-1/2}$ local misspecified data generating processes (DGPs). DiTraglia (2016) also studies the averaging GMM estimator under *given* sequences of $n^{-1/2}$ local misspecification DGPs, including the averaging of OLS and IV estimators. The averaging weight is based on the focused moment selection criterion with a targeted parameter. The simulation results in the paper show that this averaging estimator does not uniformly dominate the conservative estimator. Compared to these results, our paper makes the following contributions. First, we provide a *uniform* asymptotic framework, which is different from pointwise asymptotic analysis for fixed models or local asymptotic analysis along some drifting models (e.g., the given $n^{-1/2}$ local misspecified DGPs in DiTraglia (2016) and Hansen (2017)). For this purpose, this paper defines a large set of drifting models and show that a uniform result on the risk of this shrinkage estimator requires the study of all of them. In particular, this large set of models not only includes the crucial $n^{-1/2}$ local misspecified models that are considered by DiTraglia (2016) and Hansen (2017), but it also includes many more severely misspecified models. This uniform analysis is similar to those studied in Andrews and Guggenberger (2010) and Andrews, Cheng, and Guggenberger (2011) for uniform size control for inference in non-standard problems, but the present paper is for estimation rather than inference and focuses on a misspecification issue that is not studied in these papers. Second, the present paper extends the study of the Stein's phenomenon to non-Gaussian semiparametric nonlinear GMM models. The weight in our averaging estimator is close to those studied in Hjort and Claeskens (2003) and Liu (2015) in the sense that we first consider the optimal weight and then obtain its empirical analog.

The uniform dominance property of the averaging estimator does not contradict the unbounded risk property of the post-model-selection estimator found in Yang (2005) and Leeb and Pötscher (2008). Measured by the MSE, the post-model-selection estimator based on consistent model selection procedure usually does better than the unrestricted estimator in part of the parameter space and worse than the latter in other part of the parameter space. One standard example is the Hodge's estimator, whose scaled maximal MSE diverges to infinity with the growth of the sample size (see, e.g., Lehmann and Casella, 1998). Similar unbounded risk results for other post-model-selection estimators are established in Yang (2005) and Leeb and Pötscher (2008). The post-model-selection estimator has unbounded (scaled) maximal MSE because it is based on a non-smooth transition rule between the restricted and unrestricted estimators and a consistent model

selection procedure is employed in the transition rule.[7] However, the averaging estimator proposed in this paper is based on a smooth combination of the restricted and unrestricted estimators and no model selection procedure is used in the smooth combination. Hence our averaging estimator is essentially different from the post-model-selection estimator.

There is a large literature studying the validity of GMM moment conditions. Many methods can be applied to detect the validity, including the over-identification tests (see, e.g., Sargan, 1958; Hansen, 1982; and Eichenbaum, Hansen and Singleton 1988), the information criteria (see, e.g., Andrews, 1999; Andrews and Lu, 2003; Hong, Preston and Shum, 2003), and the penalized estimation methods (see, e.g., Liao, 2013 and Cheng and Liao, 2014). Recently, misspecified moments and their consequences are considered by Ashley (2009), Berkowitz, Caner, and Fang (2012), Conley, Hansen, and Rossi (2012), Doko Tchatoka and Dufour (2008, 2014), Guggenberger (2012), Nevo and Rosen (2012), Kolesar, Chetty, Friedman, Glaeser, Imbens (2014), Small (2007), Small, Cai, Zhang, Kang (2015), among others. Moon and Schorfheide (2009) explore over-identifying moment inequalities to reduce the MSE. This paper contributes to this literature by providing new uniform results for potentially misspecified semiparametric models.

There is also a large literature studying adaptive estimation in nonparametric regression model using model averaging; see Yang (2000, 2003, 2004), Leung and Barron (2006), and the references therein. Since the unknown function can be written as a linear combination of (possibly infinitely but countably many) basis functions, the nonparametric model may be well approximated by parametric regression models in finite samples. These papers show that the averaging estimators which combine OLS estimators from different parametric models with data dependent weights may achieve the optimal convergence rate up to some logarithm factor (or the oracle inequalities). Our paper is different from these papers since the parameter of interests is a finite dimensional real value, not an unknown function, and the bias and variance trade-off is due to the possibly misspecified moment conditions. Moreover, there is a benchmark estimator in our paper, i.e., the conservative GMM estimator whose asymptotic properties are well-known and many model selection estimators proposed in the literature do not uniformly improve upon this benchmark estimator in terms of risk over different degrees of moment misspecification. Our goal is to propose an averaging estimator with smaller risk than the conservative estimator uniformly.

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[7]The post-model-selection estimator based on conservative model selection procedure (e.g., hypothesis test with fixed critical value or Akaike information criterion) may not have unbounded (scaled) maximal MSE. However its asymptotic maximal MSE is not guaranteed to be less than or equal to the benchmark estimator (e.g., the conservative GMM estimator in the framework of this paper). The pre-test estimator in our simulation studies is a good example, since it is based on the J-test with nominal size 0.01.

3 Parameter Space and Uniform Dominance

Let $g_{2,j}(w, \theta)$ $(j = 1, \ldots, r_2)$ denote the j-th component function of $g_2(w, \theta)$. We assume that $g_{2,j}(w, \theta)$ for $j = 1, \ldots, r_2$ is twice continuously differentiable with respect to θ for any $w \in \mathcal{W}$. The first and second order derivatives of $g_2(w, \theta)$ with respect to θ are denoted by

$$g_{2,\theta}(w, \theta) \equiv \begin{pmatrix} \frac{\partial g_{2,1}(w,\theta)}{\partial \theta'} \\ \vdots \\ \frac{\partial g_{2,r_2}(w,\theta)}{\partial \theta'} \end{pmatrix} \text{ and } g_{2,\theta\theta}(w, \theta) \equiv \begin{pmatrix} \frac{\partial^2 g_{2,1}(w,\theta)}{\partial \theta \partial \theta'} \\ \vdots \\ \frac{\partial^2 g_{2,r_2}(w,\theta)}{\partial \theta \partial \theta'} \end{pmatrix}, \tag{3.1}$$

respectively.[8] Let $\mathcal{F}$ be a set of distribution functions of W. For $k = 1$ and 2, define the expectation of the moment functions, the Jacobian matrix and the variance-covariance matrix as

$$\begin{aligned} M_{k,F} &\equiv \mathbb{E}_F\left[g_k(W, \theta_F)\right], \\ G_{k,F} &\equiv \mathbb{E}_F\left[g_{k,\theta}(W, \theta_F)\right], \text{ and} \\ \Omega_{k,F} &\equiv \mathbb{E}_F\left[g_k(W, \theta_F)g_k(W, \theta_F)'\right] - M_{k,F}M_{k,F}' \end{aligned} \tag{3.2}$$

for any $F \in \mathcal{F}$ respectively. The moments above exist by Assumption 3.2 below.

Let

$$Q_F(\theta) \equiv \mathbb{E}_F[g_2(W, \theta)]'\Omega_{2,F}^{-1}\mathbb{E}_F[g_2(W, \theta)] \tag{3.3}$$

for any $\theta \in \Theta$, which denotes the population criterion of the GMM estimation in (1.4). For any $\theta \in \Theta$, define $B_\varepsilon^c(\theta) = \{\theta^* \in \Theta : ||\theta^* - \theta|| \geq \varepsilon\}$. We consider the risk difference between two estimators uniformly over $F \in \mathcal{F}$ that satisfies Assumptions 3.1-3.3 below.

Assumption 3.1 *The following conditions hold:*
(i) *for any $F \in \mathcal{F}$, $\mathbb{E}_F\left[g_1(W, \theta_F)\right] = 0_{r_1 \times 1}$ for some $\theta_F \in int(\Theta)$;*
(ii) *for any $\varepsilon > 0$, $\inf_{F \in \mathcal{F}} \inf_{\theta \in B_\varepsilon^c(\theta_F)} ||\mathbb{E}_F\left[g_1(W, \theta)\right]|| > 0$;*
(iii) *for any $F \in \mathcal{F}$, there is $\theta_F^* \in int(\Theta)$ such that*

$$\inf_{F \in \mathcal{F}} \inf_{\theta \in B_\varepsilon^c(\theta_F^*)} [Q_F(\theta) - Q_F(\theta_F^*)] > 0 \text{ for any } \varepsilon > 0;$$

(iv) $\inf_{\{F \in \mathcal{F} : ||\delta_F|| > 0\}} \frac{||G_{2,F}'\Omega_{2,F}^{-1}\delta_{2,F}||}{||\delta_{2,F}||^\tau} > 0$ *where $\delta_{2,F}' = (0_{1 \times r_1}, \delta_F')$ and $\tau > 0$ is a fixed constant;*
(v) $0_{r^* \times 1} \in int(\Delta_\delta)$ *where $\Delta_\delta = \{\delta_F : F \in \mathcal{F}\}$.*

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[8]By definition, $g_{1,\theta}(w, \theta)$ and $g_{1,\theta\theta}(w, \theta)$ are the leading $r_1 \times d_\theta$ and $(r_1 d_\theta) \times d_\theta$ submatrices of $g_{2,\theta}(w, \theta)$ and $g_{2,\theta\theta}(w, \theta)$, respectively.

Assumptions 3.1.(i)-(ii) require that the true unknown parameter θ_F is uniquely identified by the moment conditions $\mathbb{E}_F[g_1(W,\theta_F)] = 0_{r_1 \times 1}$ for any DGP $F \in \mathcal{F}$. Assumption 3.1.(iii) implies that for any $F \in \mathcal{F}$, a pseudo true value θ_F^* is identified by the unique minimizer of the population GMM criterion $Q_F(\theta)$ under possible misspecification. Assumption 3.1.(iv) requires that $\delta_{2,F}$ is not orthogonal to $\Omega_{2,F}^{-1}G_{2,F}$, which rules out the special case that θ_F may be consistently estimable even with severely misspecified moment conditions. Assumption 3.1.(v) implies that the set of distribution functions $\mathcal{F}$ is rich such that it includes the distributions under which the extra moment conditions are correctly specified. Uniform dominance is of interest only if we allow for different degrees of misspecification in the parameter space. If we only allow for correctly specified models or severely misspecified models, the desired dominance results hold trivially following a pointwise analysis. Assumption 3.1.(v) ensures that the extra moment conditions can be correctly specified, locally misspecified or severely misspecified.

Assumption 3.2 *The following conditions hold:*

(i) *For $j = 1, \ldots, r_2$, $g_{2,j}(w,\theta)$ is twice continuously differentiable with respect to θ for any $w \in \mathcal{W}$;*

(ii) $\sup_{F \in \mathcal{F}} \mathbb{E}_F \left[\sup_{\theta \in \Theta} \left(||g_2(W,\theta)||^{2+\gamma} + ||g_{2,\theta}(W,\theta)||^{2+\gamma} + ||g_{2,\theta\theta}(W,\theta)||^{2+\gamma} \right) \right] < \infty$ *for some $\gamma > 0$;*

(iii) $\inf_{F \in \mathcal{F}} \rho_{\min}(\Omega_{2,F}) > 0$;

(iv) $\inf_{F \in \mathcal{F}} \rho_{\min}(G_{1,F}' G_{1,F}) > 0$.

Assumption 3.2.(i) requires that the moment functions are smooth. Assumption 3.2.(ii) imposes $2+\gamma$ finite moment conditions on the GMM moment functions and their first and second derivatives. Assumptions 3.2.(iii) and 3.2.(iv) are important sufficient conditions for the local identification of the unknown parameter in GMM with valid moment conditions.

The next assumption is on the nuisance parameters of the DGP $F \in \mathcal{F}$. Write

$$v_F = \left(\text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})', \delta_F' \right) \tag{3.4}$$

for any $F \in \mathcal{F}$. It is clear that v_F includes the Jacobian matrix, the variance-covariance matrix, and the measure of misspecification of the moment conditions $\mathbb{E}_F[g^*(W,\theta_F)] = 0_{r^* \times 1}$. Let

$$\bar{v}_F = \left(\text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})' \right) \tag{3.5}$$

for any $F \in \mathcal{F}$.

Assumption 3.3 *The following conditions hold:*

(i) *For any $F \in \mathcal{F}$ with $\delta_F = 0_{r^* \times 1}$, there exists a constant $\varepsilon_F > 0$ such that for any $\widetilde{\delta} \in \mathbb{R}^{r^*}$ with $0 \leq ||\widetilde{\delta}|| < \varepsilon_F$, there is $\widetilde{F} \in \mathcal{F}$ with $\delta_{\widetilde{F}} = \widetilde{\delta}$ and $\left\| \bar{v}_{\widetilde{F}} - \bar{v}_F \right\| \leq C||\widetilde{\delta}||^\kappa$ for some $\kappa > 0$;*

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(ii) *The set $\Lambda \equiv \{v_F \colon F \in \mathcal{F}\}$ is closed.*

Assumption 3.3.(i) requires that for any $F \in \mathcal{F}$ such that $\mathbb{E}_F[g_2(W, \theta_F)] = 0_{r^* \times 1}$ is valid, there are many DGPs $\widetilde{F} \in \mathcal{F}$ which are close to F. Here the closeness of any two DGPs F and $\widetilde{F}$ is measured by the distance between v_F and $v_{\widetilde{F}}$. Assumption 3.3 (i) and (ii) are useful to derive the exact expression of the asymptotic risk of the GMM estimator.

Example 3.1 (Linear IV Model) We study a linear IV model and provide a set of low-level conditions that imply Assumptions 3.1, 3.2 and 3.3. The parameters of interest θ_0 are the coefficients of the endogenous regressors X in

$$Y = X'\theta_0 + U, \tag{3.6}$$

with some valid IVs $Z_1 \in \mathbb{R}^{r_1}$ and some potentially misspecified IVs $Z^* \in \mathbb{R}^{r^*}$ such that

$$\mathbb{E}_{F^*}[U] = 0, \ \mathbb{E}_{F^*}[Z_1 U] = 0_{r_1 \times 1}, \ \text{and} \tag{3.7}$$

$$Z^* = U\delta_0 + V, \ \text{with} \ \mathbb{E}_{F^*}[V] = 0_{r^* \times 1} \ \text{and} \ \mathbb{E}_{F^*}[VU] = 0_{r^* \times 1}, \tag{3.8}$$

where F^* denotes the joint distribution of $(X', Z_1', V', U)'$. In the reduced-form equation (3.8), δ_0 is a $r^* \times 1$ real vector which characterizes the degree of misspecification. Let $\mathcal{F}^*$ denote a class of distributions containing F^*, and let Θ and Δ_δ denote the parameter spaces of θ_0 and δ_0 respectively. The joint distribution of $W = (Y, Z_1', Z^{*\prime}, X')'$ is denoted as F which is determined by θ_0, δ_0 and F^* through the linear equations in (3.6) and (3.8).

For ease of discussion, we further assume that the random vector $(X', Z_1', V', U)'$ follows the normal distribution with mean ϕ and variance-covariance matrix Ψ.[9] Under the normal assumption, each distribution F^* corresponds to a pair of ϕ and Ψ.

For notational simplicity, in Lemma *3.1* below, for any finite dimensional random vectors a_1 and a_2, let $\phi_{a_j} = \mathbb{E}_{F^*}[a_j]$ for $j = 1, 2$, $\Gamma_{a_1 a_2} = \mathbb{E}_{F^*}[a_1 a_2']$, and $\Omega_{a_1 a_2} = \mathbb{E}_{F^*}[a_1 a_2'] - \phi_{a_1} \phi_{a_2}'$.

Lemma 3.1 *Let $\mathcal{F}^*$ denote the set of normal distributions which satisfies:*
(i) $\phi_u = 0$, $\Gamma_{z_1 u} = 0_{r_1 \times 1}$ and $\Gamma_{vu} = 0_{r^* \times 1}$;
(ii) $\inf_{F^* \in \mathcal{F}^*} \rho_{\min}(\Gamma_{xz_1} \Gamma_{z_1 x}) > 0$, $\sup_{F^* \in \mathcal{F}^*} ||\phi||^2 < \infty$ and
$0 < \inf_{F^* \in \mathcal{F}^*} \rho_{\min}(\Psi) \leq \sup_{F^* \in \mathcal{F}^*} \rho_{\max}(\Psi) < \infty$;
(iii) $\inf_{F^* \in \mathcal{F}^*} \inf_{\{||\delta|| \geq \varepsilon\}} ||\delta||^{-1} ||(\Gamma_{xz_1} \Gamma_{z_1 z_1}^{-1} \Gamma_{z_1 v} - \Gamma_{xv})\delta - \Gamma_{xu}|| > 0$ for some $\varepsilon > 0$ that is small enough;[10]

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[9]In Section B of the Supplemental Appendix, we provide sufficient low-level conditions for Assumptions 3.1, 3.2 and 3.3 in the linear IV model without the normal assumption.

[10]The constant ε depend on the infimum and supremum in Condition (ii) and it is given in (A.3) in the Appendix.

(iv) $\theta_0 \in int(\Theta)$ and Θ is compact and large enough such that the pseudo-true value $\theta_F^* \in int(\Theta)$;[11]

(v) $\Delta_\delta = [c_{1,\Delta}, C_{1,\Delta}] \times \cdots \times [c_{r^*,\Delta}, C_{r^*,\Delta}]$ where $\{c_{j,\Delta}, C_{j,\Delta}\}_{j=1}^{r^*}$ is a set of finite constants with $c_{j,\Delta} < 0 < C_{j,\Delta}$ for $j = 1, \ldots, r^*$,

then Assumptions 3.1, 3.2 and 3.3 hold.

Condition (i) lists the moment conditions in *(3.7)* and *(3.8)*. The inequality in Condition (ii) rules out DGPs under which $\rho_{\min}(\Gamma_{xz_1}\Gamma_{z_1x})$ may be close to zero and (part of) the unknown parameter θ_0 is weakly identified. Condition (ii) also requires that the mean of the random vector $(X', Z_1', V', U)'$ is uniformly bounded and the eigenvalues of its variance-covariance matrix are uniformly finite and bounded away from 0. Condition (iii) requires that the projection residual of the vector Γ_{xu} on the subspace spanned by the matrix $\Gamma_{xz_1}\Gamma_{z_1z_1}^{-1}\Gamma_{z_1v} - \Gamma_{xv}$ is bounded away from zero. It is a sufficient condition for Assumption 3.1.(iv), which ensures that the aggressive estimator is inconsistent under severe misspecification. Condition (iv) is needed to derive the limit of the aggressive estimator under misspecification. The compactness assumption of Θ is not needed for the linear IV model. However, it is useful to verify Assumptions *3.1, 3.2* and *3.3* which do not assume any special structure on the model. Condition (v) specifies that the parameter space of δ_0 is a product space.

Lemma *3.1* provides simple conditions on θ_0, δ_0 and $\mathcal{F}^*$ on which uniformity results are subsequently established.[12]

∎

Now we get back to the general set up. For a generic estimator $\widehat{\theta}$ of θ, consider a weighted quadratic loss function

$$\ell(\widehat{\theta}, \theta) = n(\widehat{\theta} - \theta)'\Upsilon(\widehat{\theta} - \theta), \tag{3.9}$$

where Υ is a $d_\theta \times d_\theta$ pre-determined positive semi-definite matrix. For example, if $\Upsilon = I_{d_\theta}$, $\mathbb{E}_F[\ell(\widehat{\theta}, \theta_F)]$ is the MSE of $\widehat{\theta}$. If $\Upsilon = (\Sigma_{1,F} - \Sigma_{2,F})^{-1}$ where $\Sigma_{k,F}$ ($k = 1, 2$) is defined in (4.4), the weighting matrix Υ rescales $\widehat{\theta}$ by the scale of variance reduction due to the additional moments. If $\Upsilon = \mathbb{E}_F[X_iX_i']$ for regressors X_i, $\mathbb{E}_F[\ell(\widehat{\theta}, \theta_F)]$ is the MSE of $X_i'\widehat{\theta}$, an estimator of $X_i'\theta$.

Below we compare the averaging estimator $\widehat{\theta}_{eo}$ and the conservative estimator $\widehat{\theta}_1$. We are

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[11]Specific restrictions on Θ which ensures that $\theta_F^* \in int(\Theta)$ are given in *(B.8)* and Assumption *B.1.*(vi) in the Supplemental Appendix.

[12]Similar results have been established in Section B of the Supplemental Appendix for the linear IV model when the normal assumption on $(X', Z_1', V', U)'$ is relaxed. Section B of the Supplemental Appendix also provides proof for Lemma *3.1* with and without the normal assumption.

interested in the bounds of the truncated finite sample risk difference

$$\underline{RD}_n(\widehat{\theta}_{eo}, \widehat{\theta}_1; \zeta) \equiv \inf_{F \in \mathcal{F}} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)] \text{ and}$$
$$\overline{RD}_n(\widehat{\theta}_{eo}, \widehat{\theta}_1; \zeta) \equiv \sup_{F \in \mathcal{F}} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)], \quad (3.10)$$

where

$$\ell_\zeta(\widehat{\theta}, \theta_F) \equiv \min\{\ell(\widehat{\theta}, \theta_F), \zeta\} \quad (3.11)$$

denotes the truncated loss function with an arbitrarily large trimming ζ. The truncated loss function is employed to facilitate the asymptotic analysis of the bounds of the risk difference. The finite-sample bounds in (3.10) are approximated by

$$Asy\underline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) \equiv \liminf_{\zeta \to \infty} \liminf_{n \to \infty} \underline{RD}_n(\widehat{\theta}_{eo}, \widehat{\theta}_1; \zeta) \text{ and}$$
$$Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) \equiv \limsup_{\zeta \to \infty} \limsup_{n \to \infty} \overline{RD}_n(\widehat{\theta}_{eo}, \widehat{\theta}_1; \zeta) \quad (3.12)$$

which are called lower and upper bounds of the asymptotic risk difference respectively in this paper. The averaging estimator $\widehat{\theta}_{eo}$ asymptotically uniformly dominates the conservative estimator $\widehat{\theta}_1$ if

$$Asy\underline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) < 0 \text{ and } Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) \leq 0. \quad (3.13)$$

The bounds of the asymptotic risk difference build the uniformity over $F \in \mathcal{F}$ into the definition by taking $\inf_{F \in \mathcal{F}}$ and $\sup_{F \in \mathcal{F}}$ before $\liminf_{n \to \infty}$ and $\limsup_{n \to \infty}$ respectively. Uniformity is crucial for the asymptotic results to give a good approximation to their finite-sample counterparts. These uniform bounds are different from pointwise results which are obtained under a fixed DGP. The sequence of DGPs $\{F_n\}$ along which the supremum or the infimum are approached often varies with the sample size.[13] Therefore, to determine the bounds of the asymptotic risk difference, one has to derive the asymptotic distributions of these estimators under various sequences $\{F_n\}$. Under $\{F_n\}$, the observations $\{W_{n,i}\}_{i=1}^n$ form a triangular array. For notational simplicity, $W_{n,i}$ is abbreviated to W_i throughout the paper.

To study the bounds of asymptotic risk difference, we consider sequences of DGPs $\{F_n\}$ such that δ_{F_n} satisfies[14]

$$\text{(i) } n^{1/2}\delta_{F_n} \to d \in \mathbb{R}^{r^*} \text{ or (ii) } ||n^{1/2}\delta_{F_n}|| \to \infty. \quad (3.14)$$

Case (i) models mild misspecification, where δ_{F_n} is a $n^{-1/2}$-local deviation from $0_{r^* \times 1}$. Case (ii)

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[13]In the rest of the paper, we use $\{F_n\}$ to denote $\{F_n \in \mathcal{F} : n = 1, 2, ...\}$.

[14]Since $F_n \in \mathcal{F}$, by Assumption 3.2.(ii), the sequence δ_{F_n} in (3.14) should satisfy $||\delta_{F_n}|| \leq C$ for any n.

includes the severe misspecification where $\|\delta_{F_n}\|$ is bounded away from 0 as well as the intermediate case in which $\delta_{F_n} \to 0$ and $\|n^{1/2}\delta_{F_n}\| \to \infty$. To obtain a uniform approximation, all of these sequences are necessary. Once we study the bounds of asymptotic risk difference along each of these sequences, we show that we can glue them together to obtain the bounds of asymptotic risk difference.

4 Averaging Weight

We start by deriving the joint asymptotic distribution of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ under different degrees of misspecification. We consider sequences of DGPs $\{F_n\}$ in $\mathcal{F}$ such that (i) $n^{1/2}\delta_{F_n} \to d \in \mathbb{R}^{r^*}$ or $\|n^{1/2}\delta_{F_n}\| \to \infty$; and (ii) G_{2,F_n}, Ω_{2,F_n} and M_{2,F_n} converges to $G_{2,F}$, $\Omega_{2,F}$ and $M_{2,F}$ for some $F \in \mathcal{F}$.
[15]

For $k = 1, 2$ and any $F \in \mathcal{F}$, define

$$\Gamma_{k,F} = - \left(G'_{k,F}\Omega^{-1}_{k,F}G_{k,F} \right)^{-1} G'_{k,F}\Omega^{-1}_{k,F}. \tag{4.1}$$

Let $\mathcal{Z}_{2,F}$ denote a zero mean normal random vector with variance-covariance matrix $\Omega_{2,F}$ and $\mathcal{Z}_{1,F}$ denote its first r_1 components.

Lemma 4.1 *Suppose Assumptions 3.1 and 3.2 hold. Consider any sequence of DGPs $\{F_n\}$ such that $v_{F_n} \to v_F$ for some $F \in \mathcal{F}$, and $n^{1/2}\delta_{F_n} \to d$ for $d \in \mathbb{R}^{r^*}_\infty$.*
(a) *If $d \in \mathbb{R}^{r^*}$, then*

$$\begin{pmatrix} n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \\ n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \end{pmatrix} \to_D \begin{pmatrix} \xi_{1,F} \\ \xi_{2,F} \end{pmatrix} \equiv \begin{pmatrix} \Gamma_{1,F}\mathcal{Z}_{1,F} \\ \Gamma_{2,F}(\mathcal{Z}_{2,F} + d_0) \end{pmatrix},$$

where $d_0 = (\mathbf{0}_{1 \times r_1}, d')'$.
(b) *If $\|d\| = \infty$, then $n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \to_D \xi_{1,F}$ and $\|n^{1/2}(\widehat{\theta}_2 - \theta_{F_n})\| \to_p \infty$.*

Given the joint asymptotic distribution of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, it is straightforward to study $\widehat{\theta}(\omega) = (1 - \omega)\widehat{\theta}_1 + \omega\widehat{\theta}_2$ if ω is deterministic. Following Lemma 4.1.(a),

$$n^{1/2}(\widehat{\theta}(\omega) - \theta_{F_n}) \to_D \xi_F(\omega) \equiv (1 - \omega)\xi_{1,F} + \omega\xi_{2,F} \tag{4.2}$$

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[15]The requirement on the convergence of G_{2,F_n}, Ω_{2,F_n} and M_{2,F_n} is not restrictive. Lemma A.7 in Appendix A.1 shows that the sequences G_{2,F_n}, Ω_{2,F_n} and M_{2,F_n} have subsequences that converge to $G_{2,F}$, $\Omega_{2,F}$ and $M_{2,F}$, respectively, for some $F \in \mathcal{F}$. The general result on the lower and upper bounds of the asymptotic risk difference, Lemma A.14 in Appendix A.2, only requires to consider the subsequence $\{F_{p_n}\}$ such that $G_{2,F_{p_n}}$, $\Omega_{2,F_{p_n}}$ and $M_{2,F_{p_n}}$ are convergent, where $\{p_n\}$ is a subsequence of $\{n\}$. The asymptotic properties of the GMM estimators established in this section under the full sequence of DGPs $\{F_n\}$ holds trivially for its subsequence.

for $n^{1/2}\delta_{F_n} \to d$, where $d \in \mathbb{R}^{r^*}$. In Section C of the Supplemental Appendix, a simple calculation shows that the asymptotic risk of $\widehat{\theta}(\omega)$ is minimized at the infeasible optimal weight

$$\omega_F^* \equiv \frac{\operatorname{tr}(\Upsilon\,(\Sigma_{1,F} - \Sigma_{2,F}))}{d_0'\left(\Gamma_{2,F} - \Gamma_{1,F}^*\right)' \Upsilon\left(\Gamma_{2,F} - \Gamma_{1,F}^*\right) d_0 + \operatorname{tr}(\Upsilon\,(\Sigma_{1,F} - \Sigma_{2,F}))}, \tag{4.3}$$

where Υ is the matrix specified in the loss function,

$$\Sigma_{k,F} \equiv \left(G_{k,F}' \Omega_{k,F}^{-1} G_{k,F}\right)^{-1} \text{ for } k = 1, 2 \text{ and } \Gamma_{1,F}^* \equiv [\Gamma_{1,F}, \mathbf{0}_{d_\theta \times r^*}]. \tag{4.4}$$

To gain some intuition, consider the case where $\Upsilon = I_{d_\theta}$ such that the MSE of $\widehat{\theta}(\omega)$ is minimized at ω_F^*. In this case, the infeasible optimal weight ω_F^* yields the ideal bias and variance trade off. However, the bias depends on d, which cannot be consistently estimated. Hence, ω_F^* cannot be consistently estimated. Our solution to this problem follows the popular approach in the literature which replaces d by an estimator whose asymptotic distribution is centered at d, see Liu (2015) and Charkhi, Claeskens, and Hansen (2016) for similar estimators in the least square estimation and maximum likelihood estimation problems, respectively. Moreover, we show that the resulting averaging estimator reduces the MSE for any value of d.

The empirical analog of ω_F^* is constructed as follows. First, for $k = 1$ and 2, replace $\Sigma_{k,F}$ by its consistent estimator $\widehat{\Sigma}_k \equiv (\widehat{G}_k' \widehat{\Omega}_k^{-1} \widehat{G}_k)^{-1}$,[16] where

$$\widehat{G}_k \equiv n^{-1} \sum_{i=1}^n g_{k,\theta}(W_i, \widehat{\theta}_1) \text{ and } \widehat{\Omega}_k \equiv n^{-1} \sum_{i=1}^n g_k(W_i, \widehat{\theta}_1) g_k(W_i, \widehat{\theta}_1)' - \overline{g}_k(\widehat{\theta}_1) \overline{g}_k(\widehat{\theta}_1)'. \tag{4.5}$$

Note that $\widehat{G}_k$ and $\widehat{\Omega}_k$ are based on the conservative GMM estimator $\widehat{\theta}_1$. Hence they are consistent regardless of the degree of misspecification of the moment conditions in (1.3). Second, replace $(\Gamma_{2,F} - \Gamma_{1,F}^*)d_0$ by its asymptotically unbiased estimator $n^{1/2}(\widehat{\theta}_2 - \widehat{\theta}_1)$ because

$$n^{1/2}(\widehat{\theta}_2 - \widehat{\theta}_1) \to_D (\Gamma_{2,F} - \Gamma_{1,F}^*)(\mathcal{Z}_{2,F} + d_0), \tag{4.6}$$

for $d_0 = (\mathbf{0}_{1 \times r_1}, d')'$ and $d \in \mathbb{R}^{r^*}$ following Lemma 4.1(a). Then the empirical optimal weight takes the form

$$\widetilde{\omega}_{eo} \equiv \frac{\operatorname{tr}(\Upsilon(\widehat{\Sigma}_1 - \widehat{\Sigma}_2))}{n(\widehat{\theta}_2 - \widehat{\theta}_1)' \Upsilon(\widehat{\theta}_2 - \widehat{\theta}_1) + \operatorname{tr}(\Upsilon(\widehat{\Sigma}_1 - \widehat{\Sigma}_2))}, \tag{4.7}$$

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[16]The consistency of $\widehat{\Sigma}_k$ is proved in the proof of Lemma 4.2.

and the averaging GMM estimator takes the form

$$\widehat{\theta}_{eo} = (1 - \widetilde{\omega}_{eo})\widehat{\theta}_1 + \widetilde{\omega}_{eo}\widehat{\theta}_2. \tag{4.8}$$

Next we consider the asymptotic distribution of $\widehat{\theta}_{eo}$ under different degrees of misspecification.

Lemma 4.2 *Suppose that Assumptions 3.1-3.3 hold. Consider $\{F_n\}$ such that $v_{F_n} \to v_F$ for some $F \in \mathcal{F}$, and $n^{1/2}\delta_{F_n} \to d$ for $d \in \mathbb{R}^{r^*}_{\infty}$.*
(a) *If $d \in \mathbb{R}^{r^*}$, then*

$$\widetilde{\omega}_{eo} \to_D \overline{\omega}_F \equiv \frac{\mathrm{tr}(\Upsilon(\Sigma_{1,F} - \Sigma_{2,F}))}{(\mathcal{Z}_{2,F} + d_0)'(\Gamma_{2,F} - \Gamma^*_{1,F})'\Upsilon(\Gamma_{2,F} - \Gamma^*_{1,F})(\mathcal{Z}_{2,F} + d_0) + \mathrm{tr}(\Upsilon(\Sigma_{1,F} - \Sigma_{2,F}))}$$

and

$$n^{1/2}(\widehat{\theta}_{eo} - \theta_{F_n}) \to_D \overline{\xi}_F \equiv (1 - \overline{\omega}_F)\xi_{1,F} + \overline{\omega}_F\xi_{2,F}.$$

(b) *If $||d|| = \infty$, then $\widetilde{\omega}_{eo} \to_p 0$ and $n^{1/2}(\widehat{\theta}_{eo} - \theta_{F_n}) \to_D \xi_{1,F}$.*

To study the bounds of asymptotic risk difference between $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$, it is important to take into account the data-dependent nature of $\widetilde{\omega}_{eo}$. Unlike $\widehat{\Sigma}_1$ and $\widehat{\Sigma}_2$, the randomness in $\widetilde{\omega}_{eo}$ is non-negligible in the mild misspecification case (a) of Lemma 4.2. In consequence, $\widehat{\theta}_{eo}$ does not achieve the same bounds of asymptotic risk difference as the ideal averaging estimator $(1 - \omega^*_F)\widehat{\theta}_1 + \omega^*_F\widehat{\theta}_2$ does. Nevertheless, below we show that $\widehat{\theta}_{eo}$ is insured against potentially misspecified moments because it uniformly dominates $\widehat{\theta}_1$.

5 Bounds of Asymptotic Risk Difference under Misspecification

In this section, we study the bounds of the asymptotic risk difference defined in (3.12). Note that the asymptotic distributions of $\widehat{\theta}_1$ and $\widehat{\theta}_{eo}$ in Lemma 4.1 and 4.2 only depend on d, $G_{2,F}$ and $\Omega_{2,F}$. For notational convenience, define

$$h_{F,d} = (d', \mathrm{vec}(G_{2,F})', \mathrm{vech}(\Omega_{2,F})') \tag{5.1}$$

for any $F \in \mathcal{F}$ and any $d \in \mathbb{R}^{r^*}_{\infty}$. For the mild misspecification case, define the parameter space of $h_{F,d}$ as

$$H = \{h_{F,d} : d \in \mathbb{R}^{r^*} \text{ and } F \in \mathcal{F} \text{ with } \delta_F = 0_{r^* \times 1}\} \tag{5.2}$$

where δ_F is defined by (1.5) for a given F.

Theorem 5.1 *Suppose that Assumptions 3.1-3.3 hold. The bounds of the asymptotic risk difference satisfy*

$$Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = \max\left\{\sup_{h\in H}\left[g(h)\right], 0\right\},$$

$$Asy\underline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = \min\left\{\inf_{h\in H}\left[g(h)\right], 0\right\},$$

where $g(h) \equiv \mathbb{E}[\overline{\xi}'_F \Upsilon \overline{\xi}_F - \xi'_{1,F}\Upsilon\xi_{1,F}]$ and the expectation is taken under the joint normal distribution of $\mathcal{Z}_{2,F}$.

The upper (or lower) bound of the asymptotic risk difference is determined by the maximum between $\sup_{h\in H}\left[g(h)\right]$ and zero (or the minimum between $\inf_{h\in H}\left[g(h)\right]$ and zero), where $\sup_{h\in H}\left[g(h)\right]$ (or $\inf_{h\in H}\left[g(h)\right]$) is related to the mildly misspecified DGPs and the zero component is associated with the severely misspecified DGPs. Since the GMM averaging estimator has the same asymptotic distribution as the conservative GMM estimator $\widehat{\theta}_1$ under the severely misspecified DGPs, their asymptotic risk difference is zero.

To show that $\widehat{\theta}_{eo}$ uniformly dominates $\widehat{\theta}_1$, Theorem 5.1 implies that it is sufficient to show that $\inf_{h\in H}\left[g(h)\right] < 0$ and $\sup_{h\in H}\left[g(h)\right] \leq 0$. We can investigate $\inf_{h\in H} g(h)$ and $\sup_{h\in H} g(h)$ by simulating $g(h)$. In practice, we replace $G_{2,F}$ and $\Omega_{2,F}$ by their consistent estimators and plot $g(h)$ as a function of d. Even if the uniform dominance condition does not hold, $\min\left\{\inf_{h\in H}\left[g(h)\right], 0\right\}$ and $\max\left\{\sup_{h\in H}\left[g(h)\right], 0\right\}$ quantify the most- and least-favorable scenarios for the averaging estimator.

Theorem 5.2 *Let $A_F \equiv \Upsilon\left(\Sigma_{1,F} - \Sigma_{2,F}\right)$ for any $F \in \mathcal{F}$. Suppose that Assumptions 3.1-3.3 hold. If $\mathrm{tr}(A_F) > 0$ and $\mathrm{tr}(A_F) \geq 4\rho_{\max}(A_F)$ for any $F \in \mathcal{F}$ with $\delta_F = 0$, we have*

$$Asy\underline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) < 0 \text{ and } Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = 0.$$

Thus, $\widehat{\theta}_{eo}$ uniformly dominates $\widehat{\theta}_1$.

Theorem 5.2 indicates that: (i) there exists $\varepsilon_1 < 0$ and some finite integer n_{ε_1} such that the minimum risk difference between $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$ is less than ε_1 for any n larger than n_{ε_1}; (ii) for any $\varepsilon_2 > 0$, there exists a finite integer n_{ε_2} such that the maximum risk difference between $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$ is less than ε_2 for any n larger than n_{ε_2}. Pre-test estimators fail to satisfy both properties (i) and (ii) above at the same time. Take the pre-test estimator based on the J-test for example[17] and consider three scenarios: (a) the critical value is fixed for any sample size; (b) the critical value diverges to infinity; and (c) the critical value converges to zero. In the pointwise asymptotic framework,

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[17]See Section D in Supplemental Appendix for definition and analysis of this estimator.

the J-test based on the critical values in (a), (b) and (c) leads to inconsistent (but conservative) model selection, consistent model selection and no model selection results respectively. The pre-test estimator based on the J-test violates property (ii) in scenarios (a) and (b), and violates property (i) in scenario (c).

Different from the finite-sample results for the JS estimator established for the Gaussian location model, our comparison of the two estimators $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$ is based on the asymptotic bounds of the risk differences. For a given sample size n, we do not provide results on this asymptotic approximation error, and therefore our results do not state how the finite-sample upper bound $\overline{RD}_n(\widehat{\theta}_{eo}, \widehat{\theta}_1; \zeta)$ approaches to zero as $n \to \infty$ and then $\zeta \to \infty$ (e.g., from above or from below). For the Gaussian location model, the asymptotically uniform dominance here is weaker than the classical finite-sample results established for the JS estimator. However, this asymptotic results apply to general nonlinear econometric models with non-normal random variables.[18]

To shed light on the sufficient conditions in Theorem 5.2, let us consider a scenario similar to the JS estimator: $\Sigma_{1,F} = \sigma_{1,F}^2 I_{d_\theta}$, $\Sigma_{2,F} = \sigma_{2,F}^2 I_{d_\theta}$, and $\Upsilon = I_{d_\theta}$. In this case, the sufficient conditions become $\sigma_{1,F} > \sigma_{2,F}$ and $d_\theta \geq 4$. The first condition $\mathrm{tr}(A_F) > 0$, which is reduced to $\sigma_{1,F} > \sigma_{2,F}$, requires that the additional moments $\mathbb{E}_F[g^*(W_i, \theta_F)] = 0$ are non-redundant in the sense that they lead to a more efficient estimator of θ_F. The second condition $\mathrm{tr}(A_F) \geq 4\rho_{\max}(A_F)$, which is reduced to $d_\theta \geq 4$, requires that we are interested in the total risk of several parameters rather than that of a single one. In a more general case where $\Sigma_{1,F}$ and $\Sigma_{2,F}$ are not proportional to the identity matrix, the sufficient conditions are reduced to $\Sigma_{1,F} > \Sigma_{2,F}$ and $d_\theta \geq 4$ under the choice $\Upsilon = (\Sigma_{1,F} - \Sigma_{2,F})^{-1}$, which rescales $\widehat{\theta}$ by the variance reduction $\Sigma_{1,F} - \Sigma_{2,F}$. In a simple linear IV model (Example 3.1) where Z_i^* is independent of $Z_{1,i}$ and the regression error U_i is homoskedastic conditional on the IVs, $\Sigma_{1,F} > \Sigma_{2,F}$ requires that $\mathbb{E}_{F^*}[Z_i^* X_i']$ and $\mathbb{E}_{F^*}[Z_i^* Z_i^{*\prime}]$ both have full rank.

Note that these conditions are sufficient but not necessary. If these sufficient conditions do not hold, we can still simulate the upper bounds in Theorem 5.1 to check the uniform dominance condition. In fact, simulation studies in the next session show that in many cases $\widehat{\theta}_{eo}$ has a smaller finite-sample risk than $\widehat{\theta}_1$ even if these sufficient conditions are violated. Nevertheless, these analytical sufficient conditions can be checked easily before the simulation-based methods are adopted.

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[18] In Section F of the Supplemental Appendix, we show that the averaging GMM estimator has similar finite sample dominace results in the Gaussian location model.

6 Local Uniform Dominance

In this section, we provide a *local uniform dominance* result that strengthens $Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = 0$ in Theorem 5.2 to

$$Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) < 0, \tag{6.1}$$

at the cost of comparing $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$ in a smaller parameter space that only allows for local misspecification up to $n^{-1/2}$. The result is uniform over this shrinking parameter space. Showing $Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) < 0$ is desirable because it implies that the finite sample maximum risk difference between $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$ is bounded above from zero for all large n, whereas $Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = 0$ allows this finite samples maximum risk difference to be a shrinking, but positive value for large n. A smaller parameter space is necessary for this stronger result because $\widehat{\theta}_{eo}$ cannot strictly improve upon $\widehat{\theta}_1$ when the additional moments are severely misspecified and thus do not provide any useful information.

Assumption 6.1 *For each n let $\mathcal{F}_n$ denote a set of DGPs. The following conditions hold:*

(i) *for any n and any $F \in \mathcal{F}_n$ there is $\theta_F \in int(\Theta)$ such that $\mathbb{E}_F[g_1(W, \theta_F)] = 0_{r_1 \times 1}$;*

(ii) *for any $\varepsilon > 0$, $\inf_n \inf_{F \in \mathcal{F}_n} \inf_{\theta \in B_\varepsilon^c(\theta_F)} ||\mathbb{E}_F[g_1(W, \theta)]|| > 0$;*

(iii) *for any n and any $F \in \mathcal{F}_n$ there is $d_F \in \mathbb{R}^{r^*}$ such that $\mathbb{E}_F[g_2(W, \theta_F)] = n^{-1/2}d_F$;*

(iv) *Assumption 3.2 holds for $\mathcal{F}_n$ uniformly over n;*

(v) *$\overline{\Lambda} \equiv \{\overline{v}_F: F \in \mathcal{F}_n \text{ for some } n\}$ is closed;*

(vi) *$\|d_F\| \leq C_L$ for some fixed constant C_L.*

Assumption 6.1.(i) and (ii) are similar to Assumptions 3.1.(i)-(ii), which ensures the unique identification of θ_F for any $F \in \mathcal{F}_n$ and any n. Assumptions 6.1.(iii) implies that the extra moment conditions are mildly misspecified which together with Assumptions 6.1.(i) and (ii) and Assumption 3.2 ensures that the aggressive GMM estimator $\widehat{\theta}_2$ has a normal asymptotic distribution as shown in Lemma 4.1.(a). As a result, Assumptions 3.1.(iii)-(iv) are not needed here. Assumption 3.2 contains some regularity conditions for showing the asymptotic properties of the GMM estimator and it is maintained in Assumption 6.1.(iv). Assumption 6.1.(v) is a reduced version of Assumption 3.3.(i). Assumption 6.1.(vi) is an important condition to show the local uniform dominance result.

To introduce the local uniform dominance result, we define

$$H_{C_L} = \{h_{F,d} : d \in \mathbb{R}^{r^*} \text{ with } \|d\| \leq C_L \text{ and } F \in \mathcal{F}_n \text{ for some } n\}. \tag{6.2}$$

In the local misspecification framework, the set of DGPs $\mathcal{F}_n$ may change with the sample size n. The upper bound of the finite sample risk difference between $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$ should be defined as

$$\overline{RD}_n(\widehat{\theta}_{eo}, \widehat{\theta}_1; \zeta) = \sup_{F \in \mathcal{F}_n} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)], \tag{6.3}$$

which is approximated by

$$Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = \lim_{\zeta \to \infty} \limsup_{n \to \infty} \sup_{F \in \mathcal{F}_n} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)]. \tag{6.4}$$

To show the local uniform dominance result, it is sufficient to study the upper bound of the risk difference $Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1.)$

Lemma 6.1 *Suppose that Assumption 6.1 hold. The upper bound of the asymptotic risk difference satisfies*

$$Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) \leq \sup_{h \in H_{C_L}} [g(h)], \tag{6.5}$$

where $g(h) = \mathbb{E}[\overline{\xi}_F' \Upsilon \overline{\xi}_F - \xi_{1,F}' \Upsilon \xi_{1,F}]$ is defined in Theorem 5.1.

Lemma 6.1 provides an upper bound to the maximum risk difference between $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$. The criterion function $g(h)$ in (6.5) is the same as its counterpart in Theorem 5.1. To show the local uniform dominance result in (6.1), it is sufficient to show that $\sup_{h \in H_{C_L}} [g(h)]$ is bounded away from zero. This is proved in the following Theorem.

Theorem 6.1 *Suppose that Assumption 6.1 hold. If $\text{tr}(A_F) > 0$ and $\text{tr}(A_F) \geq 4\rho_{\max}(A_F)$ for any $F \in \mathcal{F}_n$ and for any n, we have $\sup_{h \in H_{C_L}} [g(h)] < 0$ for any finite constant C_L.*

Combining the results in Lemma 6.1 and Theorem 6.1, we immediately obtain (6.1). The sufficient conditions to ensure that the upper bound $\sup_{h \in H_{C_L}} [g(h)]$ is bounded away from zero are the same as those in Theorem 5.2.

7   Simulation Studies

In this section, we investigate the finite sample performance of our averaging GMM estimator in linear IV models. In addition to the empirical optimal weight $\widetilde{\omega}_{eo}$, we consider another averaging

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estimator based on the JS type of weight. First consider the positive part of the JS weight[19]:

$$\omega_{JS} = 1 - \left(1 - \frac{\text{tr}(\widehat{A}) - 2\rho_{\max}(\widehat{A})}{n(\widehat{\theta}_2 - \widehat{\theta}_1)'\Upsilon(\widehat{\theta}_2 - \widehat{\theta}_1)}\right)_+ \tag{7.1}$$

where $(x)_+ = \max\{0, x\}$ and $\widehat{A}$ is the estimator of A_F using $\widehat{\Sigma}_k$ for $k = 1, 2$. The alternative averaging estimator uses the restricted JS weight

$$\omega_{R,JS} = (\omega_{JS})_+ . \tag{7.2}$$

By construction, $\omega_{JS} \leq 1$ and $0 \leq \omega_{R,JS} \leq 1$. We compare the finite-sample MSEs of our proposed averaging estimator with the empirical optimal weight, the JS type of averaging estimator with the restricted weight in (7.2), the conservative GMM estimator $\widehat{\theta}_1$, and the pre-test GMM estimator based on the J-test. The finite-sample MSE of the conservative GMM estimator is normalized to be 1.

We consider a linear regression model with i.i.d. observed data

$$W_i = (Y_i, X_{1,i}, \ldots, X_{6,i}, Z_{1,i}, \ldots, Z_{12,i}, Z^*_{1,i}, \ldots, Z^*_{6,i})' \text{ for } i = 1, \ldots, n, \tag{7.3}$$

where Y is the dependent variable, $(X_1, \ldots, X_6)$ are 6 endogenous regressors, $(Z_1, \ldots, Z_{12})$ are 12 valid IVs, and $(Z^*_1, \ldots, Z^*_6)$ are 6 invalid IVs. The data is generated as follows. The regression model is

$$Y = \sum_{j=1}^{6} \theta_j X_j + u, \tag{7.4}$$

where X_j are generated by

$$X_j = (Z_j + Z_{j+6})\beta_j + Z_{j+12} + \varepsilon_j \text{ for } j = 1, \ldots, 6. \tag{7.5}$$

We first draw $(Z_1, \ldots, Z_{18}, \varepsilon_1, \ldots, \varepsilon_6, u^*)'$ from normal distribution with mean zero and variance-covariance matrix $\text{diag}(I_{18\times18}, \Sigma_{7\times7})$ where

$$\Sigma_{7\times7} = \begin{pmatrix} I_{6\times6} & 0.25 \times \mathbf{1}_{6\times1} \\ 0.25 \times \mathbf{1}_{1\times6} & 1 \end{pmatrix}. \tag{7.6}$$

To show the performance of the estimator under non-Gaussian errors, we draw η from exponential

---

[19]This formula is a GMM analog of the generalized James-Stein S type shrinkage estimator in Hansen (2016) for parametric models. The shrinkage scalar τ is set to $\text{tr}(\widehat{A}) - 2\rho_{\max}(\text{tr}(\widehat{A}))$ in a fashion similar to the original James-Stein estimator.

distribution with mean 1 and η is independent of $(Z_1, \ldots, Z_{18}, \varepsilon_1, \ldots, \varepsilon_6, u^*)$. The residual term u in (7.4) is generated by

$$u = (u^* + \eta - 1)/2, \tag{7.7}$$

where η is demeaned to ensure that the mean of u is zero. The invalid IVs are generated by

$$Z_j^* = Z_{j+12} + n^{-1/2} d_j u, \text{ for } j = 1, \ldots, 6. \tag{7.8}$$

We set $(\theta_1, \ldots, \theta_6) = 2.5 \times \mathbf{1}_{1\times6}$ and $(\beta_1, \ldots, \beta_6) = 0.5 \times \mathbf{1}_{1\times6}$.

In the main regression equation (7.4), all regressors are endogenous because

$$\mathbb{E}[X_j u] = \mathbb{E}[\varepsilon_j u^*/2] = 0.125 \text{ for } j = 1, \ldots, 6. \tag{7.9}$$

From the expression of Z_j^* above, we see that increasing the magnitude of d_j will enlarge the correlation coefficient between Z_j^* and u and hence the endogeneity of Z_j^*.

Given the sample size n, we consider different DGPs of the simulated data $\{W_i : i = 1, ..., n\}$ by changing the values of the location parameters $(d_1, ..., d_6)$. We consider the following parametrization

$$(d_1, ..., d_6) = r_0(c_1, ..., c_6) \tag{7.10}$$

where r_0 is a scalar that takes values on the grid points between 0 and 25 with the grid length 0.5, $(c_1, \ldots, c_6)$ is parametrized in two different ways. In the first one, we set $c_j = 0$ or 1 for $j = 1, \ldots, 6$ and rule out the case that $c_j = 0$ for all j (since this is the same as the case which sets $r_0 = 0$). In the second one, we consider the polar transformation and set

$$
\begin{aligned}
c_1 &= \sin(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)\sin(\alpha_4)\sin(\alpha_5), \\
c_2 &= \cos(\alpha_1)\sin(\alpha_2)\sin(\alpha_3)\sin(\alpha_4)\sin(\alpha_5), \\
c_3 &= \cos(\alpha_2)\sin(\alpha_3)\sin(\alpha_4)\sin(\alpha_5), \\
c_4 &= \cos(\alpha_3)\sin(\alpha_4)\sin(\alpha_5), \\
c_5 &= \cos(\alpha_4)\sin(\alpha_5), \\
c_6 &= \cos(\alpha_5), \tag{7.11}
\end{aligned}
$$

where $\alpha_1 \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ and $\alpha_j \in \{\pi/4, 3\pi/4\}$ for $j = 2, \ldots, 5$. Therefore, there are 127 different values for $(c_1, \ldots, c_6)$ which together with 51 different values of r_0 produces 6477 different DGPs in the simulation studies. For each DGP, we consider sample size $n = 50, 100, 250, 500$ and

Figure 2. Finite Sample MSEs of the Pre-test and Averaging GMM Estimators
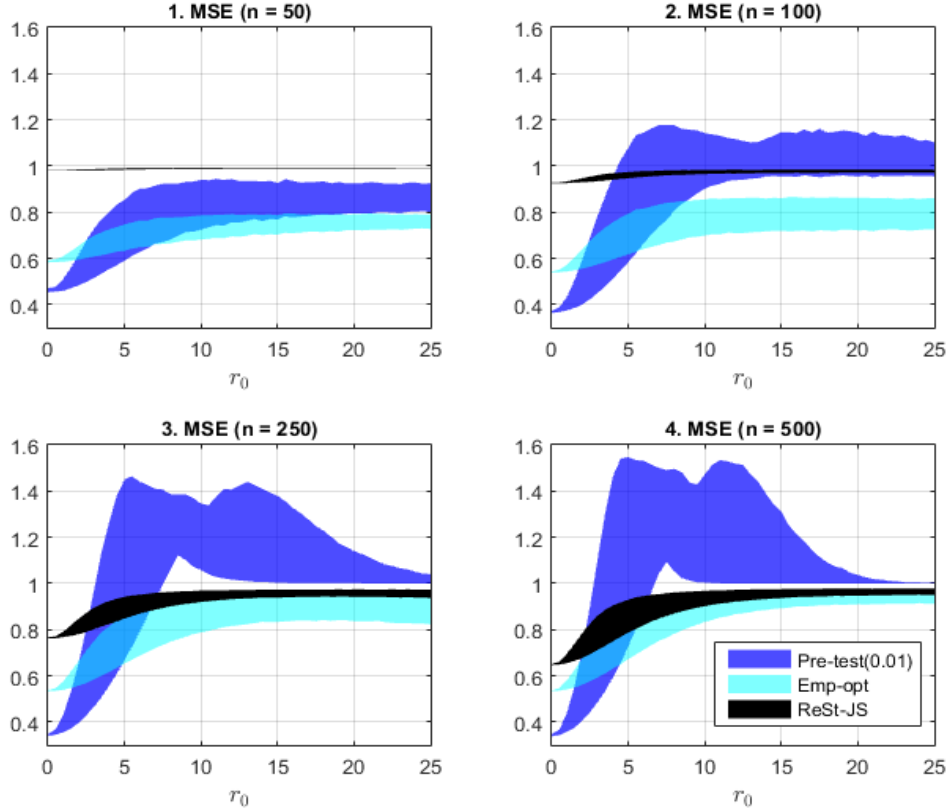
Note: "Pre-test(0.01)" refers to the pre-test GMM estimator based on the J-test with nominal size 0.01; "Emp-opt" refers to the averaging GMM estimator based on the empirical optimal weight; "ReSt-JS" refers to the averaging estimators based on the restricted James-Stein weight, respectively. For each estimator, the upper envelop of the shaded area is the maximum finite sample MSE among the 127 DGPs described in Section 7, and the lower envelop is the minimum. Both maximum and minimum are functions of r_0.

use 25,000 simulation repetitions. Given the sample size and the value of r_0, we report the minimum and maximum of the 127 values of the finite sample MSEs for each estimator.[20] Therefore given each sample size, we report a shaded area for each estimator where the upper envelope represents the estimator's maximum finite sample MSE and the lower envelope is its minimum finite sample MSE. For each estimator, both the upper envelope and lower envelope are functions of r_0.

The MSEs of various estimators of the parameters in (7.4) are included in Figure 2.[21] The two panels on the top of Figure 2 present the MSEs of estimators with sample size $n = 50$ and

---

[20]In this simulation study, we also consider the truncated risk function with $\zeta = 1000$. The simulation results are identical to what we get without truncation. These extra simulation results are available upon request.

[21]The finite sample biases and variances of the GMM estimators are reported in Subsection E.1 of the Supplemental Appendix.

100 respectively, while the two panels on the bottom provides their MSEs with $n = 250$ and 500 respectively. Our findings in the simulation studies are summarized as follows. First, the averaging GMM estimator $\widehat{\theta}_{eo}$ has smaller MSE than $\widehat{\theta}_1$ uniformly over d in all sample sizes considered, which is predicted by our theory because the key sufficient condition is satisfied in this model.[22] Second, the pre-test GMM estimator does not dominate the conservative GMM estimator $\widehat{\theta}_1$ when the sample size becomes slightly large (e.g., $n \geq 100$). For example, when $n = 500$ and the location parameter r_0 is close to zero, the pre-test GMM estimator has relative MSE as low as 0.35. However, its relative MSE is above 1 when r_0 is between 5 and 20. Third, comparing the two averaging estimators, we find that the restricted JS estimator does not reduce the MSE as much as the averaging estimator based on $\widetilde{\omega}_{eo}$ for different sample sizes and different DGPs considered in this simulation study.

8   Conclusion

This paper studies the averaging GMM estimator that combines the conservative estimator and the aggressive estimator with a data-dependent weight. The averaging weight is the sample analog of an optimal non-random weight. We provide a sufficient class of drifting DGPs under which the pointwise asymptotic results combine to yield uniform approximations to the finite-sample risk difference between two estimators. Using this asymptotic approximation, we show that the proposed averaging GMM estimator uniformly dominates the conservative GMM estimator.

Inference based on the averaging estimator is an interesting and challenging problem. As pointed out in Pötscher (2006), the finite sample density of the averaging estimator can not be consistently estimated, which implies that directly applying an estimator of the finite-sample density may not yield uniformly valid inference. In addition to the uniform validity, a desirable confidence set should have smaller volume than that obtained from the conservative moments alone. We leave the inference issue to future investigation.

---

[22]It is easy to show that in this simulation design, when $\delta_F = 0$ we have $\mathrm{tr}(A_F) = 4$ and $\mathrm{tr}(A_F) - 4\rho_{\max}(A_F) = 4/3$.

A   Appendix

In Lemma 3.1, define

$$
\begin{aligned}
c_\rho &\equiv \min_{F^* \in \mathcal{F}^*} \left\{ \rho_{\min}(\Gamma_{xz_1}\Gamma_{z_1 x}), \rho_{\min}(\Psi) \right\}, \\
C_\rho &\equiv \max_{F^* \in \mathcal{F}^*} \left\{ \|\phi\|^2, \rho_{\max}(\Psi) \right\}. \\
C_\Delta &\equiv \sup_{\delta_0 \in \Delta_\delta} \|\delta_0\|^2
\end{aligned}
\tag{A.1}
$$

Let

$$
C_{*,W} \equiv 2(d_\theta + r_2 + 1)C_\rho, \ c_{*,\rho} \equiv \min\{1, c_\rho^2\} \text{ and } C_{*,\rho} \equiv C_{*,W}^2 (2 + C_\Delta^{1/2})^2.
\tag{A.2}
$$

Then, in Lemma 3.1 (iii), the constant ε is given by

$$
\varepsilon = c_{*,\rho} C_{*,\rho}^{-1} C_\Delta^{-1},
\tag{A.3}
$$

i.e., we require the condition to hold on a set bounded away from 0 by ε. The details of the proofs are given in Section B of the Supplemental Appendix.

A.1   Proof of the Results in Section 4

Let $\mu_n(g_2(W, \theta)) = n^{-1/2} \sum_{i=1}^n (g_2(W_i, \theta) - \mathbb{E}_{F_n}[g_2(W_i, \theta)])$. In the rest of the Appendix, we use C to denote a generic fixed positive finite constant which does not depend on any $F \in \mathcal{F}$ or n.

Lemma A.1 *Suppose that Assumption 3.2.(ii) holds and Θ is compact. Then we have*
(i) $\sup_{\theta \in \Theta} \|\overline{g}_2(\theta) - \mathbb{E}_{F_n}[g_2(W_i, \theta)]\| = o_p(1)$;
(ii) $\sup_{\theta \in \Theta} \left\| n^{-1} \sum_{i=1}^n g_2(W_i, \theta)g_2(W_i, \theta)' - \mathbb{E}_{F_n}[g_2(W_i, \theta)g_2(W_i, \theta)'] \right\| = o_p(1)$;
(iii) $\sup_{\theta \in \Theta} \left\| n^{-1} \sum_{i=1}^n g_{2,\theta}(W_i, \theta) - \mathbb{E}_{F_n}[g_{2,\theta}(W_i, \theta)] \right\| = o_p(1)$;
(iv) $\mu_n(g_2(W, \theta))$ *is stochastic equicontinuous over* $\theta \in \Theta$;
(v) $\Omega_{2,F_n}^{-1/2} \mu_n(g_2(W, \theta_{F_n})) \to_D N(0_{r_2 \times 1}, I_{r_2})$.

Proof of Lemma A.1.   See Lemma 11.3-11.5 of Andrews and Cheng (2013). ∎

Define $M_{k,F}(\theta) = \mathbb{E}_F[g_k(W, \theta)]$, $G_{k,F}(\theta) = \mathbb{E}_F[g_{k,\theta}(W, \theta)]$ and $\Omega_{k,F}(\theta) = \text{Var}_F[g_k(W, \theta)]$, for any $F \in \mathcal{F}$, for any $\theta \in \Theta$ and for $k = 1, 2$. The next lemma shows that $M_{2,F}(\cdot)$, $G_{2,F}(\cdot)$ and $\Omega_{2,F}(\cdot)$ are Lipschitz continuous uniformly over $F \in \mathcal{F}$.

Lemma A.2 *Under Assumptions 3.2.(i)-(ii), for any $F \in \mathcal{F}$ and any $\theta_1, \theta_2 \in \Theta$, we have:*
(i) $\|M_{2,F}(\theta_1) - M_{2,F}(\theta_2)\| \leq C \|\theta_1 - \theta_2\|$;

(ii) $\|G_{2,F}(\theta_1) - G_{2,F}(\theta_2)\| \leq C \|\theta_1 - \theta_2\|$;

(iii) $\|\Omega_{2,F}(\theta_1) - \Omega_{2,F}(\theta_2)\| \leq C \|\theta_1 - \theta_2\|$.

Proof of Lemma A.2 is included in Section C of the Supplemental Appendix.

Lemma A.3 *Suppose that Assumptions 3.1.(i)-(ii) and 3.2.(i)-(ii) hold. Then for any sequence of DGPs $\{F_n\}$, we have*

$$\widetilde{\theta}_1 - \theta_{F_n} = o_p(1) \ and \ \overline{\Omega}_2 = \Omega_{2,F_n} + o_p(1), \tag{A.4}$$

where $\widetilde{\theta}_1$ is a preliminary estimator defined as

$$\widetilde{\theta}_1 = \arg\min_{\theta \in \Theta} \overline{g}_1(\theta)' \overline{g}_1(\theta) \tag{A.5}$$

and $\overline{\Omega}_2$ is defined in (C.13) of the Supplemental Appendix.

Proof of Lemma A.3 is included in Section C of the Supplemental Appendix.

Lemma A.4 *Suppose that Assumptions 3.1.(i)-(ii) and 3.2 hold. Then for any sequence of DGPs $\{F_n\}$, we have*

$$n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) = \Gamma_{1,F_n} \mu_n(g_1(W, \theta_{F_n})) + o_p(1), \tag{A.6}$$

where $\Gamma_{1,F_n} \mu_n(g_1(W, \theta_{F_n})) \equiv - \left(G_{1,F_n}' \Omega_{1,F_n}^{-1} G_{1,F_n}\right)^{-1} G_{1,F_n}' \Omega_{1,F_n}^{-1} = O_p(1)$.

Proof of Lemma A.4 is included in Section C of the Supplemental Appendix.

Lemma A.5 *Suppose that Assumptions 3.1.(iii) and 3.2.(i)-(iii) hold. Then for any sequence of DGPs $\{F_n\}$, we have*

$$\widehat{\theta}_2 - \theta_{F_n}^* = o_p(1). \tag{A.7}$$

Proof of Lemma A.5 is included in Section C of the Supplemental Appendix.

Lemma A.6 *Suppose that Assumptions 3.1.(i)-(ii) and 3.2.(i)-(iii) hold. Consider any sequence of DGPs $\{F_n\}$ such that $\delta_{F_n} = o(1)$. Then we have*

$$\widehat{\theta}_2 - \theta_{F_n} = o_p(1). \tag{A.8}$$

If we further have Assumption 3.2.(iv), then

$$n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) = (\Gamma_{2,F_n} + o_p(1)) \left\{\mu_n(g_2(W, \theta_{F_n})) + n^{1/2}\delta_{2,F_n}\right\} + o_p(1), \tag{A.9}$$

where $\Gamma_{2,F_n} = - \left(G_{2,F_n}' \Omega_{2,F_n}^{-1} G_{2,F_n}\right)^{-1} G_{2,F_n}' \Omega_{2,F_n}^{-1}$ and $\delta_{2,F_n} = (\mathbf{0}_{1 \times r_1}, \delta_{F_n}')'$.

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Proof of Lemma A.6 is included in Section C of the Supplemental Appendix.

Lemma A.7 *Under Assumptions 3.2.(ii) and 3.3.(ii), for any sequence of DGPs $\{F_{p_n}\}$ with $F_{p_n} \in \mathcal{F}$ where $\{p_n\}$ is a subsequence of $\{n\}$, there is a subsequence $\{p_n^*\}$ of $\{p_n\}$ such that $v_{F_{p_n^*}}(\theta_{F_{p_n^*}}) \to v_F(\theta_F)$ as $p_n^* \to \infty$, where $F \in \mathcal{F}$.*

Proof of Lemma A.7. Recall that $\Lambda = \{v_F : F \in \mathcal{F}\}$. By Assumptions 3.2.(ii) and 3.3.(ii), Λ is compact. Hence for any sequence $\{v_{F_{p_n}}(\theta_{F_{p_n}})\}$ in Λ, it has a convergent subsequence $\{v_{F_{p_n^*}}(\theta_{F_{p_n^*}})\}$ such that $v_{F_{p_n^*}}(\theta_{F_{p_n^*}}) \to v_F(\theta_F)$ as $p_n^* \to \infty$, where $F \in \mathcal{F}$. ∎

Lemma A.8 *Suppose that Assumptions 3.1.(i)-(ii) and 3.2 hold. Consider any sequence of DGPs $\{F_n\}$ such that $\overline{v}_{F_n} \to \overline{v}_F$ for some $F \in \mathcal{F}$, and $n^{1/2}\delta_{F_n} \to d$ for $d \in \mathbb{R}^{r^*}$. Then*

$$
\begin{pmatrix} n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \\ n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \end{pmatrix} \to_D \begin{pmatrix} \xi_{1,F} \\ \xi_{2,F} \end{pmatrix} \equiv \begin{pmatrix} \Gamma_{1,F}\mathcal{Z}_{1,F} \\ \Gamma_{2,F}(\mathcal{Z}_{2,F} + d_0) \end{pmatrix},
$$

where $d_0 = (\mathbf{0}_{1\times r_1}, d')'$.

Proof of Lemma A.8. In the proof, we use

$$
G_{2,F_n} \to G_{2,F} \text{ and } \Omega_{2,F_n} \to \Omega_{2,F} \tag{A.10}
$$

for some $F \in \mathcal{F}$, which is assumed in the lemma. Under Assumptions 3.1.(i)-(ii) and 3.2, for the sequence of DGPs $\{F_n\}$ considered in the lemma, we can apply Lemma A.4 and Lemma A.6 to deduce that

$$
\begin{pmatrix} n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \\ n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \end{pmatrix} = \begin{pmatrix} \Gamma_{1,F_n}\mu_n(g_1(W, \theta_{F_n})) \\ (\Gamma_{2,F_n} + o_p(1))\{\mu_n(g_2(W, \theta_{F_n})) + n^{1/2}\delta_{2,F_n}\} \end{pmatrix} + o_p(1), \tag{A.11}
$$

where $\delta_{2,F_n} = (\mathbf{0}_{1\times r_1}, \delta'_{F_n})'$. By (A.10) and Assumption 3.2, we have

$$
\Gamma_{1,F_n} = \Gamma_{1,F} + o(1) \text{ and } \Gamma_{2,F_n} = \Gamma_{2,F} + o(1) \tag{A.12}
$$

where $\Gamma_{k,F} = -\left(G'_{k,F}\Omega_{k,F}^{-1}G_{k,F}\right)^{-1}G'_{k,F}\Omega_{k,F}^{-1}$ for $k = 1, 2$. Collecting the results in Lemma A.1.(v), (A.11) and (A.12), and then applying the continuous mapping theorem (CMT), we have

$$
\begin{pmatrix} n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \\ n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \end{pmatrix} \to_D \begin{pmatrix} \Gamma_{1,F}^* \\ \Gamma_{2,F} \end{pmatrix}(\mathcal{Z}_{2,F} + d_0), \tag{A.13}
$$

where $\mathcal{Z}_{2,F} \sim N(\mathbf{0}_{r_2 \times 1}, \Omega_{2,F})$, $\Gamma^*_{1,F} = (\Gamma_{1,F}, \mathbf{0}_{d_\theta \times r^*})$ and $d_0 = (\mathbf{0}_{1 \times r_1}, d')'$. The claimed result follows from (A.13) and the definitions of $\Gamma^*_{1,F}$ and $\mathcal{Z}_{2,F}$. ∎

Proof of Lemma 4.1. The claimed result in Part (a) has been proved in Lemma A.8.

We next consider the case that $n^{1/2}\delta_{F_n} \to d$ with $||d|| = \infty$. Note that the results in (A.6) and (A.12) do not depend on $||d|| < \infty$ or $||d|| = \infty$. Using (A.6), (A.12), Lemma A.1.(v) and the CMT, we have

$$n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \to_D \Gamma_{1,F}\mathcal{Z}_{1,F}. \tag{A.14}$$

To study the properties of $\widehat{\theta}_2$, we have to consider two separate scenarios: (1) $\delta_{F_n} = o(1)$; and (2) $||\delta_{F_n}|| > c_\delta$ for some $c_\delta > 0$. In scenario (1), Assumption 3.2, Lemma A.1.(v) and Lemma A.6 imply that

$$n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) = (\Gamma_{2,F_n} + o_p(1))n^{1/2}\delta_{F_n} + O_p(1). \tag{A.15}$$

By Assumption 3.1.(iv) and $||n^{1/2}\delta_{F_n}|| \to \infty$,

$$n\delta'_{F_n}\Gamma'_{2,F_n}\Gamma_{2,F_n}\delta_{F_n} \geq C^{-2}n\delta'_{F_n}\delta_{F_n} \to \infty \tag{A.16}$$

which together with (A.15) implies that $||n^{1/2}(\widehat{\theta}_2 - \theta_{F_n})|| \to_p \infty$.

Finally, we consider the scenario (2) where $||\delta_{F_n}|| > c_\delta$. By Assumption 3.1.(iv),

$$||G'_{2,F_n}\Omega^{-1}_{2,F_n}\delta_{F_n}|| > C^{-1}||\delta_{F_n}|| > c_\delta C^{-1} \tag{A.17}$$

for any n. As $\theta^*_{F_n}$ is the minimizer of $Q_{F_n}(\theta)$, it has the following first order condition

$$0_{d_\theta \times 1} = G_{2,F_n}(\theta^*_{F_n})'\Omega^{-1}_{2,F_n}M_{2,F_n}(\theta^*_{F_n}), \tag{A.18}$$

which implies that

$$\begin{aligned}
G'_{2,F_n}\Omega^{-1}_{2,F_n}\delta_{F_n} &= G_{2,F_n}(\theta_{F_n})'\Omega^{-1}_{2,F_n}M_{2,F_n}(\theta_{F_n}) - G_{2,F_n}(\theta^*_{F_n})'\Omega^{-1}_{2,F_n}M_{2,F_n}(\theta^*_{F_n}) \\
&= \left[G_{2,F_n}(\theta_{F_n}) - G_{2,F_n}(\theta^*_{F_n})\right]'\Omega^{-1}_{2,F_n}M_{2,F_n}(\theta_{F_n}) \\
&\quad + G_{2,F_n}(\theta^*_{F_n})'\Omega^{-1}_{2,F_n}\left[M_{2,F_n}(\theta_{F_n}) - M_{2,F_n}(\theta^*_{F_n})\right].
\end{aligned} \tag{A.19}$$

By Lemma A.2, the Cauchy-Schwarz inequality and Assumption 3.2.(ii)-(iii), we have

$$\begin{aligned}
&\left\|\left[G_{2,F_n}(\theta_{F_n}) - G_{2,F_n}(\theta^*_{F_n})\right]'\Omega^{-1}_{2,F_n}M_{2,F_n}(\theta_{F_n})\right\| \\
&\leq \left\|G_{2,F_n}(\theta_{F_n}) - G_{2,F_n}(\theta^*_{F_n})\right\|\left\|\Omega^{-1}_{2,F_n}M_{2,F_n}(\theta_{F_n})\right\| \leq C\left\|\theta_{F_n} - \theta^*_{F_n}\right\|,
\end{aligned} \tag{A.20}$$

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where C is a fixed constant. Similarly, we have

$$
\left\| G_{2,F_n}(\theta_{F_n}^*)' \Omega_{2,F_n}^{-1} \left[M_{2,F_n}(\theta_{F_n}) - M_{2,F_n}(\theta_{F_n}^*) \right] \right\|
$$
$$
\leq \ \left\| M_{2,F_n}(\theta_{F_n}) - M_{2,F_n}(\theta_{F_n}^*) \right\| \left\| \Omega_{2,F_n}^{-1} G_{2,F_n} \theta_{F_n}^* \right\| \leq C \left\| \theta_{F_n} - \theta_{F_n}^* \right\|. \tag{A.21}
$$

Combining the results in (A.19), (A.20) and (A.21), and using the triangle inequality, we have

$$
\left\| \theta_{F_n} - \theta_{F_n}^* \right\| \geq c_\delta C \tag{A.22}
$$

for some fixed constant C. Using $\widehat{\theta}_2 = \theta_{F_n}^* + o_p(1)$ (which is proved in Lemma A.5) and the triangle inequality, we obtain

$$
\left\| \widehat{\theta}_2 - \theta_{F_n} \right\| \geq \left| ||\widehat{\theta}_2 - \theta_{F_n}^*|| - \left\| \theta_{F_n}^* - \theta_{F_n} \right\| \right| = \left\| \theta_{F_n}^* - \theta_{F_n} \right\| (1 + o_p(1)), \tag{A.23}
$$

which together with (A.22) implies that $n^{1/2}||\widehat{\theta}_2 - \theta_{F_n}|| \to_p \infty$. This finishes the proof. ∎

Lemma A.9 (a) $\Gamma_{1,F}^* d_0 \ = \ \mathbf{0}_{d_\theta \times 1}$; (b) $\Gamma_{1,F}^* \Omega_{2,F} \Gamma_{1,F}^{*\prime} \ = \ \Sigma_{1,F}$; (c) $\Gamma_{1,F}^* \Omega_{2,F} \Gamma_{2,F}' \ = \ \Sigma_{2,F}$; (d) $\Gamma_{2,F} \Omega_{2,F} \Gamma_{2,F}' = \Sigma_{2,F}$.

Proof of Lemma A.9 is included in Section C of the Supplemental Appendix.

A.2   Proof of the Results in Section 5

We first present some generic results on the bounds of asymptotic risk difference between two estimators under some high-level conditions. Then we apply these generic results to the two specific estimators we consider in this paper: $\widehat{\theta}_{eo}$ and $\widehat{\theta}_1$. The proof uses the subsequence techniques used to show the asymptotic size of a test in Andrews, Cheng, and Guggenberger (2011) but we adapt the proof and notations to the current setup and extend results from test to estimators.

Recall that $h_{F,d} = (d', \text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})')$ and $\overline{v}_F = (\text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})')$ for any $F \in \mathcal{F}$ and any $d \in \mathbb{R}_\infty^{r^*}$. We have defined

$$
H = \{ h_{F,d} : d \in \mathbb{R}^{r^*} \text{ and } F \in \mathcal{F} \text{ with } \delta_F = 0_{r^* \times 1} \} \tag{A.24}
$$

where δ_F is defined by (1.5) for a given F. Define

$$
H_\infty^* = \{ h_{F,d} : d \in \mathbb{R}_\infty^{r^*} \text{ with } ||d|| = \infty \text{ and } F \in \mathcal{F} \}. \tag{A.25}
$$

Let $d_h = r^* + d_\theta r_2 + (r_2 + 1) r_2 / 2$. It is clear that $h_{F,d}$ is a d_h-dimensional vector.

Condition A.1 (i) *For any sequence of DGPs $\{F_{p_n}\}$ with $F_{p_n} \in \mathcal{F}$ where $\{p_n\}$ is a subsequence of $\{n\}$, there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ and some $F \in \mathcal{F}$ such that $v_{F_{p_n^*}} \to v_F$ as $p_n^* \to \infty$;*

(ii) $M_{1,F}(\theta) = 0_{r_1 \times 1}$ has a unique solution at $\theta_F \in \Theta$ for any $F \in \mathcal{F}$;

(iii) $M_{2,F}(\cdot)$ is uniform equicontinuous over $F \in \mathcal{F}$;

(iv) for any subsequence $\{p_n\}$ of $\{n\}$, if $(p_n)^{1/2}\delta_{F_{p_n}} \to d$ for $d \in \mathbb{R}_\infty^{r^*}$ and $v_{F_{p_n}} \to v_F$, then

$$\lim_{n \to \infty} \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}, \theta_{F_{p_n}})] = R_\zeta(h_{F,d}) \text{ and } \lim_{n \to \infty} \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widetilde{\theta}, \theta_{F_{p_n}})] = \widetilde{R}_\zeta(h_{F,d})$$

where $R_\zeta(h_{F,d})$ and $\widetilde{R}_\zeta(h_{F,d})$ are some non-negative functions that are bounded from above by ζ for any $F \in \mathcal{F}$ and any $d \in \mathbb{R}_\infty^{r^*}$;

(v) for any $F \in \mathcal{F}$ with $\delta_F = 0_{r^* \times 1}$, there exists a constant $\varepsilon_F > 0$ such that for any $\widetilde{\delta} \in \mathbb{R}^{r^*}$ with $0 \le ||\widetilde{\delta}|| < \varepsilon_F$, there is $\widetilde{F} \in \mathcal{F}$ with $\delta_{\widetilde{F}} = \widetilde{\delta}$ and $||\overline{v}_F - \overline{v}_{\widetilde{F}}|| \le C||\widetilde{\delta}||^\kappa$ for some $\kappa > 0$;

(vi) for any $h_{F,d} \in H_\infty^*$ and $h_{F,\widetilde{d}} \in H_\infty^*$, we have

$$R_\zeta(h_{F,d}) = R_\zeta(h_{F,\widetilde{d}}) \text{ and } \widetilde{R}_\zeta(h_{F,d}) = \widetilde{R}_\zeta(h_{F,\widetilde{d}})$$

for any $\zeta > 0$.

Condition A.1.(i) requires that for any sequence of $\{v_{F_{p_n}}\}$, it has a convergent subsequence $\{v_{F_{p_n^*}}\}$ with limit being v_F for some $F \in \mathcal{F}$. This condition is verified under Assumptions 3.2.(ii) and 3.3.(ii) in Lemma A.7. Condition A.1.(ii) is the unique identification condition of θ_F which holds under Assumptions 3.1.(i)-(ii). Condition A.1.(iii) holds under Assumption 3.2.(ii) by Lemma A.2. Condition A.1.(iv) is a key assumption to derive an explicit upper bound of asymptotic risk. This condition can be verified by using Lemma 4.1 as we shall show in the proof of Theorem 5.1. Condition A.1.(v) enables us to show that the upper bound we derived for the asymptotic risk is also a lower bound. This condition is assumed in Assumption 3.3.(i). Condition A.1.(vi), in our context, requires that the asymptotic (truncated) risk of $\widehat{\theta}$ (or $\widetilde{\theta}$) under the subsequences of DGPs $\{F_{p_n}\}$ satisfying the restrictions in Condition A.1.(iv) are identical whenever $(p_n)^{1/2}\delta_{F_{p_n}} \to d$ with $||d|| = \infty$. Condition A.1 is verified in the proof of Theorem 5.1 below.

Lemma A.10 *Under Conditions A.1.(i) - A.1.(iv), we have*

$$AsyR_\zeta(\widehat{\theta}) \le \max \left\{ \sup_{h \in H} R_\zeta(h), \sup_{h \in H_\infty^*} R_\zeta(h) \right\}, \tag{A.26}$$

where $AsyR_\zeta(\widehat{\theta}) \equiv \limsup_{n \to \infty} \sup_{F \in \mathcal{F}} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}, \theta_F)]$.

Proof of Lemma A.10. Let $\{F_n\}$ be a sequence such that

$$\limsup_{n\to\infty}\mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta},\theta_{F_n})] = \limsup_{n\to\infty}\left(\sup_{F\in\mathcal{F}}\mathbb{E}_F[\ell_\zeta(\widehat{\theta},\theta_F)]\right) \equiv AsyR_\zeta(\widehat{\theta}). \tag{A.27}$$

Such a sequence always exists by the definition of supremum. The sequence $\{\mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta},\theta_{F_n})]\colon n \geq 1\}$ may not converge. However, by the definition of limsup, there exists a subsequence of $\{n\}$, say $\{p_n\}$, such that $\{\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n}})]\colon n \geq 1\}$ converges and

$$\lim_{n\to\infty}\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n}})] = AsyR_\zeta(\widehat{\theta}). \tag{A.28}$$

Below we show that for any subsequence $\{p_n\}$ of $\{n\}$ such that $\{\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n}})]\colon n \geq 1\}$ is convergent, there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ such that

$$\lim_{n\to\infty}\mathbb{E}_{F_{p_n^*}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n^*}})] = R_\zeta(h) \text{ for some } h \in H \text{ or } H_\infty^*. \tag{A.29}$$

Because $\lim_{n\to\infty}\mathbb{E}_{F_{p_n^*}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n^*}})] = \lim_{n\to\infty}\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n}})]$, which combined with (A.28) and (A.29) implies that

$$AsyR_\zeta(\widehat{\theta}) = R_\zeta(h) \text{ for some } h \in H \text{ or } H_\infty^*. \tag{A.30}$$

The desired result in (A.26) follows immediately by (A.30).

To show that there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ such that (A.29) holds, it suffices to show that for any sequence $\{F_n\}$ and any subsequence $\{p_n\}$ of $\{n\}$, there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ for which we have

$$(p_n^*)^{1/2}\delta_{F_{p_n^*}} \to d \text{ for } d \in \mathbb{R}_\infty^{r^*} \text{ and } v_{F_{p_n^*}} \to v_F \tag{A.31}$$

for some $F \in \mathcal{F}$. If (A.31) holds, then we can use Condition A.1.(iv) to deduce that

$$\lim_{n\to\infty}\mathbb{E}_{F_{p_n^*}}[\ell_\zeta(\widehat{\theta},\theta_{F_{p_n^*}})] = R_\zeta(h_{F,d}) \tag{A.32}$$

for the sequence of DGPs $\{F_{p_n^*}\}$ that satisfies (A.31). As $d \in \mathbb{R}_\infty^{r^*}$, we have either $\|d\| < \infty$ or $\|d\| = \infty$. In the first case, $\|d\| < \infty$ together with $(p_n^*)^{1/2}\delta_{F_{p_n^*}} \to d$ and $\delta_{F_{p_n^*}} \to \delta_F$ (which is implied by $v_{F_{p_n^*}} \to v_F$) implies that $\delta_F = 0_{r^*\times 1}$, which implies that $h_{F,d} \in H$ by the definition of H. In the second case, $h_{F,d} \in H_\infty^*$ by the definition of H_∞^*. We have proved that $h_{F,d}$ in (A.32) belongs either to H or H_∞^* which together with (A.32) proves (A.29).

Finally, we show that for any sequence $\{F_n\}$ and any subsequence $\{p_n\}$ of $\{n\}$, there ex-

ists a subsequence $\{p_n^*\}$ of $\{p_n\}$ for which (A.31) holds. Let $\delta_{p_n,j}$ denote the j-th component of δ_{p_n} and $p_{1,n} = p_n$ for any $n \geq 1$. For $j = 1$, either (a) $\limsup_{n\to\infty} |p_{j,n}^{1/2} \delta_{p_{j,n},j}| < \infty$; or (b) $\limsup_{n\to\infty} |p_{j,n}^{1/2} \delta_{p_{j,n},j}| = \infty$. If (a) holds, then for some subsequence $\{p_{j+1,n}\}$ of $\{p_{j,n}\}$, $p_{j+1,n}^{1/2} \delta_{p_{j+1,n},j} \to d_j$ for some $d_j \in \mathbb{R}$. If (b) holds, then for some subsequence $\{p_{j+1,n}\}$ of $\{p_{j,n}\}$, $p_{j+1,n}^{1/2} \delta_{p_{j+1,n},j} \to \infty$ or $-\infty$. As r^* is a fixed positive integer, we can apply the same arguments successively for $j = 1,...,r^*$ to obtain a subsequence $\{p_{r^*,n}\}$ of $\{p_n\}$ such that $(p_{r^*,n})^{1/2} \delta_{p_{r^*,n}} \to d \in \mathbb{R}_\infty^{r^*}$. By Condition A.1.(i), we know that there exists a subsequence $\{p_n^*\}$ of $\{p_{r^*,n}\}$ such that $v_{p_n^*} \to v_F$ for some $F \in \mathcal{F}$, which finishes the proof of (A.31). ∎

Lemma A.11 *Suppose that Condition A.1.(v) holds. Then (i) for any $h_{F,d} \in H$, there exists a sequence of DGPs $\{F_n\}$ with $F_n \in \mathcal{F}$ such that*

$$n^{1/2} \delta_{F_n} \to d, \ G_{2,F_n} \to G_{2,F} \ and \ \Omega_{2,F_n} \to \Omega_{2,F};\tag{A.33}$$

(ii) for any $h_{F,d} \in H_\infty^*$, there exists a sequence of DGPs $\{F_n\}$ with $F_n \in \mathcal{F}$ such that

$$||n^{1/2} \delta_{F_n}|| \to \infty, \ G_{2,F_n} \to G_{2,F}, \ \Omega_{2,F_n} \to \Omega_{2,F} \ and \ \delta_{F_n} \to \delta_F.\tag{A.34}$$

Proof of Lemma A.11. (i) By the definition of H, we have $\delta_F = 0_{r^* \times 1}$ for any F such that $h_{F,d} \in H$. Let N_{ε_F} be the smallest n such that $||d|| n^{-1/2} < \varepsilon_F$. By Condition A.1.(v), for any $n \geq N_{\varepsilon_F}$ we can find a DGP F_n such that

$$\delta_{F_n} = n^{-1/2} d \ and \ ||\overline{v}_{F_n} - \overline{v}_F|| \leq n^{-\kappa/2} C ||d||^\kappa.\tag{A.35}$$

For any $n < N_{\varepsilon_F}$ such that $||d|| n^{-1/2} \geq \varepsilon_F$, we let $F_n = F$. The desired properties in (A.33) holds under the constructed sequence of DGPs $\{F_n\}$ by (A.35), because C is a fixed constant and $\kappa > 0$.

(ii) For any $h_{F,d} \in H_\infty^*$, we have either $\delta_F = 0_{r^* \times 1}$ or $||\delta_F|| > 0$. We first consider the case that $\delta_F = 0_{r^* \times 1}$. Let $1_{r^* \times 1}$ denote the $r^* \times 1$ vector of ones. Let N_{ε_F} be the smallest n such that $n^{-1/4} (r^*)^{1/2} < \varepsilon_F$. By Condition A.1.(v), for any $n \geq N_{\varepsilon_F}$ we can find a DGP F_n such that

$$\delta_{F_n} = n^{-1/4} 1_{r^* \times 1} \ and \ ||\overline{v}_{F_{p_n}} - \overline{v}_F|| \leq C n^{-\kappa/4} (r^*)^{\kappa/2}.\tag{A.36}$$

For any $n < N_{\varepsilon_F}$ such that $n^{-1/4} (r^*)^{1/2} \geq \varepsilon_F$, we let $F_n = F$. The desired properties in (A.34) holds under the constructed sequence of DGPs $\{F_n\}$ by (A.36), because C is a fixed constant and $\kappa > 0$. When $||\delta_F|| > 0$, we define a trivial sequence of DGPs $\{F_n\}$ as $F_n = F$ for any n. It is clear that (A.34) holds trivially in this case. ∎

Lemma A.12 *Under Condition A.1, we have*

$$AsyR_\zeta(\widehat{\theta}) = \max \left\{ \sup_{h \in H} R_\zeta(h), \ \sup_{h \in H_\infty^*} R_\zeta(h) \right\}. \tag{A.37}$$

Proof of Lemma A.12. In view of the upper bound in (A.26) in Lemma A.10, it is sufficient to show that

$$AsyR_\zeta(\widehat{\theta}) \geq \max \left\{ \sup_{h \in H} R_\zeta(h), \ \sup_{h \in H_\infty^*} R_\zeta(h) \right\}. \tag{A.38}$$

First, we note that for any $h_{d,F} = (d', \text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})') \in H$, there exists a sequence $\{F_n \in \mathcal{F} : n \geq 1\}$ such that

$$n^{1/2}\delta_{F_n} \to d \in \mathbb{R}^{r^*} \text{ and } v_{F_n} \to v_F \tag{A.39}$$

by Lemma A.11.(i). The sequence $\mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta}, \theta_{F_n})]$ may not be convergent, but there exists a subsequence $\{p_n\}$ of n such that $\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}, \theta_{F_{p_n}})]$ is convergent and

$$\lim_{n \to \infty} \mathbb{E}_{F_{p_n}}[\ell(\widehat{\theta}, \theta_{F_{p_n}})] = \limsup_{n \to \infty} \mathbb{E}_{F_n}[\ell(\widehat{\theta}, \theta_{F_n})]. \tag{A.40}$$

As $\{p_n\}$ is a subsequence of $\{n\}$, by (A.39)

$$(p_n)^{1/2}\delta_{F_{p_n}} \to d \in \mathbb{R}^{r^*} \text{ and } v_{F_{p_n}} \to v_F. \tag{A.41}$$

By Condition A.1.(iv), we have that

$$\lim_{n \to \infty} \mathbb{E}_{F_{p_n}}[\ell(\widehat{\theta}, \theta_{F_{p_n}})] = R_\zeta(h_{F,d}), \tag{A.42}$$

which combined with (A.40) and the definition of $AsyR_\zeta(\widehat{\theta})$ gives

$$AsyR_\zeta(\widehat{\theta}) = \limsup_{n \to \infty} \sup_{F \in \mathcal{F}} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}, \theta_F)] \geq \limsup_{n \to \infty} \mathbb{E}_{F_n}[\ell(\widehat{\theta}, \theta_{F_n})] = R_\zeta(h_{F,d}). \tag{A.43}$$

Second, consider any $h_{d,F} = (d', \text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})') \in H_\infty^*$. By Lemma A.11.(ii), there exists a sequence of DGPs $\{F_n\}$ such that

$$||n^{1/2}\delta_{F_n}|| \to \infty \text{ and } v_{F_n} \to v_F. \tag{A.44}$$

Using the same arguments in proving (A.40) to (A.42), we can show that for some subsequence $\{p_n\}$ of $\{n\}$,

$$|p_n^{1/2}\delta_{F_{p_n}}|| \to \infty \text{ and } v_{p_n} \to v_F \tag{A.45}$$

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and

$$\limsup_{n\to\infty}\mathbb{E}_{F_n}[\ell(\widehat{\theta},\theta_{F_n})] = \lim_{n\to\infty}\mathbb{E}_{F_{pn}}[\ell(\widehat{\theta},\theta_{F_{pn}})] = R_\zeta(h_{F,d}). \tag{A.46}$$

for $||d|| = \infty$ by Conditions A.1.(vi). By the definition of $AsyR_\zeta(\widehat{\theta})$ and (A.46),

$$AsyR_\zeta(\widehat{\theta}) = \limsup_{n\to\infty}\sup_{F\in\mathcal{F}}\mathbb{E}_F[\ell_\zeta(\widehat{\theta},\theta_F)] \geq \limsup_{n\to\infty}\mathbb{E}_{F_n}[\ell(\widehat{\theta},\theta_{F_n})] = R_\zeta(h_{F,d}). \tag{A.47}$$

Combining the results in (A.43) and (A.47), we immediately get (A.37). $\blacksquare$

Lemma A.13 *Under Conditions A.1.(i) - A.1.(iv), the upper and lower bounds of the asymptotic risk difference between $\widehat{\theta}$ and $\widetilde{\theta}$ satisfy*

$$Asy\overline{RD}(\widehat{\theta},\widetilde{\theta}) \leq \lim_{\zeta\to\infty}\left(\max\left\{\sup_{h\in H}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right], \sup_{h\in H^*_\infty}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right]\right\}\right), \tag{A.48}$$

$$Asy\underline{RD}(\widehat{\theta},\widetilde{\theta}) \geq \lim_{\zeta\to\infty}\left(\min\left\{\inf_{h\in H}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right], \inf_{h\in H^*_\infty}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right]\right\}\right), \tag{A.49}$$

where

$$\widetilde{R}_\zeta(h) \equiv \mathbb{E}[\min\{\xi'_{1,F}\Upsilon\xi_{1,F},\zeta\}] \text{ and } R_\zeta(h) \equiv \begin{cases} \mathbb{E}\left[\min\left\{\overline{\xi}'_F\Upsilon\overline{\xi}_F,\zeta\right\}\right], & ||d|| < \infty \\ \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F},\zeta\}\right], & ||d|| = \infty \end{cases}$$

for any $h \in H \cup H^*_\infty$.

Proof of Lemma A.13. Define

$$\overline{R}_\zeta(H,H^*_\infty) \equiv \max\left\{\sup_{h\in H}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right], \sup_{h\in H^*_\infty}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right]\right\}, \tag{A.50}$$

$$\underline{R}_\zeta(H,H^*_\infty) \equiv \min\left\{\inf_{h\in H}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right], \inf_{h\in H^*_\infty}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right]\right\}. \tag{A.51}$$

By the definition of $Asy\overline{RD}(\widehat{\theta},\widetilde{\theta})$, to show (A.48) it is sufficient to show that for any $\zeta > 0$

$$\limsup_{n\to\infty}\sup_{F\in\mathcal{F}}\mathbb{E}_F[\ell_\zeta(\widehat{\theta},\theta_F) - \ell_\zeta(\widetilde{\theta},\theta_F)] \leq \overline{R}_\zeta(H,H^*_\infty), \tag{A.52}$$

which can be proved using the same arguments in the proof of Lemma A.10 (but replacing $\ell_\zeta(\widehat{\theta},\theta_F)$ and $R_\zeta(h)$ by $\ell_\zeta(\widehat{\theta},\theta_F) - \ell_\zeta(\widetilde{\theta},\theta_F)$ and $R_\zeta(h) - \widetilde{R}_\zeta(h)$ respectively). Similarly by the definition of $Asy\underline{RD}(\widehat{\theta},\widetilde{\theta})$, for (A.49) it is sufficient to show that for any $\zeta > 0$

$$\liminf_{n\to\infty}\inf_{F\in\mathcal{F}}\mathbb{E}_F[\ell_\zeta(\widehat{\theta},\theta_F) - \ell_\zeta(\widetilde{\theta},\theta_F)] \geq \underline{R}_\zeta(H,H^*_\infty), \tag{A.53}$$

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which can be proved using the same arguments in the proof of Lemma A.10 (but replacing $\limsup_n$, $\sup_{F \in \mathcal{F}}$, $\ell_\zeta(\widehat{\theta}, \theta_F)$ and $R_\zeta(h)$ by $\liminf_n$, $\inf_{F \in \mathcal{F}}$, $\ell_\zeta(\widehat{\theta}, \theta_F) - \ell_\zeta(\widetilde{\theta}, \theta_F)$ and $R_\zeta(h) - \widetilde{R}_\zeta(h)$ respectively). ∎

Lemma A.14 *Under Condition A.1, the upper and lower bounds of the asymptotic risk difference between $\widehat{\theta}$ and $\widetilde{\theta}$ have the following representations:*

$$Asy\overline{RD}(\widehat{\theta}, \widetilde{\theta}) \ = \ \lim_{\zeta \to \infty} \left(\max \left\{ \sup_{h \in H} \left[R_\zeta(h) - \widetilde{R}_\zeta(h) \right], \sup_{h \in H_\infty^*} \left[R_\zeta(h) - \widetilde{R}_\zeta(h) \right] \right\} \right), \quad \text{(A.54)}$$

$$Asy\underline{RD}(\widehat{\theta}, \widetilde{\theta}) \ = \ \lim_{\zeta \to \infty} \left(\min \left\{ \inf_{h \in H} \left[R_\zeta(h) - \widetilde{R}_\zeta(h) \right], \inf_{h \in H_\infty^*} \left[R_\zeta(h) - \widetilde{R}_\zeta(h) \right] \right\} \right). \quad \text{(A.55)}$$

Proof of Lemma A.14. By Lemma A.13, it is sufficient to show that

$$\limsup_{n \to \infty} \sup_{F \in \mathcal{F}} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}, \theta_F) - \ell_\zeta(\widetilde{\theta}, \theta_F)] \ \geq \ \overline{R}_\zeta(H, H_\infty^*), \quad \text{(A.56)}$$

$$\liminf_{n \to \infty} \inf_{F \in \mathcal{F}} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}, \theta_F) - \ell_\zeta(\widetilde{\theta}, \theta_F)] \ \leq \ \underline{R}_\zeta(H, H_\infty^*), \quad \text{(A.57)}$$

for any $\zeta > 0$. (A.56) can be proved using the same arguments in the proof of Lemma A.12 by replacing $\ell_\zeta(\widehat{\theta}, \theta_F)$ and $R_\zeta(h)$ by $\ell_\zeta(\widehat{\theta}, \theta_F) - \ell_\zeta(\widetilde{\theta}, \theta_F)$ and $R_\zeta(h) - \widetilde{R}_\zeta(h)$ respectively. Similarly, (A.57) can be proved using the same arguments in the proof of Lemma A.12 by replacing $\limsup_n$, $\sup_{F \in \mathcal{F}}$, $\ell_\zeta(\widehat{\theta}, \theta_F)$ and $R_\zeta(h)$ by $\liminf_n$, $\inf_{F \in \mathcal{F}}$, $\ell_\zeta(\widehat{\theta}, \theta_F) - \ell_\zeta(\widetilde{\theta}, \theta_F)$ and $R_\zeta(h) - \widetilde{R}_\zeta(h)$ respectively. ∎

Lemma A.15 *Under Assumptions 3.2.(ii) and 3.2.(iv), we have*

$$\sup_{h \in H} \mathbb{E}[(\xi_{1,F}' \Upsilon \xi_{1,F})^2] \leq C \ \text{ and } \ \sup_{h \in H} \mathbb{E}[(\overline{\xi}_F' \Upsilon \overline{\xi}_F)^2] \leq C. \quad \text{(A.58)}$$

Lemma A.16 *Let $g_\zeta(h) \equiv \mathbb{E}\left[\min\{\overline{\xi}_F' \Upsilon \overline{\xi}_F, \zeta\} - \min\left\{\xi_{1,F}' \Upsilon \xi_{1,F}, \zeta\right\} \right]$. Under Assumptions 3.2.(ii) and 3.2.(iv), we have*

$$\lim_{\zeta \to \infty} \sup_{h \in H} [|g_\zeta(h) - g(h)|] = 0 \quad \text{(A.59)}$$

where $\sup_{h \in H} [|g(h)|] \leq C$.

Proof of Theorem 5.1. The proof consists of two steps. The first step is to apply Lemma A.14 to show (A.60) and (A.61) below, and the second step is to apply Lemma A.16 to show (A.75) and (A.76) below.

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In the first step, we apply Lemma A.14 with $\widehat{\theta} = \widehat{\theta}_{eo}$ and $\widetilde{\theta} = \widehat{\theta}_1$ to show that

$$Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = \lim_{\zeta \to \infty} \max \left\{ \sup_{h \in H} [g_\zeta(h)], 0 \right\} \text{ and} \tag{A.60}$$

$$Asy\underline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) = \lim_{\zeta \to \infty} \min \left\{ \inf_{h \in H} [g_\zeta(h)], 0 \right\}. \tag{A.61}$$

To prove (A.60) and (A.61), we now verify Condition A.1 under Assumptions 3.1-3.3. Condition A.1.(i) is verified by Lemma A.7 under Assumptions 3.2.(ii) and 3.3.(ii). Condition A.1.(ii) is implied by Assumptions 3.1.(i) and 3.1.(ii). Condition A.1.(iii) is implied by Assumptions 3.2.(i)-(ii) as a result of Lemma A.2. Condition A.1.(v) is assumed in Assumption 3.3.(ii). We next verify Conditions A.1.(iv) and A.1.(vi).

Consider any sequence of DGPs $\{F_{p_n}\}$ with

$$(p_n)^{1/2} \delta_{F_{p_n}} \to d \text{ for } d \in \mathbb{R}_\infty^{r^*} \text{ and } v_{F_{p_n}} \to v_F \tag{A.62}$$

for some $F \in \mathcal{F}$, where $\{p_n\}$ is a subsequence of $\{n\}$. First, we consider the case that $d \in \mathbb{R}^{r^*}$. By Lemma 4.1.(a) and 4.2.(a),

$$(p_n)^{1/2}(\widehat{\theta}_1 - \theta_{F_{p_n}}) \to_D \xi_{1,F} \text{ and } (p_n)^{1/2}(\widehat{\theta}_{eo} - \theta_{F_{p_n}}) \to_D \overline{\xi}_F \tag{A.63}$$

which combined with the continuous mapping theorem implies that

$$\ell(\widehat{\theta}_1, \theta_{F_{p_n}}) \to_D \xi'_{1,F} \Upsilon \xi_{1,F} \text{ and } \ell(\widehat{\theta}_{eo}, \theta_{F_{p_n}}) \to_D \overline{\xi}'_F \Upsilon \overline{\xi}_F. \tag{A.64}$$

Since Υ is positive semi-definite, $\xi'_{1,F} \Upsilon \xi_{1,F}$ and $\overline{\xi}'_F \Upsilon \overline{\xi}_F$ are both non-negative. The function $f_\zeta(x) = \min\{x, \zeta\}$ is a bounded continuous function for $x \geq 0$. By (A.64) and the Portmanteau Lemma (see Lemma 2.2 in van der Vaart (1998)),

$$\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}})] \to \mathbb{E}\left[\min\{\overline{\xi}'_F \Upsilon \overline{\xi}_F, \zeta\}\right] \text{ and } \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})] \to \mathbb{E}\left[\min\{\xi'_{1,F} \Upsilon \xi_{1,F}, \zeta\}\right].$$
$$\tag{A.65}$$

Second, we consider the case that $\|d\| = \infty$. Then under Lemma 4.1.(b) and 4.2.(b),

$$(p_n)^{1/2}(\widehat{\theta}_1 - \theta_{F_{p_n}}) \to_D \xi_{1,F} \text{ and } (p_n)^{1/2}(\widehat{\theta}_{eo} - \theta_{F_{p_n}}) \to_D \xi_{1,F}. \tag{A.66}$$

Using the same arguments in showing (A.65), we get

$$\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}})] \to \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right] \text{ and } \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})] \to \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right].$$
(A.67)

Define

$$\widetilde{R}_\zeta(h_{F,d}) = \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right] \text{ and } R_\zeta(h_{F,d}) = \begin{cases} \mathbb{E}[\min\{\overline{\xi}'_F\Upsilon\overline{\xi}_F, \zeta\}], & \|d\| < \infty \\ \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right], & \|d\| = \infty \end{cases}.$$
(A.68)

Collecting the results in (A.65) and (A.67), we deduce that under the sequence of DGPs $\{F_{p_n}\}$ satisfying (A.62),

$$\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}})] \to R_\zeta(h_{F,d}) \text{ and } \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})] \to \widetilde{R}_\zeta(h_{F,d}),$$
(A.69)

where $R_\zeta(h_{F,d})$ and $\widetilde{R}_\zeta(h_{F,d})$ are non-negative and bounded from above by ζ for any $d \in \mathbb{R}^{r^*}_\infty$ and any $F \in \mathcal{F}$. This verifies Condition A.1.(iv).

By definition, $\widetilde{R}_\zeta(h_{F,d})$ in (A.68) does not depend on d for any F. Moreover, for any d and $\widetilde{d}$ with $\|d\| = \infty$ and $\|\widetilde{d}\| = \infty$, by the definition of $R_\zeta(h_{F,d})$ in (A.69),

$$R_\zeta(h_{F,d}) = \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right] = R_\zeta(h_{F,\widetilde{d}}).$$
(A.70)

Hence, Condition A.1.(vi) is also verified.

We next apply Lemma A.14 to get (A.60) and (A.61) above. By (A.68),

$$R_\zeta(h) - \widetilde{R}_\zeta(h) = \mathbb{E}[\min\{\overline{\xi}'_F\Upsilon\overline{\xi}_F, \zeta\}] - \mathbb{E}[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}] \text{ for any } h \in H$$
(A.71)

and

$$R_\zeta(h) - \widetilde{R}_\zeta(h) = \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right] - \mathbb{E}\left[\min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right] = 0 \text{ for any } h \in H^*_\infty.$$
(A.72)

By Lemma A.14, (A.71) and (A.72), we have

$$\begin{aligned} Asy\overline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) &= \lim_{\zeta \to \infty} \max\left\{ \sup_{h \in H}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right], \sup_{h \in H^*_\infty}\left[R_\zeta(h) - \widetilde{R}_\zeta(h)\right] \right\} \\ &= \lim_{\zeta \to \infty} \max\left\{ \sup_{h \in H}\mathbb{E}\left[\min\{\overline{\xi}'_F\Upsilon\overline{\xi}_F, \zeta\} - \min\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\}\right], 0 \right\} \end{aligned}$$
(A.73)

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and

$$
\begin{aligned}
Asy\underline{RD}(\widehat{\theta}_{eo}, \widehat{\theta}_1) &= \lim_{\zeta \to \infty} \min \left\{ \inf_{h \in H} \left[R_\zeta(h) - \widetilde{R}_\zeta(h) \right], \inf_{h \in H_\infty^*} \left[R_\zeta(h) - \widetilde{R}_\zeta(h) \right] \right\} \\
&= \lim_{\zeta \to \infty} \min \left\{ \inf_{h \in H} \mathbb{E} \left[\min\{\overline{\xi}_F' \Upsilon \overline{\xi}_F, \zeta\} - \min\{\xi_{1,F}' \Upsilon \xi_{1,F}, \zeta\} \right], 0 \right\}, \quad \text{(A.74)}
\end{aligned}
$$

which proves (A.60) and (A.61).

In the second step, we show that

$$
\lim_{\zeta \to \infty} \max \left\{ \sup_{h \in H} [g_\zeta(h)], 0 \right\} = \max \left\{ \sup_{h \in H} [g(h)], 0 \right\}, \text{ and} \quad \text{(A.75)}
$$

$$
\lim_{\zeta \to \infty} \min \left\{ \inf_{h \in H} [g_\zeta(h)], 0 \right\} = \min \left\{ \inf_{h \in H} [g(h)], 0 \right\}. \quad \text{(A.76)}
$$

By Lemma A.16,

$$
\lim_{\zeta \to \infty} \sup_{h \in H} [g_\zeta(h)] = \sup_{h \in H} [g(h)] \text{ and } \lim_{\zeta \to \infty} \inf_{h \in H} [g_\zeta(h)] = \inf_{h \in H} [g(h)], \quad \text{(A.77)}
$$

where $\sup_{h \in H} [g(h)]$ and $\inf_{h \in H} [g(h)]$ are finite real numbers. Let $\overline{f}(x) = \max(x, 0)$ and $\underline{f}(x) = \min(x, 0)$. It is clear that $\overline{f}(x)$ and $\underline{f}(x)$ are continuos function on $\mathbb{R}$. The asserted results in (A.75) and (A.76) follow by (A.77), and the continuity of $\overline{f}(x)$ and $\underline{f}(x)$. ∎

Proof of Theorem 5.2. For any $F \in \mathcal{F}$, define

$$
D_F = (\Gamma_{2,F} - \Gamma_{1,F}^*)' \Upsilon \Gamma_{1,F}^*. \quad \text{(A.78)}
$$

Recall that we have defined

$$
A_F = \Upsilon (\Sigma_{1,F} - \Sigma_{2,F}) \text{ and } B_F = (\Gamma_{2,F} - \Gamma_{1,F}^*)' \Upsilon (\Gamma_{2,F} - \Gamma_{1,F}^*) \quad \text{(A.79)}
$$

in Theorem 5.2 and (**??**) respectively. By the definition of $\overline{\xi}_F$,

$$
\mathbb{E}[\overline{\xi}_F' \Upsilon \overline{\xi}_F] = \text{tr}(\Upsilon \Sigma_{1,F}) + 2\text{tr}(A_F) J_{1,F} + \text{tr}(A_F)^2 J_{2,F} \quad \text{(A.80)}
$$

where

$$
J_{1,F} = \mathbb{E} \left[\frac{\mathcal{Z}_{d,2,F}' D_F \mathcal{Z}_{d,2,F}}{\mathcal{Z}_{d,2,F}' B_F \mathcal{Z}_{d,2,F} + \text{tr}(A_F)} \right] \text{ and } J_{2,F} = \mathbb{E} \left[\frac{\mathcal{Z}_{d,2,F}' B_F \mathcal{Z}_{d,2,F}}{(\mathcal{Z}_{d,2,F}' B_F \mathcal{Z}_{d,2,F} + \text{tr}(A_F))^2} \right]. \quad \text{(A.81)}
$$

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We provide a upper bound for $J_{1,F}$ defined in (A.81). Define a function

$$\eta(x) \equiv \frac{x}{x'B_F x + \mathrm{tr}(A_F)} \text{ for any } x \in \mathbb{R}^{r_2}. \tag{A.82}$$

Its derivative is

$$\frac{\partial \eta(x)'}{\partial x} = \frac{1}{x'B_F x + \mathrm{tr}(A_F)}I_{r_2} - \frac{2B_F}{(x'B_F x + \mathrm{tr}(A_F))^2}xx'. \tag{A.83}$$

Then $J_{1,F} = \mathbb{E}[\eta(\mathcal{Z}_{d,2,F})'D_F\mathcal{Z}_{d,2,F}]$. Note that $D_F\mathcal{Z}_{d,2,F} = D_F\mathcal{Z}_{2,F}$ by construction because the last r^* columns of $\Gamma_{1,F}^*$ are zeros. Applying Lemma A.9 yields

$$\begin{aligned}
\mathrm{tr}\left(D_F\Omega_{2,F}\right) &= \mathrm{tr}\left((\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon\Gamma_{1,F}^*\Omega_{2,F}\right) \\
&= \mathrm{tr}(\Upsilon\left(\Gamma_{1,F}^*\Omega_{2,F}\Gamma_{2,F}' - \Gamma_{1,F}^*\Omega_{2,F}\Gamma_{1,F}^*\right)) \\
&= \mathrm{tr}(\Upsilon\left(\Sigma_{2,F} - \Sigma_{1,F}\right)) = -\mathrm{tr}(A_F).
\end{aligned} \tag{A.84}$$

By Lemma 1 of Hansen (2016), which is a matrix version of the Stein's Lemma (Stein, 1981),

$$J_{1,F} = \mathbb{E}\left(\eta(\mathcal{Z}_{d,2,F})'D_F\mathcal{Z}_{d,2,F}\right) = \mathbb{E}\left[\mathrm{tr}\left(\frac{\partial\eta(\mathcal{Z}_{d,2,F})'}{\partial x}D_F\Omega_{2,F}\right)\right]. \tag{A.85}$$

Plugging (A.82)-(A.84) into (A.85), we have

$$\begin{aligned}
J_{1,F} &= \mathbb{E}\left[\frac{\mathrm{tr}\left(D_F\Omega_{2,F}\right)}{\mathcal{Z}_{d,2,F}'B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - 2\mathbb{E}\left[\frac{\mathrm{tr}\left(B_F\mathcal{Z}_{d,2,F}\mathcal{Z}_{d,2,F}'D_F\Omega_{2,F}\right)}{\left(\mathcal{Z}_{d,2,F}'B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right)^2}\right] \\
&= \mathbb{E}\left[\frac{-\mathrm{tr}(A_F)}{\mathcal{Z}_{d,2,F}'B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] + 2\mathbb{E}\left[\frac{-\mathcal{Z}_{d,2,F}'D_F\Omega_{2,F}B_F\mathcal{Z}_{d,2,F}}{\left(\mathcal{Z}_{d,2,F}'B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right)^2}\right]
\end{aligned} \tag{A.86}$$

where the second equality is by (A.84). By definition and Lemma A.9

$$\begin{aligned}
&-\mathcal{Z}_{d,2,F}'D_F\Omega_{2,F}B_F\mathcal{Z}_{d,2,F} \\
&= -\mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon\Gamma_{1,F}^*\Omega_{2,F}(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)\mathcal{Z}_{d,2,F} \\
&= \mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon(\Sigma_{1,F} - \Sigma_{2,F})\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)\mathcal{Z}_{d,2,F} \\
&\leq \rho_{\max}(\Upsilon^{1/2}(\Sigma_{1,F} - \Sigma_{2,F})\Upsilon^{1/2})(\mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)\mathcal{Z}_{d,2,F}) \\
&= \rho_{\max}(A_F)\mathcal{Z}_{d,2,F}'B_F\mathcal{Z}_{d,2,F},
\end{aligned} \tag{A.87}$$

where the last equality is by $\rho_{\max}(\Upsilon^{1/2}(\Sigma_{1,F} - \Sigma_{2,F})\Upsilon^{1/2}) = \rho_{\max}(\Upsilon(\Sigma_{1,F} - \Sigma_{2,F}))$. Combining the

results in (A.86) and (A.87), we get

$$
\begin{aligned}
J_{1,F} \leq & \mathbb{E}\left[\frac{-\mathrm{tr}(A_F)}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] + 2\mathbb{E}\left[\frac{\rho_{\max}(A_F)\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F}}{\left(\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right)^2}\right] \\
= & \mathbb{E}\left[\frac{-\mathrm{tr}(A_F)}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] \\
& + 2\mathbb{E}\left[\frac{\left[\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A)\right]\rho_{\max}(A_F) - \mathrm{tr}(A_F)\rho_{\max}(A_F)}{\left(\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right)^2}\right] \\
= & \mathbb{E}\left[\frac{2\rho_{\max}(A_F) - \mathrm{tr}(A_F)}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - \mathbb{E}\left[\frac{2\rho_{\max}(A_F)\mathrm{tr}(A_F)}{\left(\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right)^2}\right]. \quad (A.88)
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
J_{2,F} = & \mathbb{E}\left[\frac{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F}}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right] \\
= & \mathbb{E}\left[\frac{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F) - \mathrm{tr}(A_F)}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right] \\
= & \mathbb{E}\left[\frac{1}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - \mathbb{E}\left[\frac{\mathrm{tr}(A_F)}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right]. \quad (A.89)
\end{aligned}
$$

Combining (A.80), (A.88), (A.89) and the definition of $g(h)$ (in Theorem 5.1), we obtain that

$$
\begin{aligned}
g(h_{d,F}) = & 2\mathrm{tr}(A_F)J_{1,F} + \mathrm{tr}(A_F)^2 J_{2,F} \\
\leq & 2\mathrm{tr}(A_F)\left(\mathbb{E}\left[\frac{2\rho_{\max}(A_F) - \mathrm{tr}(A_F)}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - \mathbb{E}\left[\frac{2\mathrm{tr}(A_F)\rho_{\max}(A_F)}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right]\right) \\
& + \mathrm{tr}(A)^2\left(\mathbb{E}\left[\frac{1}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - \mathbb{E}\left[\frac{\mathrm{tr}(A_F)}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right]\right) \\
= & \mathbb{E}\left[\frac{\mathrm{tr}(A_F)\left(4\rho_{\max}(A_F) - \mathrm{tr}(A_F)\right)}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - \mathbb{E}\left[\frac{\mathrm{tr}(A_F)^2\left(4\rho_{\max}(A_F) + \mathrm{tr}(A_F)\right)}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right]. \quad (A.90)
\end{aligned}
$$

For all G_2 and Ω_2 such that $h = (d, \mathrm{vec}(G_2)', \mathrm{vech}(\Omega_2)') \in H$, we have $G_2 = G_{2,F}$ and $\Omega_2 = $

$\Omega_{2,F}$ for some $F \in \mathcal{F}$ by the definition of H. If $\text{tr}(A_F) > 0$, then $\rho_{\max}(A_F) > 0$ and thus the second term in the right-hand side of the last equality of (A.90) will be negative. If in addition $\text{tr}(A_F) \geq 4\rho_{\max}(A_F)$, then the first term in the right-hand side of the last equality of (A.90) will be non-negative. As a result, when $\text{tr}(A_F) > 0$ and $4\rho_{\max}(A_F) - \text{tr}(A_F) \leq 0$ for $\forall F \in \mathcal{F}$, we have $\sup_{h \in H}[g(h)] < 0$. This combined with Theorem 5.1 implies the results of this theorem. ∎

A.3   Proof of the Results in Section 6

Lemma A.17 *Suppose that Assumption 6.1 holds. Consider $\{F_n\}$ such that $\overline{v}_{F_n} \to \overline{v}_F$ for some $\overline{v}_F \in \overline{\Lambda}$ and $n^{1/2}\delta_{F_n} \to d$ with $\|d\| < \infty$. We have*

$$\lim_{\zeta \to \infty} \lim_{n \to \infty} \mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_n}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_n})] = g(h_{F,d})$$

where $g(h_{F,d}) = \mathbb{E}\left[\overline{\xi}'_F \Upsilon \overline{\xi}_F - \xi'_{1,F}\Upsilon\xi_{1,F}\right]$.

Proof of Lemma A.17.   For the sequence of DGPs $\{F_n\}$ considered in the lemma, by Assumptions 6.1.(i), 6.1.(ii) and 6.1.(iv), we can use Lemma A.8 to deduce that

$$\begin{pmatrix} n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \\ n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \end{pmatrix} \to_D \begin{pmatrix} \xi_{1,F} \\ \xi_{2,F} \end{pmatrix} \tag{A.91}$$

where $d_0 = (\mathbf{0}_{1 \times r_1}, d')'$. In the proof of Lemma 4.2, we have show that

$$\widehat{\Omega}_k = \Omega_{2,F} + o_p(1) \text{ and } \widehat{G}_k = G_{k,F} + o_p(1) \tag{A.92}$$

under $\overline{v}_{F_n} \to \overline{v}_F$, Assumptions 6.1.(i), 6.1.(ii) and 6.1.(iv). By (A.92), Assumption 6.1.(iv) and the Slutsky Theorem, $\widehat{\Sigma}_{1,F}$ and $\widehat{\Sigma}_{2,F}$ are consistent estimators of $\Sigma_{1,F}$ and $\Sigma_{2,F}$ respectively. By the consistency of $\widehat{\Sigma}_{1,F}$ and $\widehat{\Sigma}_{2,F}$, the weak convergence in (A.91) and the CMT, we deduce that

$$n^{1/2}(\widehat{\theta}_{eo} - \theta_{F_n}) \to_D \overline{\xi}_F. \tag{A.93}$$

Collecting the results in (A.91) and (A.93), and then applying the CMT and the Portmanteau Lemma, we get

$$\lim_{n \to \infty} \mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_n}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_n})] \to g_\zeta(h) \tag{A.94}$$

where $g_\zeta(h) = \mathbb{E}\left[\min\{\overline{\xi}'_F \Upsilon \overline{\xi}_F, \zeta\} - \min\left\{\xi'_{1,F}\Upsilon\xi_{1,F}, \zeta\right\}\right]$. The asserted result follows by Lemma A.16 and (A.94). ∎

Proof of Lemma 6.1. Let $\{F_n\}$ be a sequence such that

$$\limsup_{n\to\infty} \mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_n}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_n})] = \limsup_{n\to\infty} \left(\sup_{F\in\mathcal{F}_n} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)] \right). \tag{A.95}$$

Such a sequence always exists by the definition of supremum. The sequence $\{\mathbb{E}_{F_n}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_n}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_n})]: n \geq 1\}$ may not converge. However, by the definition of limsup, there exists a subsequence of $\{n\}$, say $\{p_n\}$, such that $\{\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})]: n \geq 1\}$ converges and

$$\lim_{n\to\infty} \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})] = \limsup_{n\to\infty} \left(\sup_{F\in\mathcal{F}_n} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)] \right). \tag{A.96}$$

Below we show that for any subsequence $\{p_n\}$ of $\{n\}$ such that $\{\mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})]:$ $n \geq 1\}$ is convergent, there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ such that

$$\lim_{n\to\infty} \mathbb{E}_{F_{p_n^*}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n^*}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n^*}})] = g_\zeta(h) \text{ for some } h \in H_{C_L} \tag{A.97}$$

Because $\lim_{n\to\infty} \mathbb{E}_{F_{p_n^*}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n^*}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n^*}})] = \lim_{n\to\infty} \mathbb{E}_{F_{p_n}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n}})]$, which combined with (A.96) and (A.97) implies that

$$\limsup_{n\to\infty} \left(\sup_{F\in\mathcal{F}_n} \mathbb{E}_F[\ell_\zeta(\widehat{\theta}_{eo}, \theta_F) - \ell_\zeta(\widehat{\theta}_1, \theta_F)] \right) = g_\zeta(h) \text{ for some } h \in H_{C_L}. \tag{A.98}$$

The desired result in (6.5) follows immediately by (A.98).

To show that there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ such that (A.97) holds, it suffices to show that for any sequence $\{F_n\}$ and any subsequence $\{p_n\}$ of $\{n\}$, there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ for which we have

$$(p_n^*)^{1/2}\delta_{F_{p_n^*}} \to d \text{ for } \|d\| \leq C_L \text{ and } \overline{v}_{F_{p_n^*}} \to \overline{v}_F \tag{A.99}$$

for some $F \in \mathcal{F}$. By (A.99), we can use Lemma A.17 to deduce that

$$\lim_{n\to\infty} \mathbb{E}_{F_{p_n^*}}[\ell_\zeta(\widehat{\theta}_{eo}, \theta_{F_{p_n^*}}) - \ell_\zeta(\widehat{\theta}_1, \theta_{F_{p_n^*}})] = g_\zeta(h_{F,d}) \tag{A.100}$$

for the sequence of DGPs $\{F_{p_n^*}\}$ satisfies (A.99). Moreover, we have $h_{F,d} \in H_{C_L}$ by the definition of H_{C_L}, which together with (A.100) proves (A.97).

Finally, we show that for any sequence $\{F_n\}$ and any subsequence $\{p_n\}$ of $\{n\}$, there exists a subsequence $\{p_n^*\}$ of $\{p_n\}$ for which (A.99) holds. Let $\delta_{p_n,j}$ denote the j-th component of δ_{p_n} and $p_{1,n} = p_n$ for any $n \geq 1$. For $j = 1$, we have $|p_{j,n}^{1/2}\delta_{p_{j,n},j}| \leq C_L$ for any n by Assumption 6.1.(vi).

Hence there is some subsequence $\{p_{j+1,n}\}$ of $\{p_{j,n}\}$, $p_{j+1,n}^{1/2}\delta_{p_{j+1,n},j} \to d_j$ for some $|d_j| \le C_L$. As r^* is a fixed positive integer, we can apply the same arguments successively for $j = 1, \ldots, r^*$ to obtain a subsequence $\{p_{r^*,n}\}$ of $\{p_n\}$ such that $(p_{r^*,n})^{1/2}\delta_{p_{r^*,n}} \to d$ with $|d_j| \le C_L$ for $j = 1, \ldots, r^*$. Since $\left\|(p_{r^*,n})^{1/2}\delta_{p_{r^*,n}}\right\| \le C_L$ for any n by Assumption 6.1.(vi), we have $\|d\| \le C_L$. By Assumptions 3.2.(ii) and 6.1.(v), $\overline{\Lambda}$ is a compact set. Hence, there is a subsequence $\{p_n^*\}$ of $\{p_{r^*,n}\}$ such that $\overline{v}_{F_{p_n^*}} \to \overline{v}_F$, which finishes the proof of (A.99). ∎

Proof of Theorem 6.1. By (A.90) in the proof of Theorem 5.2

$$g(h_{d,F}) \le \mathbb{E}\left[\frac{\mathrm{tr}(A_F)\left(4\rho_{\max}(A_F) - \mathrm{tr}(A_F)\right)}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] - \mathbb{E}\left[\frac{\mathrm{tr}(A_F)^2\left(4\rho_{\max}(A_F) + \mathrm{tr}(A_F)\right)}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right]. \quad (A.101)$$

By Jensen's inequality,

$$\mathbb{E}\left[\frac{1}{\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}\right] \ge \frac{1}{\mathrm{tr}(d'B_Fd) + 2\mathrm{tr}(A_F)}, \quad (A.102)$$

and similarly

$$\mathbb{E}\left[\frac{1}{\left|\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)\right|^2}\right] \ge \frac{1}{\left|\mathrm{tr}(d'B_Fd) + 2\mathrm{tr}(A_F)\right|^2}. \quad (A.103)$$

(A.102) and (A.103), combined with $\mathrm{tr}(A_F) \ge 4\rho_{\max}(A_F)$ and $\rho_{\max}(A_F) > 0$, imply that

$$\begin{aligned}
g(h) &\le \frac{\mathrm{tr}(A_F)\left(4\rho_{\max}(A_F) - \mathrm{tr}(A_F)\right)}{\mathrm{tr}(d'B_Fd) + 2\mathrm{tr}(A_F)} - \frac{\mathrm{tr}(A_F)^2\left(4\rho_{\max}(A_F) + \mathrm{tr}(A_F)\right)}{\left|\mathrm{tr}(d'B_Fd) + 2\mathrm{tr}(A_F)\right|^2} \\
&\le \frac{\mathrm{tr}(A_F)\left(4\rho_{\max}(A_F) - \mathrm{tr}(A_F)\right)}{C_L^2\rho_{\max}(B_F) + 2\mathrm{tr}(A_F)} - \frac{\mathrm{tr}(A_F)^2\left(4\rho_{\max}(A_F) + \mathrm{tr}(A_F)\right)}{\left|C_L^2\rho_{\max}(B_F) + 2\mathrm{tr}(A_F)\right|^2}. \quad (A.104)
\end{aligned}$$

Since Υ is positive semi-definite, by definition of B_F and Assumption 6.1.(iv), $0 \le \rho_{\max}(B_F) < \infty$. Moreover, by Assumptions 6.1.(iv) and 6.1.(vi), $\mathrm{tr}(A_F) < \infty$ and $C_L < \infty$. Therefore (A.104) immediately implies that $g(h) < 0$ for any $h \in H_{C_L}$. ∎

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Supplemental Appendix of
"An Averaging GMM Estimator Robust to Misspecification"

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In this supplemental appendix, we present supporting materials for Cheng, Liao and Shi (2017) (cited as CLS hereafter in this Appendix):

- Section B provides primitive conditions for Assumptions 3.1, 3.2 and 3.3 and the proof of Lemma 3.1 of CLS.

- Section C provides the proof of (4.3) in Section 4 and the proof of some Lemmas in Appendix A.1 of CLS.

- Section D studies the bounds of asymptotic risk difference of the pre-test GMM estimator presented in Figure 2 of CLS.

- Section E includes extra simulation studies.

- Section F presents the uniform dominance result in a Gaussian location model.

B   Primitive Conditions for Assumptions 3.1, 3.2 and 3.3 and Proof of Lemma 3.1 of CLS

In this section, we provide primitive conditions for Assumption Assumptions 3.1, 3.2 and 3.3 in the linear IV model presented in Example 3.1 of CLS.

We first provide a set of sufficient conditions without imposing the normal distribution assumption on $(X', Z_1', V', U)'$ in Lemma B.1. Then, we impose the normal assumptions and show that these conditions can be simplified to those in Lemma 3.1 of CLS under normality.

For ease of notations, we define $\Gamma_{z_1 v u^2} \equiv \mathbb{E}_{F^*}[Z_1 V' U^2]$, $\Omega_{z_1 z_1 u^2} \equiv \mathbb{E}_{F^*}[Z_1 Z_1' U^2]$ and $\Omega_{vvu^2} \equiv \mathbb{E}_{F^*}[VV'U^2]$. The Jacobian matrices are

$$G_{1,F} = -\mathbb{E}_F[Z_1 X'] \text{ and } G_{2,F} = \begin{pmatrix} -\mathbb{E}_F[Z_1 X'] \\ -\mathbb{E}_F[Z^* X'] \end{pmatrix}. \tag{B.1}$$

Let $Z_2 = (Z_1', Z^{*\prime})'$. The variance-covariance matrix of the moment conditions is

$$\Omega_{2,F} = \mathbb{E}_F[Z_2 Z_2' (Y - X'\theta_0)^2] - \mathbb{E}_F[(Y - X'\theta_0)Z_2]\mathbb{E}_F[(Y - X'\theta_0)Z_2']. \tag{B.2}$$

By definition, $\Omega_{1,F}$ is the leading $r_1 \times r_1$ submatrix of $\Omega_{2,F}$.

Let F denote the joint distribution of $W = (Y, Z_1', Z^{*\prime}, X')'$ induced by θ_0, δ_0 and F^*. By definition, we can write

$$\delta_F = \Omega_{uu}\delta_0, \; G_{2,F} = \begin{pmatrix} -\Gamma_{z_1 x} \\ -\delta_0 \Gamma_{ux} - \Gamma_{vx} \end{pmatrix}, \; \Omega_{2,F} = \begin{pmatrix} \Omega_{z_1 z_1 u^2} & \Omega_{2,1r,F} \\ \Omega_{2,r1,F} & \Omega_{2,rr,F} \end{pmatrix} \tag{B.3}$$

where

$$\Omega_{2,1r,F} = \Gamma_{z_1u^3}\delta_0' + \Gamma_{z_1vu^2} = \Omega_{2,r1,F}', \text{ and}$$
$$\Omega_{2,rr,F} = \Omega_{u^2u^2}\delta_0\delta_0' + \delta_0\Gamma_{u^3v} + \Gamma_{vu^3}\delta_0' + \Omega_{vvu^2}. \tag{B.4}$$

Therefore, the parameter v_F defined in (3.4) depends on F through F^* and δ_0, and its dependence on F^* is through v_{*,F^*}, where

$$v_{*,F^*} = \begin{pmatrix} \Omega_{u^2u^2}, \Omega_{uu}, \text{vec}(\Gamma_{z_1x})', \text{vec}(\Gamma_{ux})', \text{vec}(\Gamma_{vx})', \text{vech}(\Omega_{z_1z_1u^2})', \\ \text{vec}(\Gamma_{z_1u^3})', \text{vec}(\Gamma_{z_1vu^2})', \text{vec}(\Gamma_{u^3v})', \text{vech}(\Omega_{vvu^2})' \end{pmatrix}. \tag{B.5}$$

Define

$$\rho_{2,\max} \equiv \max\{\sup_{F\in\mathcal{F}} \rho_{\max}(\Omega_{2,F}), \sup_{F\in\mathcal{F}} \rho_{\max}(G_{2,F}G_{2,F}')\},$$
$$\rho_{2,\min} \equiv \min\{\inf_{F\in\mathcal{F}} \rho_{\min}(\Omega_{2,F}), \inf_{F\in\mathcal{F}} \rho_{\min}(G_{2,F}G_{2,F}')\},$$
$$C_W \equiv \sup_{F^*\in\mathcal{F}^*} \mathbb{E}_{F^*}[||(X', Z_1', V', U)||^2] \text{ and } C_\Delta \equiv \sup_{\delta_0\in\Delta_\delta} ||\delta_0||^2. \tag{B.6}$$

In the proof of Lemma B.1 below, we show that $\rho_{2,\max} < \infty$ (see $(B.14)$ and $(B.18)$). Moreover, we have $\rho_{2,\min} > 0$, $C_W < \infty$ and $C_\Delta < \infty$ by Assumptions $B.1.$(iii), $B.1.$(ii) and $B.1.$(vii) respectively. Define

$$B_{\rho_2}^c \equiv \{\delta \in \mathbb{R}^{r^*} : ||\delta|| \geq \rho_{2,\min}\rho_{2,\max}^{-1}C_\Delta^{-1/2}\}. \tag{B.7}$$

Let Θ_0 be a non-empty set in $\mathbb{R}^{d_\theta}$. Define

$$B_{\Theta_0} \equiv \{\theta \in \mathbb{R}^{d_\theta} : ||\theta - \theta_0|| \leq \rho_{2,\min}^{-4}\rho_{2,\max}^3 C_\Delta C_W^2 \text{ for any } \theta_0 \in \Theta_0\}. \tag{B.8}$$

Let $\{c_{j,\Delta}, C_{j,\Delta}\}_{j=1}^{r^*}$ be a set of finite constants. We next provide the low-level sufficient conditions for Assumptions 3.1, 3.2 and 3.3.

Assumption B.1 *The following conditions hold:*
(i) $\mathbb{E}_{F^*}[V] = 0$, $\mathbb{E}_{F^*}[U] = 0$, $\mathbb{E}_{F^*}[Z_1U] = 0_{r_1\times1}$ *and* $\mathbb{E}_{F^*}[VU] = 0_{r^*\times1}$ *for any* $F^* \in \mathcal{F}^*$;
(ii) $\sup_{F^*\in\mathcal{F}^*} \mathbb{E}_{F^*}[||X||^{4+\gamma} + ||Z_1||^{4+\gamma} + ||V||^{4+\gamma} + U^6] < \infty$ *for some* $\gamma > 0$;
(iii) $\inf_{F^*\in\mathcal{F}^*} \mathbb{E}_{F^*}[U^2] > 0$, $\inf_{F^*\in\mathcal{F}^*} \rho_{\min}(\Gamma_{xz_1}\Gamma_{z_1x}) > 0$ *and* $\inf_{F\in\mathcal{F}} \rho_{\min}(\Omega_{2,F}) > 0$;
(iv) $\inf_{F^*\in\mathcal{F}^*} \inf_{\delta\in B_{\rho_2}^c} ||\delta||^{-1} ||(\Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1vu^2} - \Gamma_{xv})\delta + \Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1u^3} - \Gamma_{xu}|| > 0$;
(v) *the set* $\{v_{*,F^*} : F^* \in \mathcal{F}^*\}$ *is closed*;
(vi) $\theta_0 \in \Theta_0$, $B_{\Theta_0} \subset int(\Theta)$ *and* Θ *is compact*;
(vii) $\Delta_\delta = [c_{1,\Delta}, C_{1,\Delta}] \times \cdots \times [c_{r^*,\Delta}, C_{r^*,\Delta}]$ *where* $c_{j,\Delta} < 0 < C_{j,\Delta}$ *for* $j = 1, \ldots, r^*$.

Lemma B.1 *Suppose that* $\{W_i\}_{i=1}^n$ *are i.i.d. and generated by the linear model (3.6) and (3.8) in CLS. Then under Assumption* B.1, $\mathcal{F}$ *satisfies Assumptions 3.1, 3.2 and 3.3.*

For the linear IV model, Lemma B.1 provides simple conditions on θ_0, δ_0 and $\mathcal{F}^*$ on which uniformity results are subsequently established.

Proof of Lemma B.1. By Assumption $B.1.$(i) and the definition of $G_{1,F}$,

$$\mathbb{E}_F[g_1(W, \theta)] = \mathbb{E}_{F^*}[Z_1(U - X'(\theta - \theta_0))] = G_{1,F}(\theta - \theta_0), \tag{B.9}$$

which together with Assumption $B.1.$(iii) implies that $\theta_F = \theta_0$ and hence $\mathbb{E}_F[g_1(W, \theta_F)] = 0_{r_1 \times 1}$. Also $\theta_F \in int(\Theta)$ holds by $\theta_F = \theta_0$ and Assumption $B.1.$(vi). This verifies Assumption 3.1.(i).

By (B.9) for any $\theta \in \Theta$ with $||\theta - \theta_F|| \geq \varepsilon$ and any $F \in \mathcal{F}$

$$||\mathbb{E}_F[g_1(W, \theta)]|| \geq \rho_{\min}^{1/2}(G'_{1,F} G_{1,F}) ||\theta_F - \theta|| \geq \varepsilon \rho_{\min}^{1/2}(G'_{1,F} G_{1,F}) \tag{B.10}$$

which combined with Assumption $B.1.$(iii) and $G_{1,F} = -\Gamma'_{xz_1,F^*}$ implies that

$$\inf_{F \in \mathcal{F}} \inf_{\theta \in B_\varepsilon^c(\theta_F)} ||\mathbb{E}_F[g_1(W, \theta)]|| > 0. \tag{B.11}$$

This verifies Assumption 3.1.(ii).

Next, we show Assumption 3.1.(iii). Let $Z_2 \equiv (Z'_1, Z^{*\prime})'$. By the Lyapunov inequality, Assumptions $B.1.$(i)-(ii) and $B.1.$(vii),

$$\begin{aligned} \sup_{F \in \mathcal{F}} \mathbb{E}_F[||Z_2||^2] &\leq \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||Z_1||^2] + 2 \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||V||^2] \\ &\quad + 2 \sup_{\delta_0 \in \Delta_\delta} ||\delta_0||^2 \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[U^2] < \infty. \end{aligned} \tag{B.12}$$

By $(B.12)$, the Hölder inequality, the Lyapunov inequality and Assumption $B.1.$(ii),

$$\sup_{F \in \mathcal{F}} ||G_{2,F}|| = \sup_{F \in \mathcal{F}} ||\mathbb{E}_F[Z_2 X']|| \leq \sup_{F \in \mathcal{F}} (\mathbb{E}_F[||Z_2||^2])^{1/2} \sup_{F^* \in \mathcal{F}^*} (\mathbb{E}_{F^*}[||X||^2])^{1/2} < \infty, \tag{B.13}$$

which together with the definition of $G_{2,F}$ and the Cauchy-Schwarz inequality implies that

$$\sup_{F \in \mathcal{F}} ||G'_{2,F} G_{2,F}|| < \infty. \tag{B.14}$$

Similarly by the Cauchy-Schwarz inequality, the Lyapunov inequality, Assumptions $B.1.$(ii) and $B.1.$(vii), we have

$$\begin{aligned} \sup_{F \in \mathcal{F}} \mathbb{E}_F[||Z_2||^4] &= \sup_{F \in \mathcal{F}} \mathbb{E}_F[(||Z_1||^2 + ||Z^*||^2)^2] \\ &\leq 2 \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||Z_1||^4] + 2 \sup_{F \in \mathcal{F}} \mathbb{E}_F[||Z^*||^4] \\ &\leq 2 \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||Z_1||^4] + 8 \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||V||^4] \\ &\quad + 8 \sup_{\delta_0 \in \Delta_\delta} ||\delta_0||^4 \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[U^4] < \infty. \end{aligned} \tag{B.15}$$

By $(B.12)$, $(B.15)$, Assumption $B.1.$(ii), the Lyapunov inequality and the Hölder inequality, we have

$$\sup_{F \in \mathcal{F}} \left\| \mathbb{E}_F[Z_2 Z_2'(Y - X'\theta_0)^2] \right\|$$

$$\leq \sup_{F \in \mathcal{F}} \mathbb{E}_F[||Z_2||^2 (Y - X'\theta_0)^2]$$

$$\leq \sup_{F \in \mathcal{F}} (\mathbb{E}_F[||Z_2||^4])^{1/2} \sup_{F^* \in \mathcal{F}^*} (\mathbb{E}_{F^*}[U^4])^{1/2} < \infty, \tag{B.16}$$

and

$$\sup_{F \in \mathcal{F}} \left\| \mathbb{E}_F[(Y - X'\theta_0)Z_2] \right\| \leq \sup_{F \in \mathcal{F}} (\mathbb{E}_F[||Z_2||^2])^{1/2} \sup_{F^* \in \mathcal{F}^*} (\mathbb{E}_{F^*}[U^2])^{1/2} < \infty. \tag{B.17}$$

By the definition of $\Omega_{2,F}$, the triangle inequality, the Cauchy-Schwarz inequality and the results in (B.16) and (B.17),

$$\sup_{F \in \mathcal{F}} \|\Omega_{2,F}\| < \infty. \tag{B.18}$$

We then show that $\theta_F^* \in int(\Theta)$. By the triangle inequality, the Cauchy-Schwarz inequality and the Hölder inequality,

$$\begin{aligned}
\|G_{2,F}\| &\leq \|\Gamma_{xz_1}\| + \|\delta_0\| \|\Gamma_{xu}\| + \|\Gamma_{xv}\| \\
&\leq (\mathbb{E}_{F^*}[||X||^2])^{1/2}(\mathbb{E}_{F^*}[||Z_1||^2])^{1/2} \\
&\quad + \|\delta_0\| (\mathbb{E}_{F^*}[||X||^2])^{1/2}(\mathbb{E}_{F^*}[U^2])^{1/2} \\
&\quad + (\mathbb{E}_{F^*}[||X||^2])^{1/2}(\mathbb{E}_{F^*}[||V||^2])^{1/2} \\
&\leq C_W(2 + C_\Delta^{1/2}), \tag{B.19}
\end{aligned}$$

for any $F \in \mathcal{F}$, where $C_W < \infty$ by Assumptions $B.1.$(ii) and (vii). Since $G_{2,F}' = (G_{1,F}', G_{r*,F}')$ where $G_{r*,F} = -\delta_0 \mathbb{E}_{F^*}[UX'] - \mathbb{E}_{F^*}[VX']$, we have

$$G_{2,F}' G_{2,F} = G_{1,F}' G_{1,F} + G_{r*,F}' G_{r*,F}, \tag{B.20}$$

which implies that for any $F \in \mathcal{F}$,

$$\rho_{\min}(G_{2,F}' G_{2,F}) \geq \rho_{\min}(G_{1,F}' G_{1,F}). \tag{B.21}$$

To show Assumption 3.1.(iii), we write

$$\begin{aligned}
Q_F(\theta) &= \mathbb{E}_F[Z_2(Y - X'\theta)]' \Omega_{2,F}^{-1} \mathbb{E}_F[Z_2(Y - X'\theta)] \\
&= \theta' G_{2,F}' \Omega_{2,F}^{-1} G_{2,F} \theta + 2\theta' G_{2,F}' \Omega_{2,F}^{-1} C_F + C_F' \Omega_{2,F}^{-1} C_F, \tag{B.22}
\end{aligned}$$

where $C_F = \mathbb{E}_F[Z_2 Y]$. Since $G_{2,F}' \Omega_{2,F}^{-1} G_{2,F}$ is non-singular by (B.18), (B.21) and Assumption

B.1.(iii), $Q_F(\theta)$ is minimized at $\theta_F^* = -(G'_{2,F}\Omega_{2,F}^{-1}G_{2,F})^{-1}G'_{2,F}\Omega_{2,F}^{-1}C_F$ for any $F \in \mathcal{F}$. Therefore,

$$
\begin{aligned}
\|\theta_F^* - \theta_0\|^2 &= \left\| (G'_{2,F}\Omega_{2,F}^{-1}G_{2,F})^{-1}G'_{2,F}\Omega_{2,F}^{-1}\mathbb{E}_F\left[Z_2 U\right] \right\|^2 \\
&\leq \frac{\rho_{\max}^2(\Omega_{2,F})}{\rho_{\min}^2(G'_{2,F}G_{2,F})}\mathbb{E}_F\left[UZ'_2\right]\Omega_{2,F}^{-1}G_{2,F}G'_{2,F}\Omega_{2,F}^{-1}\mathbb{E}_F\left[Z_2 U\right] \\
&\leq \frac{\rho_{\max}^2(\Omega_{2,F})\rho_{\max}(G'_{2,F}G_{2,F})\Gamma_{uu}^2}{\rho_{\min}^2(\Omega_{2,F})\rho_{\min}^2(G'_{2,F}G_{2,F})}\|\delta_0\|^2 \\
&\leq \rho_{2,\min}^{-4}\rho_{2,\max}^3 C_\Delta C_W^2 \qquad\qquad\qquad\qquad\qquad (B.23)
\end{aligned}
$$

for any $F \in \mathcal{F}$. By Assumption B.1.(vi), $\theta_F^* \in int(\Theta)$. Moreover for any $\theta \in \Theta$ with $\|\theta - \theta_F^*\| \geq \varepsilon$,

$$
\begin{aligned}
Q_F(\theta) - Q_F(\theta_F^*) &\geq \rho_{\min}(G'_{2,F}\Omega_{2,F}^{-1}G_{2,F})\|\theta - \theta_F^*\|^2 \\
&\geq \varepsilon^2\rho_{\min}(G'_{2,F}\Omega_{2,F}^{-1}G_{2,F}) \\
&\geq \varepsilon^2\rho_{\max}^{-1}(\Omega_{2,F})\rho_{\min}(G'_{2,F}G_{2,F}), \qquad\qquad (B.24)
\end{aligned}
$$

which together with (B.18), (B.21) and Assumption B.1.(iii) implies that

$$
\inf_{F \in \mathcal{F}} \inf_{\theta \in B_\varepsilon^c(\theta_F^*)} [Q_F(\theta) - Q_F(\theta_F^*)] > 0. \qquad\qquad (B.25)
$$

This verifies Assumption 3.1.(iii).

Next, we verify Assumption 3.1.(iv). Let $\Omega_{2,F}^{(22)} = (\Omega_{2,rr,F} - \Omega'_{2,r1,F}\Omega_{z_1z_1u^2}^{-1}\Omega_{2,1r,F})^{-1}$, where $\Omega_{2,1r,F}$ and $\Omega_{2,rr,F}$ are defined in (B.4). Then

$$
\begin{aligned}
&G'_{2,F}\Omega_{2,F}^{-1}\delta_{2,F} \\
&= -(\Gamma_{xz_1}, \Gamma_{xv} + \Gamma_{xu}\delta'_0)\begin{pmatrix} -\Omega_{z_1z_1u^2}^{-1}\Omega_{2,1r,F} \\ I_{r^*} \end{pmatrix}\Omega_{2,F}^{(22)}\Omega_{uu}\delta_0 \\
&= \Omega_{uu}[(\Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1u^3} - \Gamma_{xu})\delta'_0 + \Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1vu^2} - \Gamma_{xv}]\Omega_{2,F}^{(22)}\delta_0 \\
&= \Omega_{uu}\delta'_0\Omega_{2,F}^{(22)}\delta_0(\Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1u^3} - \Gamma_{xu}) \\
&\quad + \Omega_{uu}(\Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1vu^2} - \Gamma_{xv})\Omega_{2,F}^{(22)}\delta_0, \qquad\qquad (B.26)
\end{aligned}
$$

by the formula of the inverse of partitioned matrix. For any $\delta_0 \in \Delta_\delta$ with $\|\delta_0\| > 0$, we have

$$
\frac{\delta'_0(\Omega_{2,F}^{(22)})^2\delta_0}{(\delta'_0\Omega_{2,F}^{(22)}\delta_0)^2} \geq \frac{(\rho_{\min}(\Omega_{2,F}^{(22)}))^2}{(\rho_{\max}(\Omega_{2,F}^{(22)}))^2}\frac{1}{\delta'_0\delta_0} \geq \frac{\rho_{2,\min}^2}{C_\Delta\rho_{2,\max}^2} \qquad\qquad (B.27)
$$

and

$$
\delta'_0\Omega_{2,F}^{(22)}\delta_0 = \frac{\delta'_0\Omega_{2,F}^{(22)}\delta_0(\delta'_0(\Omega_{2,F}^{(22)})^2\delta_0)^{1/2}}{(\delta'_0(\Omega_{2,F}^{(22)})^2\delta_0)^{1/2}} \geq \frac{\|\delta_0\|}{\rho_{2,\max}}\frac{\delta'_0\Omega_{2,F}^{(22)}\delta_0}{(\delta'_0(\Omega_{2,F}^{(22)})^2\delta_0)^{1/2}} \qquad (B.28)
$$

where the last inequality in (B.27) and the inequality in (B.28) are due to

$$\rho_{\min}(\Omega_{2,F}^{(22)}) \geq \rho_{\min}(\Omega_{2,F}^{-1}) = \rho_{2,\max}^{-1}$$

and

$$\rho_{\max}(\Omega_{2,F}^{(22)}) \leq \rho_{\max}(\Omega_{2,F}^{-1}) = \rho_{2,\min}^{-1}.$$

Therefore, for any $F \in \mathcal{F}$ with $\delta_{2,F} = \Omega_{uu}(0_{1 \times r_1}, \delta_0')'$ and $\|\delta_0\| > 0$,

$$
\begin{aligned}
\frac{\left\| G_{2,F}' \Omega_{2,F}^{-1} \delta_{2,F} \right\|}{\|\delta_{2,F}\|} &= \frac{\delta_0' \Omega_{2,F}^{(22)} \delta_0}{\|\delta_0\|} \left\| \begin{array}{c} (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 v u^2} - \Gamma_{xv}) \frac{\Omega_{2,F}^{(22)} \delta_0}{\delta_0' \Omega_{2,F}^{(22)} \delta_0} \\ + (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 u^3} - \Gamma_{xu}) \end{array} \right\| \\[2ex]
&\geq \frac{1}{\rho_{2,\max}} \frac{\delta_0' \Omega_{2,F}^{(22)} \delta_0}{(\delta_0' (\Omega_{2,F}^{(22)})^2 \delta_0)^{1/2}} \left\| \begin{array}{c} (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 v u^2} - \Gamma_{xv}) \frac{\Omega_{2,F}^{(22)} \delta_0}{\delta_0' \Omega_{2,F}^{(22)} \delta_0} \\ + (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 u^3} - \Gamma_{xu}) \end{array} \right\| \\[2ex]
&= \frac{1}{\rho_{2,\max}} \frac{1}{\|\widetilde{\delta}_0\|} \left\| \begin{array}{c} (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 v u^2} - \Gamma_{xv}) \widetilde{\delta}_0 \\ + (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 u^3} - \Gamma_{xu}) \end{array} \right\|
\end{aligned}
\tag{B.29}
$$

where $\widetilde{\delta}_0 \equiv \Omega_{2,F}^{(22)} \delta_0 / \delta_0' \Omega_{2,F}^{(22)} \delta_0$ and the inequality is by (B.28). By (B.28) and the definition of $B_{\rho_2}^c$, $\widetilde{\delta}_0 \in B_{\rho_2}^c$. Therefore, (B.29) implies that

$$
\frac{\left\| G_{2,F}' \Omega_{2,F}^{-1} \delta_{2,F} \right\|}{\|\delta_{2,F}\|} \geq \frac{1}{\rho_{2,\max}} \inf_{\delta \in B_{\rho_2}^c} \|\delta\|^{-1} \left\| \begin{array}{c} (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 v u^2} - \Gamma_{xv}) \delta \\ + (\Gamma_{xz_1} \Omega_{z_1 z_1 u^2}^{-1} \Gamma_{z_1 u^3} - \Gamma_{xu}) \end{array} \right\|.
\tag{B.30}
$$

Collecting the results in (B.18) and (B.30) and then applying Assumption B.1.(iv), we get

$$
\inf_{\{F \in \mathcal{F}: \|\delta_F\| > 0\}} \frac{\left\| G_{2,F}' \Omega_{2,F}^{-1} \delta_{2,F} \right\|}{\|\delta_{2,F}\|} > 0
\tag{B.31}
$$

which shows Assumption 3.1.(iv) with $\tau = 1$.

Assumption 3.1.(v) is implied by Assumption B.1.(vii). This finishes the verification of Assumption 3.1.

To verify Assumption 3.2, note that $g_2(W, \theta) = Z_2(U - X'(\theta - \theta_0))$, $g_{2,\theta}(W, \theta) = -Z_2 X'$ and $g_{2,\theta\theta}(W, \theta) = 0_{(r_2 d_\theta) \times d_\theta}$. Therefore, Assumption 3.2.(i) holds automatically. Moreover Assumption 3.2.(ii) is implied by Assumption B.1.(ii) and the assumption that Θ is bounded. Assumptions 3.2(iii)-(iv) follow from Assumption B.1.(iii).

We next verify Assumption 3.3. By definition,

$$
v_F = \left(\text{vec}(G_{2,F})', \text{vech}(\Omega_{2,F})', \delta_F \right).
\tag{B.32}
$$

Let $\Lambda_* = \{v_{*,F^*} : F^* \in \mathcal{F}^*\}$. From the expressions in (B.3), we see that $\Lambda = \{v_F : F \in \mathcal{F}\}$ is the

image of $\Lambda_* \times \Delta_\delta$ under a continuous mapping. By Assumption B.1.(ii) and the Hölder inequality, Λ_* is bounded which together with Assumption B.1.(v) implies that Λ_* is compact. Since Δ_δ is also a compact set by Assumption B.1.(vii), we know that $\Lambda_* \times \Delta_\delta$ is compact. Therefore, Λ is compact and hence closed. This verifies Assumption 3.3.(ii).

Let $\varepsilon_F = \Omega_{uu} c_\Delta$ where $c_\Delta = \min\{\min_{j \leq r^*} |c_{j,\Delta}|, \min_{j \leq r^*} |C_{j,\Delta}|\}$. Below we show that for any $\widetilde{\delta} \in \mathbb{R}^{r^*}$ with $0 \leq ||\widetilde{\delta}|| \leq \varepsilon_F$, there is $\widetilde{F} \in \mathcal{F}$ such that

$$\delta_{\widetilde{F}} = \widetilde{\delta}, \; ||G_{2,\widetilde{F}} - G_{2,F}|| \leq C_1 ||\widetilde{\delta}_F||^{1/4} \text{ and } ||\Omega_{2,\widetilde{F}} - \Omega_{2,F}|| \leq C_2 ||\widetilde{\delta}||^{1/4} \tag{B.33}$$

for some fixed constants C_1 and C_2. This verifies Assumption 3.3.(i) with $\kappa = 1/4$.

First if $\widetilde{\delta} = 0_{r^* \times 1}$, then we set $\widetilde{F}$ to be F which is induced by δ_0, θ_0 and F^* with $\delta_0 = 0_{r^* \times 1}$. By definition $G_{2,\widetilde{F}} = G_{2,F}$, $\Omega_{2,\widetilde{F}} = \Omega_{2,F}$ and $\widetilde{\delta}_{\widetilde{F}} = \delta_F = \delta_0 \Omega_{uu} = 0 = \widetilde{\delta}$ which implies that (B.33) holds.

Second consider any $\widetilde{\delta} \in \mathbb{R}^{r^*}$ with $0 < ||\widetilde{\delta}|| < \varepsilon_F$. Define $\widetilde{\delta}_0 = \widetilde{\delta} \Omega_{uu}^{-1}$. Since $||\widetilde{\delta}|| < \varepsilon_F$ and $\varepsilon_F = \Omega_{uu} c_\Delta$,

$$||\widetilde{\delta}_0|| = ||\widetilde{\delta} \Omega_{uu}^{-1}|| = ||\widetilde{\delta}|| \Omega_{uu}^{-1} < c_\Delta, \tag{B.34}$$

which combined with the definition of Δ_δ implies that $\widetilde{\delta}_0 \in \Delta_\delta$. Let $\widetilde{F}$ be the joint distribution induced by $\widetilde{\delta}_0$, θ_0 and F^*. By the definition of $\mathcal{F}$, we have $\widetilde{F} \in \mathcal{F}$. Moreover,

$$\widetilde{\delta}_{\widetilde{F}} = \widetilde{\delta}_0 \Omega_{uu} = \widetilde{\delta} \tag{B.35}$$

which verifies the equality in (B.33). By definition,

$$G_{2,\widetilde{F}} = \begin{pmatrix} -\mathbb{E}_{F^*}[Z_1 X'] \\ -\widetilde{\delta}_0 \mathbb{E}_{F^*}[U X'] - \mathbb{E}_{F^*}[V X'] \end{pmatrix} \text{ and } G_{2,F} = \begin{pmatrix} -\mathbb{E}_{F^*}[Z_1 X'] \\ -\mathbb{E}_{F^*}[V X'] \end{pmatrix} \tag{B.36}$$

which together with the Cauchy-Schwarz inequality and the Hölder inequality implies that

$$\begin{aligned} ||G_{2,\widetilde{F}} - G_{2,F}|| &= ||\widetilde{\delta}_0 \mathbb{E}_{F^*}[U X']|| \leq ||\widetilde{\delta}_0|| (\Omega_{uu} \mathbb{E}_{F^*}[||X||^2])^{1/2} \\ &= ||\widetilde{\delta}_0||^{3/4} \Omega_{uu}^{1/4} (\mathbb{E}_{F^*}[||X||^2])^{1/2} ||\widetilde{\delta}_0 \Omega_{uu}||^{1/4}. \end{aligned} \tag{B.37}$$

By Assumption B.1.(ii),

$$\sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||X||^2] < \infty \text{ and } \sup_{F^* \in \mathcal{F}^*} \Omega_{uu} < \infty \tag{B.38}$$

which together with (B.34), (B.37) and the definition of $\widetilde{\delta}$ implies that

$$||G_{2,\widetilde{F}} - G_{2,F}|| \leq C_1 ||\widetilde{\delta}||^{1/4}, \tag{B.39}$$

where $C_1 = c_\Delta^{3/4} \sup_{F^* \in \mathcal{F}^*} (\mathbb{E}_{F^*}[||X||^2])^{1/2} \sup_{F^* \in \mathcal{F}^*} \Omega_{uu}^{1/4}$ is finite.

To show the last inequality in (B.33), note that by definition $\theta_{\widetilde{F}} = \theta_0 = \theta_F$ and hence

$$\mathbb{E}_{\widetilde{F}}[Z_1 Z_1'(Y - X'\theta_{\widetilde{F}})^2] = \mathbb{E}_{F^*}[Z_1 Z_1' U^2] = \mathbb{E}_F[Z_1 Z_1'(Y - X'\theta_F)^2]. \tag{B.40}$$

Under $\widetilde{F}$,

$$\mathbb{E}_{\widetilde{F}}[Z_1 Z^{*\prime}(Y - X'\theta_{\widetilde{F}})^2] = \mathbb{E}_{F^*}[Z_1 (U\widetilde{\delta}_0 + V)' U^2] = \mathbb{E}_{F^*}[U^3 Z_1]\widetilde{\delta}_0' + \mathbb{E}_{F^*}[U^2 Z_1 V'], \tag{B.41}$$

and

$$\begin{aligned}
& \mathbb{E}_{\widetilde{F}}[Z^* Z^{*\prime}(Y - X'\theta_{\widetilde{F}})^2] \\
= \ & \mathbb{E}_{F^*}[(U\widetilde{\delta}_0 + V)(U\widetilde{\delta}_0 + V)' U^2] \\
= \ & \mathbb{E}_{F^*}[U^4]\widetilde{\delta}_0 \widetilde{\delta}_0' + \widetilde{\delta}_0 \mathbb{E}_{F^*}[U^3 V'] + \mathbb{E}_{F^*}[U^3 V]\widetilde{\delta}_0' + \mathbb{E}_{F^*}[U^2 V V'].
\end{aligned} \tag{B.42}$$

Under F,

$$\mathbb{E}_F[Z_1 Z^{*\prime}(Y - X'\theta_F)^2] = \mathbb{E}_{F^*}[U^2 Z_1 V'] \text{ and } \mathbb{E}_F[Z^* Z^{*\prime}(Y - X'\theta_F)^2] = \mathbb{E}_{F^*}[U^2 V V']. \tag{B.43}$$

Collecting the results in (B.40), (B.41), (B.42) and (B.43), and applying the triangle inequality, we get

$$\begin{aligned}
& \left\| \mathbb{E}_{\widetilde{F}}[Z_2 Z_2'(Y - X'\theta_{\widetilde{F}})^2] - \mathbb{E}_F[Z_2 Z_2'(Y - X'\theta_F)^2] \right\| \\
\leq \ & \left\| \mathbb{E}_{F^*}[U^3 Z_1]\widetilde{\delta}_0' \right\| + \left\| \mathbb{E}_{F^*}[U^4]\widetilde{\delta}_0 \widetilde{\delta}_0' \right\| \\
& + \left\| \widetilde{\delta}_0 \mathbb{E}_{F^*}[U^3 V'] \right\| + \left\| \mathbb{E}_{F^*}[U^3 V]\widetilde{\delta}_0' \right\|.
\end{aligned} \tag{B.44}$$

By Assumption B.1.(ii) and the Lyapunov inequality

$$\sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[|U|^5] < \infty, \quad \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||Z_1||^4] < \infty \text{ and } \sup_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[||V||^4] < \infty. \tag{B.45}$$

By the Hölder inequality,

$$\begin{aligned}
\left\| \mathbb{E}_{F^*}[U^3 Z_1] \right\| & \leq (\mathbb{E}_{F^*}[|U|||Z_1||^2] \mathbb{E}_{F^*}[|U|^5])^{1/2} \\
& \leq (\mathbb{E}_{F^*}[|U|^5])^{1/2} (\mathbb{E}_{F^*}[||Z_1||^4])^{1/4} (\mathbb{E}_{F^*}[U^2])^{1/4} \\
& = \Omega_{uu}^{1/4} (\mathbb{E}_{F^*}[|U|^5])^{1/2} (\mathbb{E}_{F^*}[||Z_1||^4])^{1/4}.
\end{aligned} \tag{B.46}$$

Similarly, we can show that

$$\left\| \mathbb{E}_{F^*}[U^3 V'] \right\| \leq \Omega_{uu}^{1/4} (\mathbb{E}_{F^*}[|U|^5])^{1/2} (\mathbb{E}_{F^*}[||V||^4])^{1/4} \tag{B.47}$$

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and

$$\mathbb{E}_{F^*}[U^4] \leq (\mathbb{E}_{F^*}[U^2]\mathbb{E}_{F^*}[U^6])^{1/2} = \Omega_{uu}^{1/4} \sup_{F^* \in \mathcal{F}^*} (\mathbb{E}_{F^*}[U^6])^{1/2}. \tag{B.48}$$

Let $C_{2,0} = \sup_{F^* \in \mathcal{F}^*}\{(\mathbb{E}_{F^*}[|U|^5])^{1/2}[(\mathbb{E}_{F^*}[||Z_1||^4])^{1/4} + (\mathbb{E}_{F^*}[||V||^4])^{1/4}] + (\mathbb{E}_{F^*}[U^6])^{1/2}\}$. Combining the results in (B.44), (B.46), (B.47) and (B.48), and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left\|\mathbb{E}_{\widetilde{F}}[Z_2 Z_2'(Y - X'\theta_{\widetilde{F}})^2] - \mathbb{E}_F[Z_2 Z_2'(Y - X'\theta_F)^2]\right\| \\
&\leq \quad 3C_{2,0}\Omega_{uu}^{1/4}||\widetilde{\delta}_0|| + C_{2,0}\Omega_{uu}^{1/4}||\widetilde{\delta}_0||^2 \\
&= \quad (3C_{2,0}||\widetilde{\delta}_0||^{3/4} + C_{2,0}||\widetilde{\delta}_0||^{7/4})\Omega_{uu}^{1/4}||\widetilde{\delta}_0||^{1/4} \leq C_{2,1}||\widetilde{\delta}||^{1/4}
\end{aligned} \tag{B.49}$$

where $C_{2,1} = C_{2,0}(3c_\Delta^{3/4} + c_\Delta^{7/4})$, the second inequality is by (B.34) and the definition of $\widetilde{\delta}$. By (B.45), Assumption B.1.(ii) and the definition of c_Δ,

$$C_{2,1} < \infty. \tag{B.50}$$

Next note that

$$\mathbb{E}_{\widetilde{F}}[Z_2(Y - X'\theta_{\widetilde{F}})] = \begin{pmatrix} \mathbb{E}_{F^*}[Z_1 U] \\ \widetilde{\delta}_0 \Omega_{uu} \end{pmatrix} \text{ and } \mathbb{E}_F[Z_2(Y - X'\theta_F)] = \begin{pmatrix} \mathbb{E}_{F^*}[Z_1 U] \\ 0_{r^* \times 1} \end{pmatrix} \tag{B.51}$$

which implies that

$$\begin{aligned}
&\left\|\begin{array}{c} \mathbb{E}_{\widetilde{F}}[Z_2(Y - X'\theta_{\widetilde{F}})]\mathbb{E}_{\widetilde{F}}[Z_2'(Y - X'\theta_{\widetilde{F}})] \\ -\mathbb{E}_F[Z_2(Y - X'\theta_F)]\mathbb{E}_F[Z_2'(Y - X'\theta_F)] \end{array}\right\| \\
&= \quad \left\|\begin{pmatrix} 0_{r_1 \times r_1} & \Omega_{uu,F^*}\mathbb{E}_{F^*}[Z_1 U]\widetilde{\delta}_0' \\ \widetilde{\delta}_0 \mathbb{E}_{F^*}[Z_1' U]\Omega_{uu} & \widetilde{\delta}_0\widetilde{\delta}_0'\Omega_{uu}^2 \end{pmatrix}\right\| \\
&\leq \quad \Omega_{uu}\left\|\mathbb{E}_{F^*}[Z_1 U]\widetilde{\delta}_0'\right\| + \Omega_{uu}\left\|\widetilde{\delta}_0\mathbb{E}_{F^*}[Z_1 U]\right\| + \Omega_{uu}^2||\widetilde{\delta}_0||^2 \\
&\leq \quad 2\Omega_{uu}||\widetilde{\delta}_0||\,||\mathbb{E}_{F^*}[Z_1 U]|| + \Omega_{uu}^2||\widetilde{\delta}_0||^2 \\
&\leq \quad (2\Omega_{uu}^{5/4}||\widetilde{\delta}_0||^{3/4}(\mathbb{E}_{F^*}[||Z_1||^2])^{1/2} + \Omega_{uu}^{7/4}||\widetilde{\delta}_0||^{7/4})||\widetilde{\delta}||^{1/4} \leq C_{2,2}||\widetilde{\delta}||^{1/4}
\end{aligned} \tag{B.52}$$

where $C_{2,2} = \sup_{F^* \in \mathcal{F}^*}\{2\Omega_{uu}^{5/4}(\mathbb{E}_{F^*}[||Z_1||^2])^{1/2}c_\Delta^{3/4} + \Omega_{uu}^{7/4}c_\Delta^{7/4}\}$, the second inequality is by the Cauchy-Schwarz inequality, the third inequality is by the Hölder inequality. By Assumption B.1.(ii) and the definition of c_Δ,

$$C_{2,2} < \infty. \tag{B.53}$$

By the definition of $\Omega_{2,F}$ in (B.2), we can use the triangle inequality and the results in (B.49) and (B.52) to deduce that

$$\left\|\Omega_{2,\widetilde{F}} - \Omega_{2,F}\right\| \leq C_2||\widetilde{\delta}||^{1/4} \tag{B.54}$$

where $C_2 = C_{2,1} + C_{2,2}$ and $C_2 < \infty$ by (B.50) and (B.53), which proves the second inequality in (B.33). This verifies Assumption 3.3.(i) with $\kappa = 1/4$. ∎

Proof of Lemma 3.1.   Next, we apply Lemma B.1 to prove Lemma 3.1 in the paper. For convenience, the conditions of Lemma 3.1 are stated here. The proof verifies the conditions of Lemma B.1 with the following conditions in a Gaussian model. Let $\mathcal{F}^*$ denote the set of normal distributions which satisfies:

(i) $\phi_u = 0$, $\Gamma_{z_1 u} = 0_{r_1 \times 1}$ and $\Gamma_{vu} = 0_{r^* \times 1}$;

(ii) $\inf_{F^* \in \mathcal{F}^*} \rho_{\min}(\Gamma_{xz_1} \Gamma_{z_1 x}) > 0$, $\sup_{F^* \in \mathcal{F}^*} ||\phi||^2 < \infty$ and $0 < \inf_{F^* \in \mathcal{F}^*} \rho_{\min}(\Psi) \leq \sup_{F^* \in \mathcal{F}^*} \rho_{\max}(\Psi) < \infty$;

(iii) $\inf_{F^* \in \mathcal{F}^*} \inf_{\{||\delta|| \geq \varepsilon\}} ||\delta||^{-1} ||(\Gamma_{xz_1} \Gamma_{z_1 z_1}^{-1} \Gamma_{z_1 v} - \Gamma_{xv})\delta - \Gamma_{xu}|| > 0$ for some $\varepsilon > 0$ that is small enough (where ε is given in (A.3) in the Appendix of CLS);

(iv) $\theta_0 \in int(\Theta)$ and Θ is compact and large enough such that the pseudo-true value $\theta^*(F) \in int(\Theta)$;

(v) $\Delta_\delta = [c_{1,\Delta}, C_{1,\Delta}] \times \cdots \times [c_{r^*,\Delta}, C_{r^*,\Delta}]$ where $\{c_{j,\Delta}, C_{j,\Delta}\}_{j=1}^{r^*}$ is a set of finite constants with $c_{j,\Delta} < 0 < C_{j,\Delta}$ for $j = 1, \ldots, r^*$.

Specifically, we assume that condition (ii) of Lemma 3.1 holds with some constants c_ρ and C_ρ such that $c_\rho \leq \rho_{\min}(\Gamma_{xz_1} \Gamma_{z_1 x})$, $||\phi||^2 \leq C_\rho$ and $c_\rho \leq \rho_{\min}(\Psi) \leq \rho_{\max}(\Psi) \leq C_\rho$; condition (iii) of Lemma 3.1 holds with

$$\inf_{\delta \in B_\varepsilon^c} ||\delta||^{-1} ||(\Gamma_{xz_1} \Gamma_{z_1 z_1}^{-1} \Gamma_{z_1 v} - \Gamma_{xv})\delta - \Gamma_{xu}|| \geq c_\Gamma \tag{B.55}$$

for some positive constant c_Γ and

$$B_\varepsilon^c \equiv \{\delta \in \mathbb{R}^{r^*} : ||\delta|| \geq c_{*,\rho} C_{*,\rho}^{-1} C_\Delta^{-1}\}, \tag{B.56}$$

where

$$C_{*,W} \equiv 2(d_\theta + r_2 + 1)C_\rho, \ c_{*,\rho} \equiv \min\{1, c_\rho^2\} \text{ and } C_{*,\rho} \equiv C_{*,W}^2 (2 + C_\Delta^{1/2})^2 \tag{B.57}$$

and $C_\Delta \equiv \sup_{\delta_0 \in \Delta_\delta} ||\delta_0||^2$.

Assumption B.1.(i) holds under Condition (i) of Lemma 3.1. Since $(X', Z_1', V', U')'$ is a normal random vector, Assumption B.1.(ii) holds by $||\phi||^2 \leq C_\rho$ and $\rho_{\max}(\Psi) \leq C_\rho$. By $\rho_{\min}(\Psi) \geq c_\rho$ and $\phi_u = 0$, we have $\mathbb{E}_{F^*}[U^2] \geq c_\rho$ for any $F^* \in \mathcal{F}^*$ and hence $\inf_{F^* \in \mathcal{F}^*} \mathbb{E}_{F^*}[U^2] > 0$. Let F denote the distribution of W induced by F^* with mean ϕ and variance-covariance matrix Ψ. By definition, $G_{1,F} = -\mathbb{E}_{F^*}[Z_1 X'] = \Gamma_{z_1 x}$. Therefore,

$$\inf_{F \in \mathcal{F}} \rho_{\min}(G_{1,F}' G_{1,F}) \geq c_\rho > 0 \tag{B.58}$$

holds by $\rho_{\min}(\Gamma_{xz_1} \Gamma_{z_1 x}) \geq c_\rho > 0$ for any $F^* \in \mathcal{F}^*$. Since $\Gamma_{z_1 u} = 0_{r_1 \times 1}$ and $\Gamma_{vu} = 0_{r^* \times 1}$ for any $F^* \in \mathcal{F}^*$, U is independent with respect to $(Z_1', V')'$ under the normal assumption. Therefore, by

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Condition (i) of Lemma 3.1

$$
\begin{aligned}
\Omega_{2,F} &= \begin{pmatrix} \Omega_{uu}\Gamma_{z_1z_1} & \Omega_{uu}\Gamma_{z_1v} \\ \Omega_{uu}\Gamma'_{z_1v} & 2\Omega_{uu}^2\delta_0\delta'_0 + \Omega_{uu}\Gamma_{vv} \end{pmatrix} \\
&= \Omega_{uu}\begin{pmatrix} \Omega_{z_1z_1} & \Omega_{z_1v} \\ \Omega_{vz_1} & \Omega_{vv} \end{pmatrix} + \Omega_{uu}\begin{pmatrix} \phi_{z_1} \\ \phi_v \end{pmatrix}\begin{pmatrix} \phi_{z_1} \\ \phi_v \end{pmatrix}' + \begin{pmatrix} 0_{r_1\times r_1} & 0_{r_1\times r^*} \\ 0_{r^*\times r_1} & 2\Omega_{uu}^2\delta_0\delta'_0 \end{pmatrix}
\end{aligned}
\tag{B.59}
$$

which implies that $\rho_{\min}(\Omega_{2,F}) \geq \rho_{\min}^2(\Psi)$ where F is the distribution of W induced by F^* with mean ϕ and variance-covariance matrix Ψ. Since $\rho_{\min}(\Psi) \geq c_\rho > 0$, we have

$$
\inf_{F\in\mathcal{F}} \rho_{\min}(\Omega_{2,F}) \geq c_\rho^2 > 0.
\tag{B.60}
$$

This finishes the proof of Assumption $B.1.$(iii).

By $(B.59)$, Conditions (ii) and (v) of Lemma 3.1

$$
\sup_{F\in\mathcal{F}} \rho_{\max}(\Omega_{2,F}) \leq \rho_{\max}^2(\Psi) + \rho_{\max}(\Psi)\|\phi\|^2 + 2\rho_{\max}^2(\Psi)C_\Delta \leq 2C_\rho^2(1 + C_\Delta).
\tag{B.61}
$$

By (B.19) in the proof of Lemma B.1,

$$
\|G_{2,F}\| \leq 2C_\rho(d_\theta + r_2 + 1)(2 + C_\Delta^{1/2})
$$

which implies that

$$
\sup_{F\in\mathcal{F}} \rho_{\max}(G'_{2,F}G_{2,F}) \leq 4C_\rho^2(d_\theta + r_2 + 1)^2(2 + C_\Delta^{1/2})^2.
\tag{B.62}
$$

By (B.58) and (B.60),

$$
\min\left\{ \inf_{F\in\mathcal{F}} \rho_{\min}(\Omega_{2,F}), \inf_{F\in\mathcal{F}} \rho_{\min}(G'_{2,F}G_{2,F}) \right\} \geq \min\{1, c_\rho^2\}.
\tag{B.63}
$$

By (B.61) and (B.62),

$$
\max\left\{ \sup_{F\in\mathcal{F}} \rho_{\max}(\Omega_{2,F}), \sup_{F\in\mathcal{F}} \rho_{\max}(G'_{2,F}G_{2,F}) \right\} \leq 4C_\rho^2(d_\theta + r_2 + 1)^2(2 + C_\Delta^{1/2})^2.
\tag{B.64}
$$

From $(B.63)$, $(B.64)$, the definitions of $c_{*,\rho}$, $C_{*,\rho}$ and $B_{N,\rho}^c$, we have $B_{\rho_2}^c \subset B_{N,\rho}^c$ where $B_{\rho_2}^c$ is defined in $(B.7)$. Moreover, by $\phi_u = 0$, the normal assumption and the independence between U and $(Z'_1, V')'$, we have $\Omega_{z_1z_1u^2} = \Omega_{uu}\Gamma_{z_1z_1}$, $\Gamma_{z_1vu^2} = \Omega_{uu}\Gamma_{z_1v}$ and $\Gamma_{z_1u^3} = 0_{r_1\times 1}$, which implies that

$$
\begin{aligned}
&\|(\Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1vu^2} - \Gamma_{xv})\delta + \Gamma_{xz_1}\Omega_{z_1z_1u^2}^{-1}\Gamma_{z_1u^3} - \Gamma_{xu}\| \\
&= \|(\Gamma_{xz_1}\Gamma_{z_1z_1}^{-1}\Gamma_{z_1v} - \Gamma_{xv})\delta - \Gamma_{xu}\|.
\end{aligned}
\tag{B.65}
$$

Assumption $B.1.$(iv) follows by $B_{\rho_2}^c \subset B_{N,\rho}^c$, $(B.65)$ and Condition (iii) of the lemma.

We next show that Assumption $B.1.$(v) holds. Define

$$\overline{v}_{*,F^*} = \left(\begin{array}{c} \Omega_{uu}, \mathrm{vec}(\Gamma_{xz_1})', \mathrm{vec}(\Gamma_{xu})', \mathrm{vec}(\Gamma_{xv})', \\ \mathrm{vec}(\Gamma_{z_1 v})', \mathrm{vech}(\Gamma_{z_1 z_1})', \mathrm{vech}(\Gamma_{vv})' \end{array} \right).$$

Under Condition (i) of Lemma 3.1 and the normal assumption, $\Gamma_{u^2 u^2} = 3\Omega_{uu}^2$, $\Gamma_{z_1 u^3} = 0_{r_1 \times 1}$, $\Gamma_{vu^3} = 0_{r^* \times 1}$, $\Omega_{z_1 z_1 u^2} = \Omega_{uu}\Gamma_{z_1 z_1}$, $\Gamma_{z_1 v u^2} = \Omega_{uu}\Gamma_{z_1 v}$ and $\Omega_{vvu^2} = \Omega_{uu}\Gamma_{vv}$. Therefore to verify Assumption $B.1.$(v), it is sufficient to show that the set $\{\overline{v}_{*,F^*} : F^* \in \mathcal{F}^*\}$ is compact because the set $\{v_{*,F^*} : F^* \in \mathcal{F}^*\}$ is the image of the set $\{\overline{v}_{*,F^*} : F^* \in \mathcal{F}^*\}$ under a continuous mapping. Let $\{(\phi_n, \Psi_n)\}_n$ be a convergent sequence where (ϕ_n, Ψ_n) satisfies Conditions (i)-(iii) of Lemma 3.1 for any n. Let $\widetilde{\phi}$ and $\widetilde{\Psi}$ denote the limits of ϕ_n and Ψ_n under the Euclidean norm respectively. We first show that Conditions (i)-(iii) of Lemma 3.1 hold for $(\widetilde{\phi}, \widetilde{\Psi})$. Since $\phi_{u,n} = 0$, $\Gamma_{z_1 u,n} = 0_{r_1 \times 1}$ and $\Gamma_{vu,n} = 0_{r^* \times 1}$ for any n, we have $\widetilde{\phi}_u = 0$, $\widetilde{\Gamma}_{z_1 u} = 0_{r_1 \times 1}$ and $\widetilde{\Gamma}_{vu} = 0_{r^* \times 1}$ which shows that $(\widetilde{\phi}, \widetilde{\Psi})$ satisfies Condition (i) of Lemma 3.1. Since $\phi_n \to \widetilde{\phi}$ and $\|\phi_n\|^2 \leq C_\rho$ for any n, we have $\|\widetilde{\phi}\|^2 \leq C_\rho$. By the convergence of (ϕ_n, Ψ_n), $\Gamma_{xz_1,n} \to \widetilde{\Gamma}_{xz_1}$. Since the roots of a polynomial continuously depends on its coefficients, we have

$$\rho_{\min}(\Gamma_{xz_1,n}\Gamma'_{xz_1,n}) \to \rho_{\min}(\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}'_{xz_1}), \; \rho_{\min}(\Psi_n) \to \rho_{\min}(\widetilde{\Psi}) \text{ and } \rho_{\max}(\Psi_n) \to \rho_{\max}(\widetilde{\Psi})$$

which together with the assumption that $\Gamma_{xz_1,n}$ and Ψ_n satisfy Condition (ii) of Lemma 3.1 implies that

$$c_\rho \leq \rho_{\min}(\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}'_{xz_1}) \text{ and } c_\rho \leq \rho_{\min}(\widetilde{\Psi}) \leq \rho_{\max}(\widetilde{\Psi}) \leq C_\rho.$$

This shows that Condition (ii) of Lemma 3.1 holds for $(\widetilde{\phi}, \widetilde{\Psi})$. For any $\delta \in B_{N,\rho}^c$, by the triangle inequality, the Cauchy-Schwarz inequality and $\|\delta\| \geq c_\rho^2 C_\rho^{-2} C_\Delta^{-1}(1+C_\Delta)^{-1} 2^{-1}$,

$$
\begin{aligned}
&\|\delta\|^{-1} \|(\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}_{z_1 z_1}^{-1}\widetilde{\Gamma}_{z_1 v} - \widetilde{\Gamma}_{xv})\delta - \widetilde{\Gamma}_{xu}\| \\
\geq \; & \|\delta\|^{-1} \|(\Gamma_{xz_1,n}\Gamma_{z_1 z_1,n}^{-1}\Gamma_{z_1 v,n} - \Gamma_{xv,n})\delta - \Gamma_{xu,n}\| \\
& -\|\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}_{z_1 z_1}^{-1}\widetilde{\Gamma}_{z_1 v} - \Gamma_{xz_1,n}\Gamma_{z_1 z_1,n}^{-1}\Gamma_{z_1 v,n}\| \\
& -\|\widetilde{\Gamma}_{xv} - \Gamma_{xv,n}\| - 2C_\rho^2 C_\Delta(1+C_\Delta)c_\rho^{-2}\|\Gamma_{xu,n} - \widetilde{\Gamma}_{xu}\|
\end{aligned}
$$

which together with the convergence of (ϕ_n, Ψ_n) and Conditions (ii)-(iii) of Lemma 3.1 implies that

$$
\begin{aligned}
&\|\delta\|^{-1} \|(\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}_{z_1 z_1}^{-1}\widetilde{\Gamma}_{z_1 v} - \widetilde{\Gamma}_{xv})\delta - \widetilde{\Gamma}_{xu}\| \\
\geq \; & c_\Gamma - \|\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}_{z_1 z_1}^{-1}\widetilde{\Gamma}_{z_1 v} - \Gamma_{xz_1,n}\Gamma_{z_1 z_1,n}^{-1}\Gamma_{z_1 v,n}\| \\
& -\|\widetilde{\Gamma}_{xv} - \Gamma_{xv,n}\| - 2C_\rho^2 C_\Delta(1+C_\Delta)c_\rho^{-2}\|\Gamma_{xu,n} - \widetilde{\Gamma}_{xu}\|
\end{aligned}
$$

for any n. Let n go to infinity, we get

$$\|\delta\|^{-1} \|(\widetilde{\Gamma}_{xz_1}\widetilde{\Gamma}_{z_1 z_1}^{-1}\widetilde{\Gamma}_{z_1 v} - \widetilde{\Gamma}_{xv})\delta - \widetilde{\Gamma}_{xu}\| \geq c_\Gamma$$

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for any $\delta \in B_\varepsilon^c$. This shows that Condition (iii) of Lemma 3.1 also holds for $(\widetilde{\phi}, \widetilde{\Psi})$. Hence the set of (ϕ, Ψ) which satisfies Conditions (i)-(iii) of Lemma 3.1 is closed. By Conditions (i)-(ii) of the Lemma, we know that this set is compact because it is also bounded. Let F^* denote the normal distribution with mean ϕ and variance-covariance matrix Ψ. Then $\overline{v}_{*,F^*}$ is the image of (ϕ, Ψ) under a continuous mapping, which implies that $\{\overline{v}_{*,F^*} : F^* \in \mathcal{F}^*\}$ is compact. Therefore the set $\{v_{*,F^*} : F^* \in \mathcal{F}^*\}$ is compact and hence closed. This proves Assumption $B.1.$(v).

Assumption $B.1.$(vi) is used to show that $\theta_F \in int(\Theta)$ and $\theta_F^* \in int(\Theta)$ for any $F \in \mathcal{F}$. By $\theta_F = \theta_0$ and Condition (iv) of Lemma 3.1, we have $\theta_F \in int(\Theta)$ and $\theta_F^* \in int(\Theta)$.

Finally, Assumption $B.1.$(vii) is the same as Condition (v) of Lemma 3.1. ∎

C   Proof of Some Auxiliary Results in Sections 4, 5 and 6 of CLS

Proof of Lemma A.2.   (i) Let $g_{2,j}(w, \theta)$ denote the j-th $(j = 1, \ldots, r_2)$ component of $g_2(w, \theta)$. By the mean value expansion,

$$g_{2,j}(w, \theta_1) - g_{2,j}(w, \theta_2) = g_{2,j,\theta}(w, \widetilde{\theta}_{1,2})(\theta_1 - \theta_2) \tag{C.1}$$

for any $j = 1, \ldots, r_2$, where $\widetilde{\theta}_{1,2}$ is some vector between θ_1 and θ_2. By (C.1) and the Cauchy-Schwarz inequality

$$|\mathbb{E}_F\left[g_{2,j}(w, \theta_1) - g_{2,j}(w, \theta_2)\right]| \leq \mathbb{E}_F\left[\sup_{\theta \in \Theta} \|g_{2,\theta}(W, \theta)\|\right] \|\theta_1 - \theta_2\|, \tag{C.2}$$

for any $j = 1, \ldots, r_2$. By (C.2), we deduce that

$$\begin{aligned}
\|M_{2,F}(\theta_1) - M_{2,F}(\theta_2)\| &\leq \sqrt{r_2}\mathbb{E}_F\left[\sup_{\theta \in \Theta} \|g_{2,\theta}(W, \theta)\|\right] \|\theta_1 - \theta_2\| \\
&\leq C_{M,1}\sqrt{r_2} \|\theta_1 - \theta_2\|
\end{aligned} \tag{C.3}$$

for any $F \in \mathcal{F}$, where $C_{M,1} \equiv \sup_{F \in \mathcal{F}} \mathbb{E}_F\left[\sup_{\theta \in \Theta} \|g_{2,\theta}(W, \theta)\|\right]$ and $C_{M,1} < \infty$ by Assumption 3.2.(ii). This immediately proves the claim in (i). The claim in (ii) follows by similar argument and its proof is omitted.

(iii) By the mean value expansion,

$$\begin{aligned}
&g_{2,j_1}(w, \theta_1)g_{2,j_2}(w, \theta_1) - g_{2,j_1}(w, \theta_2)g_{2,j_2}(w, \theta_2) \\
&= \left[g_{2,j_1,\theta}(w, \widetilde{\theta}_{1,2})g_{2,j_2}(w, \widetilde{\theta}_{1,2}) + g_{2,j_1}(w, \widetilde{\theta}_{1,2})g_{2,j_2,\theta}(w, \widetilde{\theta}_{1,2})\right](\theta_1 - \theta_2)
\end{aligned} \tag{C.4}$$

for any $j_1, j_2 = 1, \ldots, r_2$, where $\widetilde{\theta}_{1,2}$ is some vector between θ_1 and θ_2 and may take different values

from the $\widetilde{\theta}_{1,2}$ in (C.1). By (C.4), the triangle inequality and the Cauchy-Schwarz inequality

$$|\mathbb{E}_F\left[g_{2,j_1}(w,\theta_1)g_{2,j_2}(w,\theta_1) - g_{2,j_1}(w,\theta_2)g_{2,j_2}(w,\theta_2)\right]|$$
$$\leq\ 2\mathbb{E}_F\left[\sup_{\theta\in\Theta}\|g_2(W,\theta)\|\,\|g_{2,\theta}(W,\theta)\|\right]\|\theta_1-\theta_2\|$$
$$\leq\ \mathbb{E}_F\left[\sup_{\theta\in\Theta}(\|g_2(W,\theta)\|^2 + \|g_{2,\theta}(W,\theta)\|^2)\right]\|\theta_1-\theta_2\| \tag{C.5}$$

for any $j_1, j_2 = 1,\ldots,r_2$, where the second inequality is by the simple inequality that $|ab| \leq (a^2+b^2)/2$. By (C.5)

$$\left\|\mathbb{E}_F\left[g_2(W,\theta_1)g_2(W,\theta_1)' - g_2(W,\theta_2)g_2(W,\theta_2)'\right]\right\|$$
$$\leq\ r_2\mathbb{E}_F\left[\sup_{\theta\in\Theta}(\|g_2(W,\theta)\|^2 + \|g_{2,\theta}(W,\theta)\|^2)\right]\|\theta_1-\theta_2\|$$
$$\leq\ r_2 C_{M,2}\|\theta_1-\theta_2\| \tag{C.6}$$

for any $F \in \mathcal{F}$, where $C_{M,2} \equiv \sup_{F\in\mathcal{F}}\mathbb{E}_F\left[\sup_{\theta\in\Theta}(\|g_2(W,\theta)\|^2 + \|g_{2,\theta}(W,\theta)\|^2)\right]$ and $C_{M,2} < \infty$ by Assumption 3.2.(ii). Using the triangle inequality, and the inequality in (C.2), we deduce that

$$|\mathbb{E}_F[g_{2,j_1}(w,\theta_1)]\mathbb{E}_F[g_{2,j_2}(w,\theta_1)] - \mathbb{E}_F[g_{2,j_1}(w,\theta_2)]\mathbb{E}_F[g_{2,j_2}(w,\theta_2)]|$$
$$\leq\ |\mathbb{E}_F[g_{2,j_1}(w,\theta_1) - g_{2,j_1}(w,\theta_2)]\mathbb{E}_F[g_{2,j_2}(w,\theta_1)]|$$
$$\quad + |\mathbb{E}_F[g_{2,j_1}(w,\theta_2)]\mathbb{E}_F[g_{2,j_2}(w,\theta_2) - g_{2,j_2}(w,\theta_1)]|$$
$$\leq\ 2\mathbb{E}_F\left[\sup_{\theta\in\Theta}\|g_2(W,\theta)\|\right]\mathbb{E}_F\left[\sup_{\theta\in\Theta}\|g_{2,\theta}(W,\theta)\|\right]\|\theta_1-\theta_2\| \tag{C.7}$$

for any $j_1, j_2 = 1,\ldots,r_2$. By (C.7)

$$\left\|\mathbb{E}_F[g_2(w,\theta_1)]\mathbb{E}_F[g_2(w,\theta_1)'] - \mathbb{E}_F[g_2(w,\theta_2)]\mathbb{E}_F[g_2(w,\theta_2)']\right\| \leq r_2 C_{M,3}\|\theta_1-\theta_2\| \tag{C.8}$$

for any $F \in \mathcal{F}$, where $C_{M,3} \equiv 2\sup_{F\in\mathcal{F}}\mathbb{E}_F\left[\sup_{\theta\in\Theta}\|g_2(W,\theta)\|\right]\mathbb{E}_F\left[\sup_{\theta\in\Theta}\|g_{2,\theta}(W,\theta)\|\right]$ and $C_{M,3} < \infty$ by Assumption 3.2.(ii).

By the definition of $\Omega_{2,F}(\theta)$, the triangle inequality and the results in (C.6) and (C.8)

$$\|\Omega_{2,F}(\theta_1) - \Omega_{2,F}(\theta_2)\| \leq r_2(C_{M,2} + C_{M,3})\|\theta_1-\theta_2\|, \tag{C.9}$$

which immediately proves the claim in (iii). ∎

Proof of Lemma A.3. By Lemma A.1.(i),

$$\bar{g}_2(\theta) = M_{2,F_n}(\theta) + \left[n^{-1}\sum_{i=1}^n g_2(W_i,\theta) - M_{2,F_n}(\theta)\right] = M_{2,F_n}(\theta) + o_p(1), \tag{C.10}$$

uniformly over $\theta \in \Theta$. As $g_1(W, \theta)$ is a subvector of $g_2(W, \theta)$, by (C.10) and Assumption 3.2.(ii),

$$\overline{g}_1(\theta)'\overline{g}_1(\theta) = M_{1,F_n}(\theta)'M_{1,F_n}(\theta) + o_p(1) \tag{C.11}$$

uniformly over $\theta \in \Theta$. By Assumptions 3.1.(i)-(ii) and $F_n \in \mathcal{F}$, $M_{1,F_n}(\theta)'M_{1,F_n}(\theta)$ is uniquely minimized at θ_{F_n}, which together with the uniform convergence in (C.11) implies that

$$\widetilde{\theta}_1 - \theta_{F_n} \to_p 0. \tag{C.12}$$

To show the consistency of $\overline{\Omega}_2$, note that

$$
\begin{aligned}
\overline{\Omega}_2 &= n^{-1}\sum_{i=1}^{n} g_2(W_i, \widetilde{\theta}_1)g_2(W_i, \widetilde{\theta}_1)' - \overline{g}_2(\widetilde{\theta}_1)\overline{g}_2(\widetilde{\theta}_1)' \\
&= \mathbb{E}_{F_n}[g_2(W, \widetilde{\theta}_1)g_2(W, \widetilde{\theta}_1)'] - M_{2,F_n}(\widetilde{\theta}_1)'M_{2,F_n}(\widetilde{\theta}_1) + o_p(1) \\
&= \Omega_{2,F_n}(\widetilde{\theta}_1) + o_p(1) = \Omega_{2,F_n} + o_p(1),
\end{aligned}
\tag{C.13}
$$

where the first equality is by the definition of $\overline{\Omega}_2$, the second equality holds by (C.10), Lemma A.1.(ii) and Assumption 3.2.(ii), the third equality follows from the definition of $\Omega_{2,F_n}(\theta)$, and the last equality holds by Lemma A.2.(iii) and (C.12). This shows the consistency of $\overline{\Omega}_2$. ∎

In the rest of the Supplemental Appendix, we use C denote a generic fixed positive finite constant whose value does not depend on F or n.

Proof of Lemma A.4. As $\overline{g}_1(\theta)$ is a subvector of $\overline{g}_2(\theta)$, and $\overline{\Omega}_{1,n}$ is a submatrix of $\overline{\Omega}_{2,n}$, using (C.10), (C.13) and Assumptions 3.2.(ii)-(iii), we have

$$\overline{g}_1(\theta)'(\overline{\Omega}_1)^{-1}\overline{g}_1(\theta) = M_{1,F_n}(\theta)'\Omega_{1,F_n}^{-1}M_{1,F_n}(\theta) + o_p(1), \tag{C.14}$$

uniformly over Θ. By Assumptions 3.2.(ii)-(iii),

$$C^{-1} \leq \rho_{\min}(\Omega_{1,F_n}^{-1}) \leq \rho_{\max}(\Omega_{1,F_n}^{-1}) \leq C \tag{C.15}$$

which together with Assumptions 3.1.(i)-(ii) implies that $M_{1,F_n}(\theta)'\Omega_{1,F_n}^{-1}M_{1,F_n}(\theta)$ is uniquely minimized at θ_{F_n}. By the standard arguments for the consistency of an extremum estimator, we have

$$\widehat{\theta}_1 - \theta_{F_n} = o_p(1). \tag{C.16}$$

Using (C.16), Lemma A.1.(iv) and Assumption 3.2.(ii), we have

$$
\begin{aligned}
\overline{g}_1(\widehat{\theta}_1) &= \overline{g}_1(\theta_{F_n}) + \left[M_{1,F_n}(\widehat{\theta}_1) - M_{1,F_n}(\theta_{F_n})\right] + o_p(n^{-1/2}) \\
&= \overline{g}_1(\theta_{F_n}) + [G_{1,F_n}(\theta_{F_n}) + o_p(1)](\widehat{\theta}_1 - \theta_{F_n}) + o_p(n^{-1/2}).
\end{aligned}
\tag{C.17}
$$

Similarly,

$$n^{-1}\sum_{i=1}^{n} g_{1,\theta}(W_i,\widehat{\theta}_1) = G_{1,F_n}(\widehat{\theta}_1) + o_p(1) = G_{1,F_n} + o_p(1), \tag{C.18}$$

where the first equality follows from Lemma A.1.(iii) and the second equality follows by (C.16) and Lemma A.2.(ii). From the first order condition for the GMM estimator $\widehat{\theta}_1$, we deduce that

$$
\begin{aligned}
0 &= \left[n^{-1}\sum_{i=1}^{n} g_{1,\theta}(W_i,\widehat{\theta}_1) \right]' (\overline{\Omega}_1)^{-1}\overline{g}_1(\widehat{\theta}_1) \\
&= (G'_{1,F_n}\Omega_{1,F_n}^{-1} + o_p(1)) \left[\overline{g}_1(\theta_{F_n}) + (G_{1,F_n} + o_p(1))(\widehat{\theta}_1 - \theta_{F_n}) + o_p(n^{-1/2}) \right]
\end{aligned} \tag{C.19}
$$

where the second equality follows from Assumptions 3.2.(ii)-(iii), (C.13), (C.17) and (C.18). By (C.19), $\mathbb{E}_{F_n}[g_1(W,\theta_{F_n})] = 0$ and Assumption 3.2,

$$n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) = (\Gamma_{1,F_n} + o_p(1))\mu_n(g_1(W,\theta_{F_n})) + o_p(1). \tag{C.20}$$

By Assumptions 3.2 and Lemma A.1.(v), $\Gamma_{1,F_n} = O(1)$ and $\mu_n(g_1(W,\theta_{F_n})) = o_p(1)$, which together with (C.20) implies that

$$n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) = \Gamma_{1,F_n}\mu_n(g_1(W,\theta_{F_n})) + O_p(1),$$

where $\Gamma_{1,F_n}\mu_n(g_1(W,\theta_{F_n})) = O_p(1)$. This finishes the proof. ∎

Proof of Lemma A.5. By (C.10), (C.13) and Assumptions 3.2.(ii)-(iii), we have

$$\overline{g}_2(\theta)'(\overline{\Omega}_2)^{-1}\overline{g}_2(\theta) = M_{2,F_n}(\theta)'\Omega_{2,F_n}^{-1}M_{2,F_n}(\theta) + o_p(1) = Q_{F_n}(\theta) + o_p(1) \tag{C.21}$$

uniformly over Θ. By Assumption 3.1.(iii), $Q_{F_n}(\theta)$ is uniquely minimized at $\theta^*_{F_n}$. The consistency result $\widehat{\theta}_2 - \theta^*_{F_n} \to_p 0$ follows from standard arguments for the consistency of an extremum estimator. ∎

Proof of Lemma A.6. By the definition of $\widehat{\theta}_2$,

$$\overline{g}_2(\widehat{\theta}_2)'(\overline{\Omega}_2)^{-1}\overline{g}_2(\widehat{\theta}_2) \le \overline{g}_2(\theta_{F_n})'(\overline{\Omega}_2)^{-1}\overline{g}_2(\theta_{F_n}), \tag{C.22}$$

which implies that

$$\|\overline{g}_2(\widehat{\theta}_2)\|^2 \le \rho_{\max}(\overline{\Omega}_2)\rho_{\min}^{-1}(\overline{\Omega}_2)\|\overline{g}_2(\theta_{F_n})\|^2. \tag{C.23}$$

By (C.13) and Assumptions 3.2.(ii)-(iii),

$$C^{-1} \le \rho_{\min}(\overline{\Omega}_2) \le \rho_{\max}(\overline{\Omega}_2) \le C \tag{C.24}$$

with probability approaching 1. By Lemma A.1.(i), $M_{1,F_n}(\theta_{F_n}) = 0_{r_1 \times 1}$ and $\delta_{F_n} = o(1)$,

$$\|\bar{g}_2(\theta_{F_n})\|^2 = o_p(1) \tag{C.25}$$

which combined with (C.23) and (C.24) implies that

$$\|\bar{g}_2(\widehat{\theta}_2)\| = o_p(1). \tag{C.26}$$

Moreover, by (C.26), Lemma A.1.(i) and the triangle inequality,

$$\|M_{2,F_n}(\widehat{\theta}_2)\| \leq \|\bar{g}_2(\widehat{\theta}_2) - M_{2,F_n}(\widehat{\theta}_2)\| + \|\bar{g}_2(\widehat{\theta}_2)\| = o_p(1) \tag{C.27}$$

which immediately implies that

$$\|M_{1,F_n}(\widehat{\theta}_2)\| = o_p(1). \tag{C.28}$$

The first result in Lemma A.6 follows by (C.28) and the unique identification of θ_{F_n} maintained by Assumptions 3.1.(i)-(ii).

Using $\widehat{\theta}_2 - \theta_{F_n} = o_p(1)$, Lemma A.1.(iv) and Assumption 3.2.(ii), we have

$$\begin{aligned}
\bar{g}_2(\widehat{\theta}_2) &= \bar{g}_2(\theta_{F_n}) + \left[M_{2,F_n}(\widehat{\theta}_2) - M_{2,F_n}(\theta_{F_n})\right] + o_p(n^{-1/2}) \\
&= \bar{g}_2(\theta_{F_n}) + [G_{2,F_n}(\theta_{F_n}) + o_p(1)](\widehat{\theta}_2 - \theta_{F_n}) + o_p(n^{-1/2}).
\end{aligned} \tag{C.29}$$

Similarly,

$$n^{-1} \sum_{i=1}^{n} g_{2,\theta}(W_i, \widehat{\theta}_2) = G_{2,F_n}(\widehat{\theta}_2) + o_p(1) = G_{2,F_n}(\theta_{F_n}) + o_p(1), \tag{C.30}$$

where the first equality follows from Lemma A.1.(iii) and the second equality follows by $\widehat{\theta}_2 - \theta_{F_n} = o_p(1)$ and Lemma A.2.(ii). From the first order condition for the GMM estimator $\widehat{\theta}_2$, we deduce that

$$\begin{aligned}
0 &= \left[n^{-1} \sum_{i=1}^{n} g_{2,\theta}(W_i, \widehat{\theta}_2)\right]' (\overline{\Omega}_2)^{-1} \bar{g}_2(\widehat{\theta}_2) \\
&= (G'_{2,F_n} \Omega_{2,F_n}^{-1} + o_p(1)) \left[\bar{g}_2(\theta_{F_n}) + (G_{2,F_n} + o_p(1))(\widehat{\theta}_2 - \theta_{F_n}) + o_p(n^{-1/2})\right]
\end{aligned} \tag{C.31}$$

where the second equality follows from Assumptions 3.2.(ii)-(iii), (C.13), (C.29) and (C.30). By (C.31) and Assumption 3.2,

$$n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) = (\Gamma_{2,F_n} + o_p(1)) \left\{\mu_n(g_2(W, \theta_{F_n})) + n^{1/2}\mathbb{E}_{F_n}[g_2(W, \theta_{F_n})]\right\} + o_p(1), \tag{C.32}$$

where $\Gamma_{2,F_n} = -\left(G'_{2,F_n} \Omega_{2,F_n}^{-1} G_{2,F_n}\right)^{-1} G'_{2,F_n} \Omega_{2,F_n}^{-1}$. ∎

Proof for the claim in equation (4.3).

Consider the case $n^{1/2}\delta_{F_n} \to d \in \mathbb{R}^{r^*}$. By Lemma 4.1,

$$n^{1/2}\left[\widehat{\theta}(\omega) - \theta_{F_n}\right] = n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) + \omega\left[n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) - n^{1/2}(\widehat{\theta}_1 - \theta_{F_n})\right]$$

$$\to_D \Gamma_{1,F}^* \mathcal{Z}_{d,2,F} + \omega(\Gamma_{2,F} - \Gamma_{1,F}^*)\mathcal{Z}_{d,2,F}, \tag{C.33}$$

where $\mathcal{Z}_{d,2,F}$ has the same distribution as $\mathcal{Z}_{2,F} + d_0$. This implies that

$$\ell(\widehat{\theta}(\omega)) = n\left[\widehat{\theta}_n(\omega) - \theta_{F_n}\right]' \Upsilon \left[\widehat{\theta}_n(\omega) - \theta_{F_n}\right] \to_D \lambda_F(\omega) \tag{C.34}$$

where

$$\begin{aligned}
\lambda_F(\omega) &= \mathcal{Z}_{d,2,F}'\Gamma_{1,F}^{*\prime}\Upsilon\Gamma_{1,F}^*\mathcal{Z}_{d,2,F} + 2\omega\mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon\Gamma_{1,F}^*\mathcal{Z}_{d,2,F} \\
&\quad + \omega^2 \mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)\mathcal{Z}_{d,2,F}.
\end{aligned}$$

Now we consider $\mathbb{E}[\lambda_F(\omega)]$ using the equalities in Lemma A.9 below. First,

$$\mathbb{E}[\mathcal{Z}_{d,2,F}'\Gamma_{1,F}^{*\prime}\Upsilon\Gamma_{1,F}^*\mathcal{Z}_{d,2,F}] = \text{tr}(\Upsilon\Sigma_{1,F}) \tag{C.35}$$

because $\Gamma_1^*\mathcal{Z}_{d,2,F} = \Gamma_{1,F}\mathcal{Z}_{1,F}$ and $\Gamma_{1,F}\mathbb{E}[\mathcal{Z}_{1,F}\mathcal{Z}_{1,F}']\Gamma_{1,F}' = \Sigma_{1,F}$ by definition. Second,

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon\Gamma_{1,F}^*\mathcal{Z}_{d,2,F}\right] \\
&= \text{tr}(\Upsilon\Gamma_{1,F}^*\mathbb{E}\left[\mathcal{Z}_{d,2,F}\mathcal{Z}_{d,2,F}'\right](\Gamma_{2,F} - \Gamma_{1,F}^*)') \\
&= \text{tr}(\Upsilon\Gamma_{1,F}^*\left[d_0 d_0' + \Omega_{2,F}\right](\Gamma_{2,F} - \Gamma_{1,F}^*)') \\
&= \text{tr}(\Upsilon(\Sigma_{2,F} - \Sigma_{1,F})), \tag{C.36}
\end{aligned}$$

where the last equality holds by Lemma A.9. Third,

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{Z}_{d,2,F}'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)\mathcal{Z}_{d,2,F}\right] \\
&= \text{tr}(\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)\left[d_0 d_0' + \Omega_{2,F}\right](\Gamma_{2,F} - \Gamma_{1,F}^*)') \\
&= d_0'\Gamma_{2,F}'\Upsilon\Gamma_{2,F}d_0 - \text{tr}(\Upsilon(\Sigma_{2,F} - \Sigma_{1,F})) \tag{C.37}
\end{aligned}$$

by Lemma A.9. Combining the results in (C.35)-(C.37), we obtain

$$\begin{aligned}
\mathbb{E}[\lambda_F(\omega)] &= \text{tr}(\Upsilon\Sigma_{1,F}) - 2\omega\text{tr}(\Upsilon(\Sigma_{1,F} - \Sigma_{2,F})) \\
&\quad + \omega^2\left[d_0'\Gamma_{2,F}'\Upsilon\Gamma_{2,F}d_0 + tr(\Upsilon(\Sigma_{1,F} - \Sigma_{2,F}))\right]. \tag{C.38}
\end{aligned}$$

Note that $d_0'\Gamma_{2,F}'\Upsilon\Gamma_{2,F}d_0 = d_0'(\Gamma_{2,F} - \Gamma_{1,F}^*)'\Upsilon(\Gamma_{2,F} - \Gamma_{1,F}^*)d_0$ because $\Gamma_{1,F}^*d_0 = \mathbf{0}_{d_\theta}$. It is clear that the optimal weight ω_F^* in (4.3) minimizes the quadratic function of ω in (C.38). $\blacksquare$

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Proof of Lemma A.9. By construction, $\Gamma_{1,F}^{*} d_0 = \mathbf{0}_{d_\theta \times 1}$. For ease of notation, we write $\Omega_{2,F}$ and $G_{2,F}$ as

$$\Omega_{2,F} = \begin{pmatrix} \Omega_{1,F} & \Omega_{1r*} \\ \Omega_{r*1,F} & \Omega_{r*,F} \end{pmatrix} \text{ and } G_{2,F} = \begin{pmatrix} G_{1,F} \\ G_{r*,F} \end{pmatrix}. \tag{C.39}$$

To prove part (b), we have

$$\Gamma_{1,F}^{*} \Omega_{2,F} \Gamma_{1,F}^{*} = [\Gamma_{1,F}, \mathbf{0}_{d_\theta \times r*}] \begin{pmatrix} \Omega_{1,F} & \Omega_{1r*} \\ \Omega_{r*1,F} & \Omega_{r*,F} \end{pmatrix} [\Gamma_{1,F}, \mathbf{0}_{d_\theta \times r*}]$$

$$= \Gamma_{1,F} \Omega_{1,F} \Gamma_{1,F}' = \left(G_{1,F}' \Omega_{1,F}^{-1} G_{1,F} \right)^{-1} = \Sigma_{1,F}. \tag{C.40}$$

To show part (c), note that

$$\Gamma_{1,F}^{*} \Omega_{2,F} \Gamma_{2,F}' = - [\Gamma_{1,F}, \mathbf{0}_{d_\theta \times r*}] \Omega_{2,F} \Omega_{2,F}^{-1} G_{2,F} \left(G_{2,F}' \Omega_{2,F}^{-1} G_{2,F} \right)^{-1}$$

$$= -\Gamma_{1,F} G_{1,F} \left(G_{2,F}' \Omega_{2,F}^{-1} G_{2,F} \right)^{-1} = \left(G_{2,F}' \Omega_{2,F}^{-1} G_{2,F} \right)^{-1} = \Sigma_{2,F} \tag{C.41}$$

because $-\Gamma_{1,F} G_{1,F} = I_{d_\theta \times d_\theta}$. Part (d) follows from the definition of $\Gamma_{2,F}$. $\blacksquare$

Proof of Lemma 4.2. We first prove the consistency of $\widehat{\Omega}_k$, $\widehat{G}_k$ and $\widehat{\Sigma}_k$ for $k = 1, 2$. By Lemma 4.1, we have $\widehat{\theta}_1 = \theta_{F_n} + o_p(1)$. Using the same arguments in showing (C.13), we can show that

$$\widehat{\Omega}_2 = \Omega_{2,F_n} + o_p(1) = \Omega_{2,F} + o_p(1), \tag{C.42}$$

where the second equality is by the assumption of the lemma that $v_{F_n} \to v_F$ for some $F \in \mathcal{F}$. As $\widehat{\Omega}_1$ is a submatrix of $\widehat{\Omega}_2$, by (C.42) we have

$$\widehat{\Omega}_1 = \Omega_{1,F_n} + o_p(1) = \Omega_{1,F} + o_p(1). \tag{C.43}$$

By the consistency of $\widehat{\theta}_1$ and the same arguments used to show (C.30), we have

$$n^{-1} \sum_{i=1}^{n} g_{2,\theta}(W_i, \widehat{\theta}_1) = G_{2,F_n}(\theta_{F_n}) + o_p(1) = G_{2,F} + o_p(1), \tag{C.44}$$

where the second equality is by (A.10) which is assumed in the lemma. As $n^{-1} \sum_{i=1}^{n} g_{1,\theta}(W_i, \widehat{\theta}_1)$ is a submatrix of $n^{-1} \sum_{i=1}^{n} g_{2,\theta}(W_i, \widehat{\theta}_1)$, by (C.44) we have

$$n^{-1} \sum_{i=1}^{n} g_{1,\theta}(W_i, \widehat{\theta}_1) = G_{1,F_n}(\theta_{F_n}) + o_p(1) = G_{1,F} + o_p(1). \tag{C.45}$$

From Assumption 3.2, (C.42), (C.43), (C.44) and (C.45), we see that $\widehat{\Omega}_k$ and $\widehat{G}_k$ are consistent

estimators of $\Omega_{k,F}$ and $G_{k,F}$ respectively for $k = 1, 2$. By the Slutsky theorem and Assumption 3.2, we know that $\widehat{\Sigma}_k$ is a consistent estimator of $\Sigma_{k,F}$ for $k = 1, 2$.

In the case where $n^{1/2}\delta_{F_n} \to d \in \mathbb{R}^{r^*}$, the desired result follows from Lemma 4.1, the consistency of $\widehat{\Sigma}_{1,F}$ and $\widehat{\Sigma}_{2,F}$, and the CMT. In the case where $||n^{1/2}\delta_{F_n}|| \to \infty$, $\widetilde{\omega}_{eo} \to_p 0$ because $n^{1/2}||\widehat{\theta}_2 - \widehat{\theta}_1|| \to_p \infty$ and

$$
\begin{aligned}
n^{1/2}(\widehat{\theta}_{eo} - \theta_{F_n}) &= n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) + \widetilde{\omega}_{eo}n^{1/2}(\widehat{\theta}_2 - \widehat{\theta}_1) \\
&= n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) + \frac{n^{1/2}(\widehat{\theta}_2 - \widehat{\theta}_1)tr\left[\Upsilon(\widehat{\Sigma}_1 - \widehat{\Sigma}_2)\right]}{n(\widehat{\theta}_2 - \widehat{\theta}_1)'\Upsilon(\widehat{\theta}_2 - \widehat{\theta}_1) + tr\left[\Upsilon(\widehat{\Sigma}_1 - \widehat{\Sigma}_2)\right]} \to_D \xi_{1,F}
\end{aligned}
\tag{C.46}
$$

by Lemma 4.1. ∎

Proof of Lemma A.15. By definition,

$$
\xi'_{1,F}\Upsilon\xi_{1,F} = \mathcal{Z}'_{1,F}\Gamma'_{1,F}\Upsilon\Gamma_{1,F}\mathcal{Z}_{1,F} = \mathcal{Z}'_1\Omega^{1/2}_{1,F}\Gamma'_{1,F}\Upsilon\Gamma_{1,F}\Omega^{1/2}_{1,F}\mathcal{Z}_1
\tag{C.47}
$$

where $\mathcal{Z}_1 \sim N(\mathbf{0}_{r_1}, I_{r_1 \times r_1})$. By Assumptions 3.2.(ii) and 3.2.(iv), and the fact that Υ is a fixed matrix,

$$
\sup_{F \in \mathcal{F}} \rho_{\max}(\Omega^{1/2}_{1,F}\Gamma'_{1,F}\Upsilon\Gamma_{1,F}\Omega^{1/2}_{1,F}) \le C.
\tag{C.48}
$$

By (C.48),

$$
\sup_{h \in H} \mathbb{E}[(\xi'_{1,F}\Upsilon\xi_{1,F})^2] \le \sup_{h \in H} \rho^2_{\max}(\Omega^{1/2}_{1,F}\Gamma'_{1,F}\Upsilon\Gamma_{1,F}\Omega^{1/2}_{1,F})\mathbb{E}[(\mathcal{Z}'_1\mathcal{Z}_1)^2] \le 3r_1 C
\tag{C.49}
$$

where the second inequality is by $\mathbb{E}[(\mathcal{Z}'_1\mathcal{Z}_1)^2] \le 3r_1 + r_1(r_1 - 1) = r_1^2 + 2r_1$ which is implied by the assumption that $\mathcal{Z}_1$ is a r_1-dimensional standard normal random vector. The first inequality of this lemma follows as the upper bound in (C.49) does not depend on F.

For any $F \in \mathcal{F}$, define

$$
B_F \equiv (\Gamma_{2,F} - \Gamma^*_{1,F})'\Upsilon(\Gamma_{2,F} - \Gamma^*_{1,F}).
$$

By the Cauchy-Schwarz inequality and the simple inequality $|ab| \le (a^2 + b^2)/2$ (for any real numbers a and b),

$$
\begin{aligned}
\overline{\xi}'_F\Upsilon\overline{\xi}_F &\le 2\left(\mathcal{Z}'_{d,2,F}\Gamma^{*\prime}_{1,F}\Upsilon\Gamma^*_{1,F}\mathcal{Z}_{d,2,F} + \overline{\omega}^2_F\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F}\right) \\
&= 2\left(\mathcal{Z}'_{1,F}\Gamma'_{1,F}\Upsilon\Gamma_{1,F}\mathcal{Z}_{1,F} + \overline{\omega}^2_F\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F}\right)
\end{aligned}
\tag{C.50}
$$

where the equality is by $\Gamma^*_{1,F}d_0 = \mathbf{0}_{d_\theta \times 1}$ (which is proved in Lemma A.9). By (C.50) and the simple inequality $(a + b)^2 \le 2(a^2 + b^2)$ (for any real numbers a and b),

$$
(\overline{\xi}'_F\Upsilon\overline{\xi}_F)^2 \le 8(\mathcal{Z}'_{1,F}\Gamma'_{1,F}\Upsilon\Gamma_{1,F}\mathcal{Z}_{1,F})^2 + 8(\overline{\omega}^2_F\mathcal{Z}'_{d,2,F}B_F\mathcal{Z}_{d,2,F})^2.
\tag{C.51}
$$

By the first inequality of this lemma, we have $\sup_{h \in H} \mathbb{E}[(\xi'_{1,F}\Upsilon\xi_{1,F})^2] \le C$. Hence by (C.51), to

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show the second inequality of this lemma, it is sufficient to prove that

$$\sup_{h \in H} \mathbb{E}[(\overline{\omega}_F^2 \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F})^2] \leq C. \tag{C.52}$$

Recall that we have defined $A_F = \Upsilon (\Sigma_{1,F} - \Sigma_{2,F})$ in Theorem 5.2. By the definition,

$$
\begin{aligned}
\overline{\omega}_F^2 \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} &= \frac{(\mathrm{tr}(A_F))^2 \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F}}{(\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F))^2} \\
&= \mathrm{tr}(A_F) \frac{\mathrm{tr}(A_F)}{\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)} \frac{\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F}}{\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)}. \tag{C.53}
\end{aligned}
$$

By Lemma 2.1 in Cheng and Liao (2015), $\mathrm{tr}(A_F) \geq 0$ for any $F \in \mathcal{F}$. This together with $\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} \geq 0$ implies that

$$\frac{\mathrm{tr}(A_F)}{\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)} \leq 1 \text{ and } \frac{\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F}}{\mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} + \mathrm{tr}(A_F)} \leq 1. \tag{C.54}$$

By (C.54) and $\mathrm{tr}(A_F) \geq 0$,

$$\overline{\omega}_F^2 \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F} \leq \mathrm{tr}(A_F) = \mathrm{tr}(\Upsilon \Sigma_{1,F}) - \mathrm{tr}(\Upsilon \Sigma_{2,F}), \tag{C.55}$$

where the equality is by $A_F = \Upsilon(\Sigma_{1,F} - \Sigma_{2,F})$. By (C.55) and the simple inequality $(a + b)^2 \leq 2(a^2 + b^2)$,

$$\mathbb{E}[(\overline{\omega}_F^2 \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F})^2] \leq 2(\mathrm{tr}(\Upsilon \Sigma_{1,F}))^2 + 2(\mathrm{tr}(\Upsilon \Sigma_{2,F}))^2. \tag{C.56}$$

By Assumptions 3.2.(ii) and 3.2.(iv),

$$\rho_{\min}(G'_{k,F} \Omega_{k,F}^{-1} G_{k,F}) \geq \rho_{\min}(\Omega_{k,F}^{-1}) \rho_{\min}(G'_{k,F} G_{k,F}) = \rho_{\min}(G'_{k,F} G_{k,F})/\rho_{\max}(\Omega_{k,F}) \geq C^{-1} \tag{C.57}$$

for any $F \in \mathcal{F}$ and for $k = 1, 2$. By (C.57) and the definition of $\Sigma_{k,F}$ ($k = 1, 2$),

$$\rho_{\max}(\Sigma_{k,F}) = \rho_{\min}^{-1}(G'_{k,F} \Omega_{k,F}^{-1} G_{k,F}) \leq C \tag{C.58}$$

for any $F \in \mathcal{F}$. As Υ and $\Sigma_{k,F}$ are positive definite symmetric matrix, by the standard trace inequality ($tr(AB) \leq tr(A)\rho_{\max}(B)$ for Hermitian matrices $A \geq 0$ and $B \geq 0$),

$$\mathrm{tr}(\Upsilon \Sigma_{k,F}) \leq \mathrm{tr}(\Upsilon) \rho_{\max}(\Sigma_{k,F}) \leq C \text{ for } k = 1, 2, \tag{C.59}$$

for any $F \in \mathcal{F}$. Collecting the results in (C.56) and (C.59), we immediately get (C.52). This finishes the proof. ∎

Proof of Lemma A.16. First note that

$$\min\{x, \zeta\} - x = (\zeta - x)I\{x > \zeta\}. \tag{C.60}$$

Hence we have

$$
\begin{aligned}
&\sup_{h \in H} \left| \mathbb{E}\left[\min\{\bar{\xi}_F' \Upsilon \bar{\xi}_F, \zeta\} - \bar{\xi}_F' \Upsilon \bar{\xi}_F\right] \right| \\
\leq\ & \sup_{h \in H} \mathbb{E}\left[\left|\zeta - \bar{\xi}_F' \Upsilon \bar{\xi}_F\right| I\{\bar{\xi}_F' \Upsilon \bar{\xi}_F > \zeta\}\right] \\
\leq\ & \zeta \sup_{h \in H} \mathbb{E}\left[I\{\bar{\xi}_F' \Upsilon \bar{\xi}_F > \zeta\}\right] + \sup_{h \in H} \mathbb{E}\left[\bar{\xi}_F' \Upsilon \bar{\xi}_F I\{\zeta^{-1} > (\bar{\xi}_F' \Upsilon \bar{\xi}_F)^{-1}\}\right] \\
\leq\ & 2\zeta^{-1} \sup_{h \in H} \mathbb{E}\left[(\bar{\xi}_F' \Upsilon \bar{\xi}_F)^2\right] \leq 2C\zeta^{-1}
\end{aligned}
\tag{C.61}
$$

where the first inequality is by the Jensen's inequality, the second inequality is by the Markov inequality, the third inequality is by the monotonicity of expectation and the last inequality is by Lemma A.15. Using the same arguments, we can show that

$$\sup_{h \in H} \left| \mathbb{E}\left[\min\{\xi_{1,F}' \Upsilon \xi_{1,F}, \zeta\} - \xi_{1,F}' \Upsilon \xi_{1,F}\right] \right| \leq 2C\zeta^{-1}. \tag{C.62}$$

Collecting the results in (C.61) and (C.62), and applying the triangle inequality, we deduce that

$$\sup_{h \in H} \left[|g_\zeta(h) - g(h)|\right] \leq 4C\zeta^{-1}. \tag{C.63}$$

The claimed result of this lemma follows by (C.63) as C is a fixed constant.

By the triangle inequality, the Jensen's inequality and Lemma A.15,

$$
\begin{aligned}
\sup_{h \in H} |g(h)| &= \sup_{h \in H} \left| \mathbb{E}[\bar{\xi}_F' \Upsilon \bar{\xi}_F - \xi_{1,F}' \Upsilon \xi_{1,F}] \right| \\
&\leq \sup_{h \in H} \mathbb{E}[\bar{\xi}_F' \Upsilon \bar{\xi}_F] + \sup_{h \in H} \mathbb{E}[\xi_{1,F}' \Upsilon \xi_{1,F}] \leq C
\end{aligned}
$$

which finishes the proof of the lemma. ∎

D   Asymptotic Risk of the Pre-test GMM Estimator

In this section, we establish similar results in Theorem 5.1 for the pre-test GMM estimator based on the J-test statistic. The pre-test estimator is defined as

$$\widehat{\theta}_{pre} = 1\{J_n > c_\alpha\}\widehat{\theta}_1 + 1\{J_n \leq c_\alpha\}\widehat{\theta}_2, \tag{D.1}$$

where $J_n = n\bar{g}_2(\widehat{\theta}_2)'(\widehat{\Omega}_2)^{-1}\bar{g}_2(\widehat{\theta}_2)$ and c_α is the $100(1-\alpha)$th quantile of the chi-squared distribution with degree of freedom $r_2 - d_\theta$.

Theorem D.1 *Suppose that Assumptions 3.1-3.3 hold. The bounds of the asymptotic risk difference satisfy*

$$Asy\underline{RD}(\widehat{\theta}_{pre}, \widehat{\theta}_1) = \min\left\{\inf_{h \in H}\left[g_p(h)\right], 0\right\},$$

$$Asy\overline{RD}(\widehat{\theta}_{pre}, \widehat{\theta}_1) = \max\left\{\sup_{h \in H}\left[g_p(h)\right], 0\right\},$$

where $g_p(h) \equiv \mathbb{E}[\overline{\xi}'_{p,F}\Upsilon\overline{\xi}_{p,F} - \xi'_{1,F}\Upsilon\xi_{1,F}]$ and $\overline{\xi}_{p,F}$ is defined in (D.3) below.

Proof of Theorem D.1. The two equalities and inequalities in the theorem follow by the same arguments in the proof of Theorem 5.1 with Lemma 4.2 for $\widehat{\theta}_{eo}$ replaced by Lemma D.1 for $\widehat{\theta}_{pre}$, Lemma A.15 replaced by Lemma D.2, and Lemma A.16 replaced by Lemma D.3. Its proof is hence omitted. ∎

By Definition,

$$\begin{aligned}
\mathbb{E}[\overline{\xi}'_{p,F}\Upsilon\overline{\xi}_{p,F}] &= \mathbb{E}[\mathcal{Z}'_{d,2,F}\Gamma^{*\prime}_{1,F}\Upsilon\Gamma^*_{1,F}\mathcal{Z}_{d,2,F}] + 2\mathbb{E}[\overline{\omega}_{p,F}\mathcal{Z}'_{d,2,F}(\Gamma_{2,F} - \Gamma^*_{1,F})'\Upsilon\Gamma^*_{1,F}\mathcal{Z}_{d,2,F}] \\
&\quad + \mathbb{E}[\overline{\omega}^2_{p,F}\mathcal{Z}'_{d,2,F}(\Gamma_{2,F} - \Gamma^*_{1,F})'\Upsilon(\Gamma_{2,F} - \Gamma^*_{1,F})\mathcal{Z}_{d,2,F}] \\
&= tr(\Upsilon\Sigma_{1,F}) + 2\mathbb{E}[\overline{\omega}_{p,F}\mathcal{Z}'_{d,2,F}(\Gamma_{2,F} - \Gamma^*_{1,F})'\Upsilon\Gamma^*_{1,F}\mathcal{Z}_{d,2,F}] \\
&\quad + \mathbb{E}[\overline{\omega}^2_{p,F}\mathcal{Z}'_{d,2,F}(\Gamma_{2,F} - \Gamma^*_{1,F})'\Upsilon(\Gamma_{2,F} - \Gamma^*_{1,F})\mathcal{Z}_{d,2,F}]
\end{aligned} \tag{D.2}$$

The asymptotic risk of the pre-test estimator $\widehat{\theta}_p$ in Figure 2 is simulated based on the formula in (D.2).

The following lemma provides the asymptotic distribution of the pre-test GMM estimator under various sequence of DGPs, which is used to show Theorem D.1.

Lemma D.1 *Suppose that Assumptions 3.1-3.3 hold. Consider $\{F_n\}$ such that $v_{F_n} \to v_F$ for some $F \in \mathcal{F}$.*
(a) If $n^{1/2}\delta_{F_n} \to d$ for some $d \in \mathbb{R}^{r^*}$, then

$$J_n \to_D J_\infty(h_{d,F}) \equiv (\mathcal{Z}_{2,F} + d_0)' L_F(\mathcal{Z}_{2,F} + d_0),$$

where $L_F \equiv \Omega^{-1}_{2,F} - \Omega^{-1}_{2,F}G_{2,F}\left(G'_{2,F}\Omega^{-1}_{2,F}G_{2,F}\right)^{-1}G'_{2,F}\Omega^{-1}_{2,F}$ and $d_0 = (0_{1\times r_1}, d')'$, and

$$n^{1/2}(\widehat{\theta}_{pre} - \theta_{F_n}) \to_D \overline{\xi}_{p,F} \equiv (1 - \overline{\omega}_{p,F})\xi_{1,F} + \overline{\omega}_{p,F}\xi_{2,F} \tag{D.3}$$

where $\overline{\omega}_{p,F} = 1\{J_\infty(h_{d,F}) \le c_\alpha\}$.
(b) If $\|n^{1/2}\delta_{F_n}\| \to \infty$, then $\overline{\omega}_{p,F} \to_p 0$ and $n^{1/2}(\widehat{\theta}_{pre} - \theta_{F_n}) \to_D \xi_{1,F}$.

Proof of Lemma D.1. (a) By Assumption 3.2.(ii), (C.29) and (C.32),

$$
\begin{aligned}
\bar{g}_2(\widehat{\theta}_2) &= \bar{g}_2(\theta_{F_n}) + [G_{2,F_n}(\theta_{F_n}) + o_p(1)](\widehat{\theta}_2 - \theta_{F_n}) + o_p(n^{-1/2}) \\
&= \bar{g}_2(\theta_{F_n}) + G_{2,F_n}\Gamma_{2,F_n}\bar{g}_2(\theta_{F_n}) + o_p(n^{-1/2}) \\
&= (I_{r_2} + G_{2,F_n}\Gamma_{2,F_n})\bar{g}_2(\theta_{F_n}) + o_p(n^{-1/2}),
\end{aligned}
\tag{D.4}
$$

which implies that

$$
J_n = n\bar{g}_2(\theta_{F_n})'L_{F_n}\bar{g}_2(\theta_{F_n}) + o_p(1)
\tag{D.5}
$$

where $L_{F_n} \equiv \Omega_{2,F_n}^{-1} - \Omega_{2,F_n}^{-1}G_{2,F_n}(G_{2,F_n}'\Omega_{2,F_n}^{-1}G_{2,F_n})^{-1}G_{2,F_n}'\Omega_{2,F_n}^{-1}$.
   By $n^{1/2}\delta_{F_n} \to d$ and Lemma A.1.(v),

$$
n^{1/2}\Omega_{2,F_n}^{-1/2}\bar{g}_2(\theta_{F_n}) = \Omega_{2,F_n}^{-1/2}\mu_n(g_2(W,\theta_{F_n})) + \Omega_{2,F_n}^{-1/2}n^{1/2}\delta_{F_n} \to_D \mathcal{Z} + \Omega_{2,F}^{-1/2}d_0
\tag{D.6}
$$

where $d_0' = (0_{1\times r_1}, d')$ and $\mathcal{Z}$ is a $r_2 \times 1$ standard normal random vector. By $v_{F_n} \to v_F$, (D.5), (D.6) and the CMT,

$$
J_n \to_D (\mathcal{Z}_{2,F} + d_0)'L_F(\mathcal{Z}_{2,F} + d_0).
\tag{D.7}
$$

Recall that Lemma 4.1.(a) implies that

$$
n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) \to_D \xi_{1,F} \text{ and } n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \to_D \xi_{2,F},
\tag{D.8}
$$

which together with (D.7) and the CMT implies that

$$
\begin{aligned}
n^{1/2}(\widehat{\theta}_{pre} - \theta_{F_n}) &= 1\{J_n > c_\alpha\}n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) + 1\{J_n \le c_\alpha\}n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \\
&\to_D (1 - \bar{\omega}_{p,F})\xi_{1,F} + \bar{\omega}_{p,F}\xi_{2,F},
\end{aligned}
\tag{D.9}
$$

which finishes the proof of the claim in (a).
   (b) There are two cases to consider: (i) $\|\delta_{F_n}\| > C^{-1}$; and (ii) $\|\delta_{F_n}\| \to 0$. We first consider case (i). As $\bar{g}_1(\widehat{\theta}_2)$ is a subvector of $\bar{g}_2(\widehat{\theta}_2)$,

$$
\begin{aligned}
J_n &= n\bar{g}_2(\widehat{\theta}_2)'(\widehat{\Omega}_2)^{-1}\bar{g}_2(\widehat{\theta}_2) \\
&\ge n\rho_{\max}^{-1}(\widehat{\Omega}_2)\bar{g}_2(\widehat{\theta}_2)'\bar{g}_2(\widehat{\theta}_2) \\
&\ge n\rho_{\max}^{-1}(\widehat{\Omega}_2)\bar{g}_1(\widehat{\theta}_2)'\bar{g}_1(\widehat{\theta}_2).
\end{aligned}
\tag{D.10}
$$

By (A.22) and (A.23) in the Appendix of CLS

$$
\left\|\widehat{\theta}_2 - \theta_{F_n}\right\| \ge C^{-1} \text{ with probability approaching 1,}
\tag{D.11}
$$

which together with Assumption 3.1.(ii) and Lemma A.1.(i) implies that

$$
\bar{g}_1(\widehat{\theta}_2) = M_{1,F}(\widehat{\theta}_2) + o_p(1) \ge C
\tag{D.12}
$$

with probability approaching 1. By (C.42) and Assumption 3.2.(ii), we have

$$\rho_{\max}(\widehat{\Omega}_2) \le C \text{ with probability approaching 1.} \tag{D.13}$$

Combining the results in (D.10), (D.12) and (D.13), we deduce that

$$J_n \ge nC^{-1} \text{ with probability approaching 1,} \tag{D.14}$$

which immediately implies that

$$\overline{\omega}_{p,F} = 1\{J_n \le c_\alpha\} = 0 \tag{D.15}$$

with probability approaching 1, as c_α is a fixed constant. By Lemma 4.1.(b), (D.15) and the assumption that Θ is bounded, we have

$$
\begin{aligned}
n^{1/2}(\widehat{\theta}_{pre} - \theta_{F_n}) &= 1\{J_n > c_\alpha\}n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) + 1\{J_n \le c_\alpha\}n^{1/2}(\widehat{\theta}_2 - \theta_{F_n}) \\
&= 1\{J_n > c_\alpha\}n^{1/2}(\widehat{\theta}_1 - \theta_{F_n}) + o_p(1) \to_D \xi_{1,F} \tag{D.16}
\end{aligned}
$$

where the convergence in distribution is by the CMT.

We next consider the case that $||\delta_{F_n}|| \to 0$ and $||n^{1/2}\delta_{F_n}|| \to \infty$. In the proof of Lemma 4.1, we have shown that $\widehat{\theta}_2 - \theta_{F_n} = o_p(1)$, and that (D.4) and (D.5) hold in this case. It is clear that

$$n^{1/2}\overline{g}_2(\theta_{F_n}) = \mu_n(g_2(W, \theta_{F_n}) + \begin{pmatrix} 0_{r_1 \times 1} \\ n^{1/2}\delta_{F_n} \end{pmatrix} \tag{D.17}$$

which implies that

$$
\begin{aligned}
n\overline{g}_2(\theta_{F_n})'L_{F_n}\overline{g}_2(\theta_{F_n}) &= [\mu_n(g_2(W, \theta_{F_n})]'L_{F_n}[\mu_n(g_2(W, \theta_{F_n})] \\
&\quad + 2\begin{pmatrix} 0_{1 \times r_1} & n^{1/2}\delta'_{F_n} \end{pmatrix}L_{F_n}[\mu_n(g_2(W, \theta_{F_n})] \\
&\quad + \begin{pmatrix} 0_{1 \times r_1} & n^{1/2}\delta'_{F_n} \end{pmatrix}L_{F_n}\begin{pmatrix} 0_{1 \times r_1} & n^{1/2}\delta'_{F_n} \end{pmatrix}'. \tag{D.18}
\end{aligned}
$$

By Lemma A.1.(v) and Assumptions 3.2.(ii)-(iii),

$$[\mu_n(g_2(W, \theta_{F_n})]'L_{F_n}[\mu_n(g_2(W, \theta_{F_n})] = O_p(1). \tag{D.19}$$

In order to bound the third term in (D.18) from below, we shall show that for any $d_0 = (0_{1 \times r_1}, d')'$ for $d \in \mathbb{R}^{r*}$ with $||d|| = 1$,

$$d_0'L_{F_n}d_0 \ge C^{-1} \tag{D.20}$$

By definition, L_{F_n} has d_θ many zero eigenvalues and $r_2 - d_\theta$ many of eigenvalues of ones. The matrix G_{2,F_n} contains the d_θ many eigenvectors of the zero eigenvalues of L_{F_n}, because

$$L_{F_n}G_{2,F_n} = 0_{r_2 \times d_\theta} \text{ and } \rho_{\min}(G'_{2,F_n}G_{2,F_n}) \ge C^{-1}. \tag{D.21}$$

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Let $G_{\perp,F_n}$ denote the orthogonal complement of G_{2,F_n} with $G'_{\perp,F_n} G_{\perp,F_n} = I_{r_2 - d_\theta}$. Then we have

$$
\begin{pmatrix} G_{1,F_n} \\ G_{r^*,F_n} \end{pmatrix} a_1 + \begin{pmatrix} G_{1,\perp,F_n} \\ G_{r^*,\perp,F_n} \end{pmatrix} a_2 = \begin{pmatrix} 0_{r_1 \times 1} \\ d \end{pmatrix} \tag{D.22}
$$

for some constant vectors $a_1 \in \mathbb{R}^{d_\theta}$ and $a_2 \in \mathbb{R}^{r_2 - d_\theta}$. As $\rho_{\min}(G'_{1,F_n} G_{1,F_n}) \geq C^{-1}$ by Assumption 3.2, we have

$$
a_1 = -(G'_{1,F_n} G_{1,F_n})^{-1} G'_{1,F_n} G_{1,\perp,F_n} a_2 \tag{D.23}
$$

and

$$
(G_{r^*,\perp,F_n} - G_{r^*,F_n}(G'_{1,F_n} G_{1,F_n})^{-1} G'_{1,F_n} G_{1,\perp,F_n}) a_2 = d. \tag{D.24}
$$

Let $H_{F_n} = G_{r^*,\perp,F_n} - G_{r^*,F_n}(G'_{1,F_n} G_{1,F_n})^{-1} G'_{1,F_n} G_{1,\perp,F_n}$. By $\rho_{\min}(G'_{1,F_n} G_{1,F_n}) \geq C^{-1}$, Assumptions 3.2.(ii), (D.24) and the Cauchy-Schwarz inequality,

$$
\|d\|^2 = a'_2 H_{F_n} H'_{F_n} a_2 \leq C \|a_2\|^2 \tag{D.25}
$$

which together with $\|d\| = 1$ implies that

$$
\|a_2\|^2 \geq C^{-1}. \tag{D.26}
$$

Using (D.21), (D.22) and (D.26), we deduce that

$$
\begin{aligned}
d'_0 L_{F_n} d_0 &= (G_{2,F_n} a_1 + G_{\perp,F_n} a_2)' L_{F_n} (G_{2,F_n} a_1 + G_{\perp,F_n} a_2) \\
&= a'_2 G'_{\perp,F_n} L_{F_n} G_{\perp,F_n} a_2 = \|a_2\|^2 \geq C^{-1}
\end{aligned} \tag{D.27}
$$

which proves (D.20). By (D.20),

$$
\begin{pmatrix} 0_{1 \times r_1} & n^{1/2} \delta'_{F_n} \end{pmatrix} L_{F_n} \begin{pmatrix} 0_{1 \times r_1} & n^{1/2} \delta'_{F_n} \end{pmatrix}' \geq C^{-1} n \| \delta_{F_n} \|^2 \tag{D.28}
$$

which together with $n \| \delta_{F_n} \|^2 \to \infty$ implies that

$$
\begin{pmatrix} 0_{1 \times r_1} & n^{1/2} \delta'_{F_n} \end{pmatrix} L_{F_n} \begin{pmatrix} 0_{1 \times r_1} & n^{1/2} \delta'_{F_n} \end{pmatrix}' \to \infty. \tag{D.29}
$$

Collecting the results in (D.18), (D.19) and (D.19), and by the Cauchy-Schwarz inequality, we deduce that $n \bar{g}_2(\theta_{F_n})' L_{F_n} \bar{g}_2(\theta_{F_n}) \to_p \infty$, which together with (D.5) implies that

$$
J_n \to_p \infty. \tag{D.30}
$$

Using the same arguments in showing (D.16), we deduce that

$$
n^{1/2}(\widehat{\theta}_{pre} - \theta_{F_n}) \to_D \xi_{1,F}. \tag{D.31}
$$

This finishes the proof. ∎

Lemma D.2 *Under Assumptions 3.2, we have*

$$\sup_{h \in H} \mathbb{E}[(\overline{\xi}'_{p,F} \Upsilon \overline{\xi}_{p,F})^2] \leq C. \tag{D.32}$$

Proof of Lemma D.2. By the same arguments in showing (C.51), we have

$$(\overline{\xi}'_{p,F} \Upsilon \overline{\xi}_{p,F})^2 \leq 8(\mathcal{Z}'_{1,F} \Gamma'_{1,F} \Upsilon \Gamma_{1,F} \mathcal{Z}_{1,F})^2 + 8(\overline{\omega}^2_{p,F} \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F})^2. \tag{D.33}$$

By the first inequality in (A.58) in the Appendix of CLS, we have $\sup_{h \in H} \mathbb{E}[(\xi'_{1,F} \Upsilon \xi_{1,F})^2] \leq C$. Hence by (D.33), to show the inequality in (D.32), it is sufficient to prove that

$$\sup_{h \in H} \mathbb{E}[(\overline{\omega}^2_{p,F} \mathcal{Z}'_{d,2,F} B_F \mathcal{Z}_{d,2,F})^2] \leq C. \tag{D.34}$$

By definition,

$$\overline{\omega}_{p,F} = I\{J_\infty(h_{d,F}) \leq c_\alpha\} = I\{\mathcal{Z}'_{d,2,F} L_F \mathcal{Z}_{d,2,F} \leq c_\alpha\}. \tag{D.35}$$

By the simple inequality $(a+b)^2 \geq a^2/2 - 2b^2$,

$$(z + d_0)' L_F (z + d_0) \geq d'_0 L_F d_0 / 2 - 2z' L_F z \tag{D.36}$$

for any $z \in \mathbb{R}$, which together with Assumption 3.2 and (D.20) implies that

$$(z + d_0)' L_F (z + d_0) \geq \|d\|^2 / C - 2z' L_F z \geq \|d\|^2 / C - C \|z\|^2. \tag{D.37}$$

Under Assumption 3.2, $\|B_F\| \leq C$ for any $F \in \mathcal{F}$ which together with the simple inequality $(a+b)^2 \leq 2(a^2 + b^2)$ implies that

$$(z + d_0)' B_F (z + d_0) \leq 2C(\|d\|^2 + \|z\|^2) \tag{D.38}$$

for any $z \in \mathbb{R}$. Collecting the results in (D.36) and (D.38), we get

$$
\begin{aligned}
& I\{(z + d_0)' L_F (z + d_0) \leq c_\alpha\} z' B_F z \\
\leq \ & 2CI\{\|d\|^2 \leq c_\alpha C + C^2 \|z\|^2\}(\|d\|^2 + \|z\|^2) \\
\leq \ & 2C(c_\alpha C + (C^2 + 1) \|z\|^2)
\end{aligned} \tag{D.39}
$$

which implies that

$$
\begin{aligned}
&\sup_{h\in H} \mathbb{E}[(\overline{\omega}_{p,F}^2 \mathcal{Z}_{d,2,F}' B_F \mathcal{Z}_{d,2,F})^2] \\
&\leq \quad 4C^2 \mathbb{E}[(c_\alpha C + (C^2+1)\mathcal{Z}_{2,F}' \mathcal{Z}_{2,F})^2] \\
&\leq \quad C(c_\alpha + \mathbb{E}[(\mathcal{Z}_{2,F}' \mathcal{Z}_{2,F})^2]) = C(c_\alpha + 3\rho_{\max}^2(\Omega_2)r_2).
\end{aligned}
\tag{D.40}
$$

This finishes the proof. $\blacksquare$

Lemma D.3 *Let* $g_{p,\zeta}(h) \equiv \mathbb{E}\left[\min\{\overline{\xi}_{p,F}' \Upsilon \overline{\xi}_{p,F}, \zeta\} - \min\{\xi_{1,F}' \Upsilon \xi_{1,F}, \zeta\}\right].$ *Under Assumptions 3.2, we have*

$$
\lim_{\zeta\to\infty} \sup_{h\in H} [|g_{p,\zeta}(h) - g_p(h)|] = 0
\tag{D.41}
$$

where $\sup_{h\in H} [|g_p(h)|] \leq C.$

Proof of Lemma D.3. The proof follows the same arguments of the proof of Lemma A.16 with the second inequality in (A.58) in the Appendix of CLS replaced by (D.32). $\blacksquare$

E   Extra Simulation Studies

E.1   Finite sample bias and variance of the GMM estimators in Section 7 of CLS

In this subsection, we report the finite sample biases and variances of the pre-test and averaging GMM estimators in the simulation design in Section 7 of CLS.

Suppose $\widehat{\theta}_n$ is a generic estimator of θ_0 with sample size n. The mean square error (MSE) of the estimator $\widehat{\theta}_n$ is

$$
MSE_n = E\left[\left\|\widehat{\theta}_n - \theta_0\right\|^2\right],
$$

which is a measure of the distance between the estimator and the true value. The bias and variance of $\widehat{\theta}_n$ are defined as

$$
B_n = E\left[\left\|E[\widehat{\theta}_n] - \theta_0\right\|^2\right] \text{ and } Var_n = E\left[\left\|\widehat{\theta}_n - E[\widehat{\theta}_n]\right\|^2\right].
$$

Then it is easy to see that

$$
MSE_n = B_n + Var_n.
$$

The MSEs of the pre-test and averaging GMM estimators are reported in Figure 2 of CLS. The finite sample biases B_n and variances Var_n of these estimator (together with the conservative GMM estimator $\widehat{\theta}_{1,n}$) are reported in Figure E.1 and Figure E.2 respectively.

In Figure E.1 and Figure E.2, the blue area, light blue area and the red line represent the pre-test GMM estimator, the averaging GMM estimator and the conservative GMM estimator respectively.

For each shaded area, the upper and lower envelopes are the maximum and minimum bias/variance of the GMM estimator respectively. For different values of r_0, the maximum and minimum bias (or variance) of the conservative GMM estimator are very close to each other. Therefore, we only present its minimum bias (and variance) in Figure E.1 (and Figure E.2 respectively).

From Figure E.1, we see the that the pre-test GMM estimator and the averaging GMM estimator have larger finite sample biases than the conservative GMM estimator. As sample size increases, the finite sample biases of the averaging and the conservative GMM estimators go to zero fast. However, the finite sample bias of the pre-test GMM estimator goes to zero very slowly which leads to larger finite sample MSE than the conservative GMM estimator as we can see from Figure 2 in Section 7 of CLS. From Figure E.2, we see that the pre-test GMM estimator has smaller finite sample variance than the conservative GMM estimator when the sample size is small (e.g., $n = 50$, 100), but it may have slightly larger variance in some area of r_0 when the sample size is slightly larger (e.g., $n = 250$, 500). In contrast, the averaging GMM estimator has smaller finite sample variances than the conservative GMM estimator for all sample sizes.

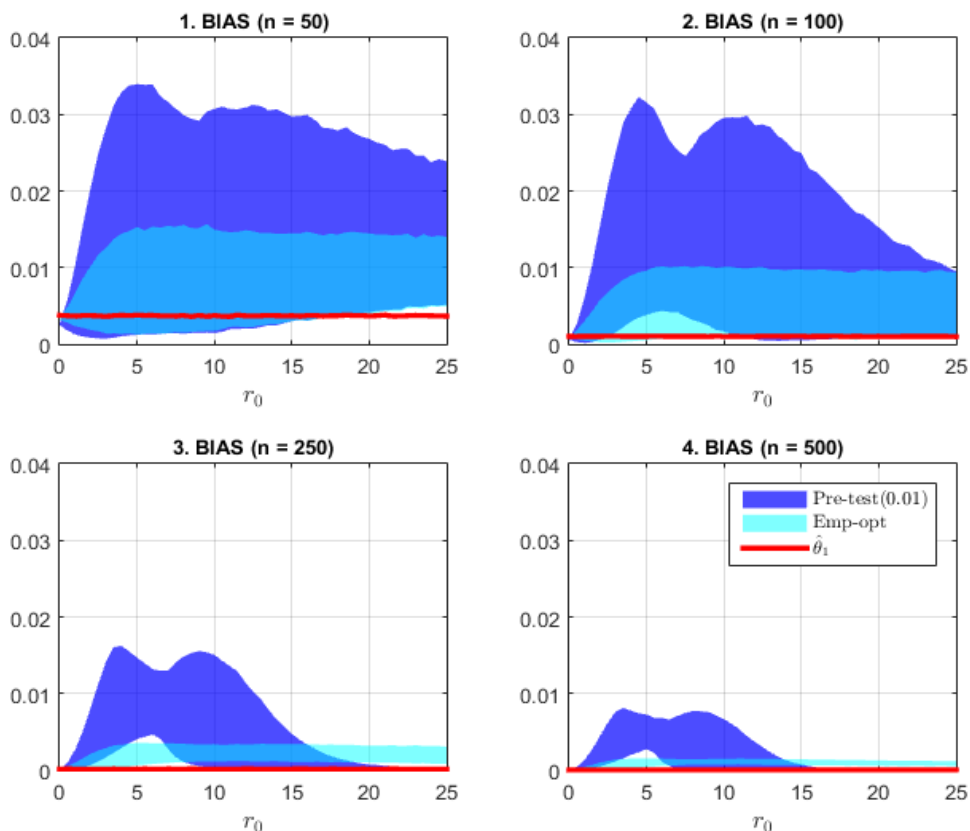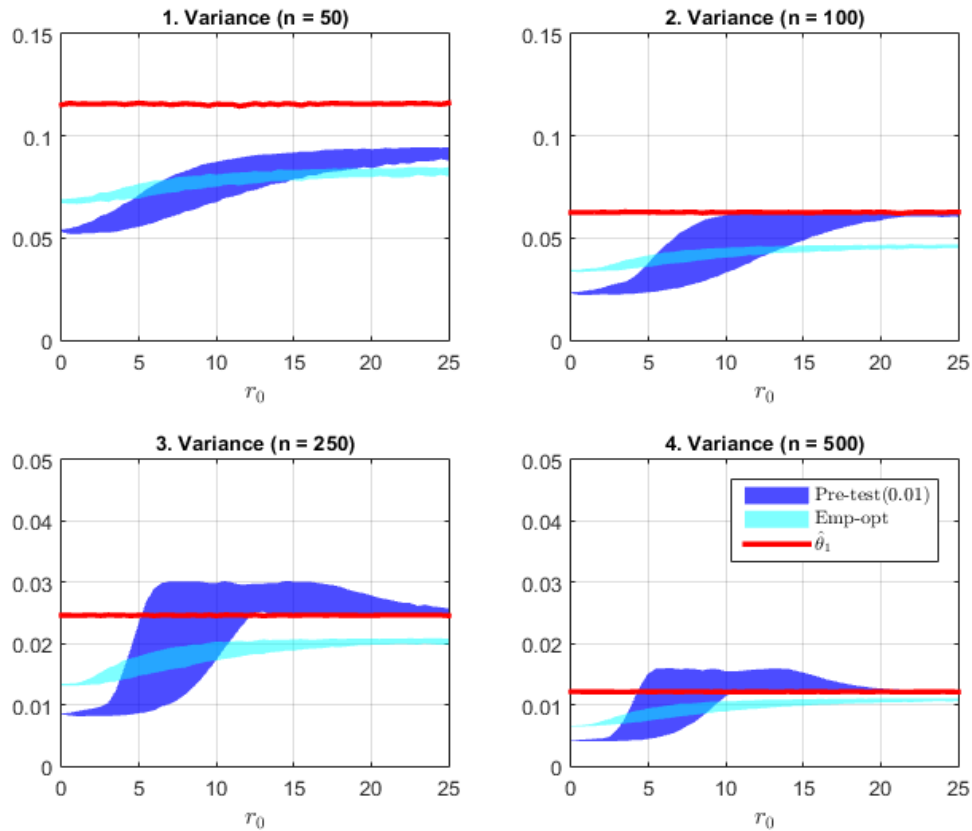Figure E.1. Finite Sample Biases of the Pre-test and Averaging GMM Estimators

Figure E.2. Finite Sample Variances of the Pre-test and Averaging GMM Estimators

E.2   Simulation under the normal distribution

In this subsection, we report the simulation results on the finite sample properties of the pre-test and averaging GMM estimators, when the residual term u in the structure equation (7.4) of CLS follows normal distribution, that is, (7.7) in CLS is replaced by

$$u = u^*.$$

In this simulation study, we consider r_0 to be a scalar that takes values on the grid points between 0 and 20 with the grid length 0.5. The rest of the simulation design is the same as what is used in Section 7 of CLS.

The simulation results are reported in Figure E.3 - E.5. For different values of r_0, the maximum and minimum bias (or variance) of the conservative GMM estimator are very close to each other. Therefore, we only present its minimum bias (and variance) in Figure E.4 (and Figure E.5 respectively). As we can see from Figure E.3 - E.5, the simulation results are quite similar to what we get in the simulation design of Section 7 of CLS.

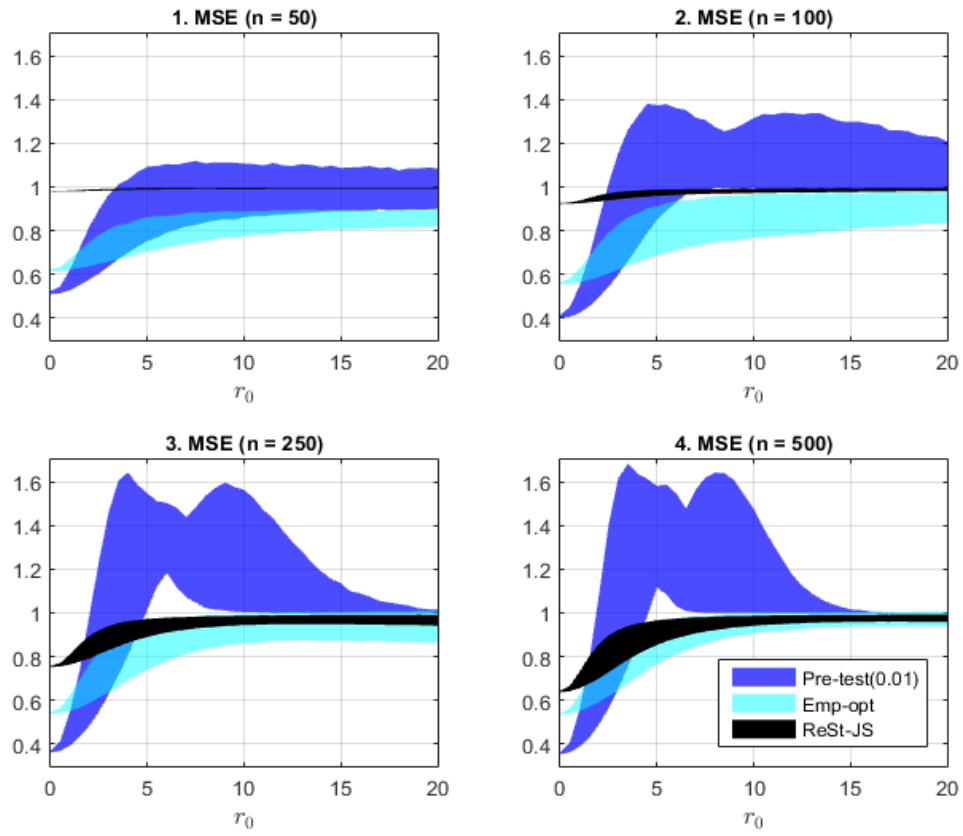Figure E.3. Finite Sample MSEs of the Pre-test and Averaging GMM Estimators

Figure E.4. Finite Sample Biases of the Pre-test and Averaging GMM Estimators
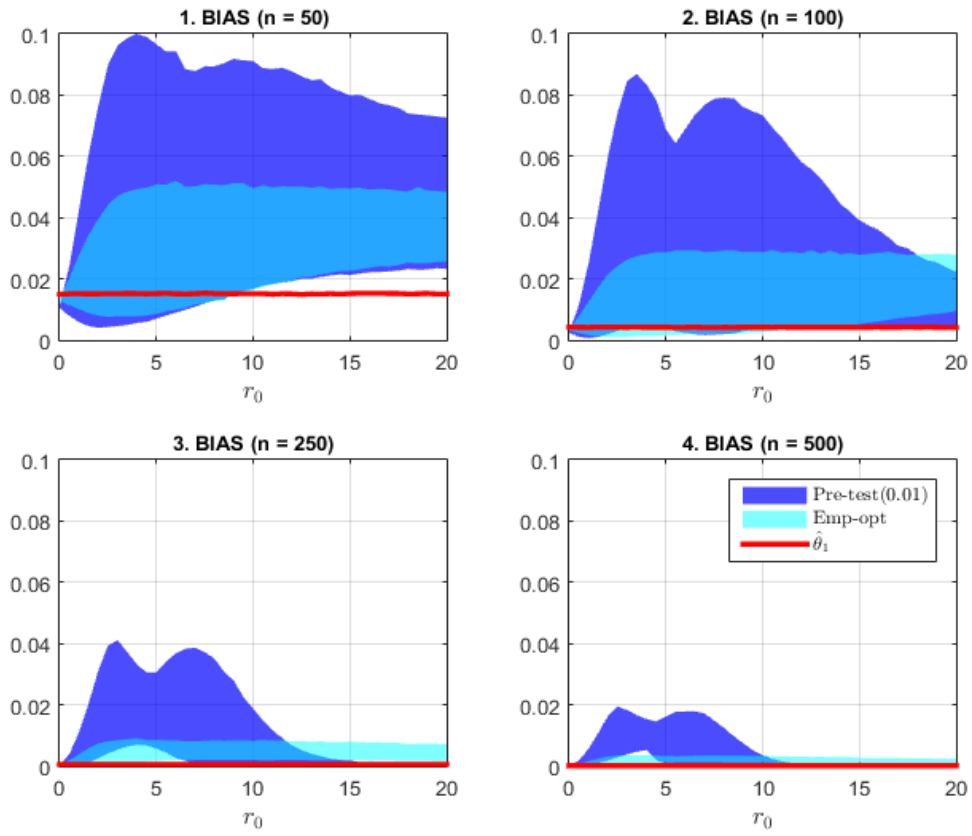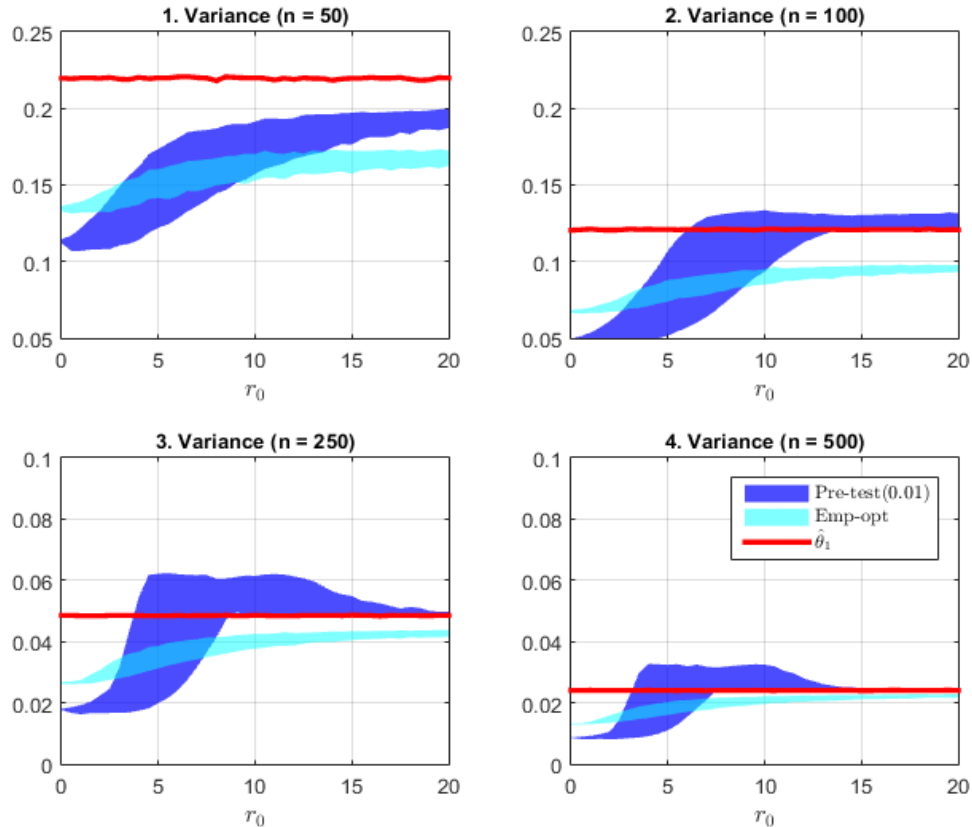
Figure E.5. Finite Sample Variances of the Pre-test and Averaging GMM Estimators

E.3 Simulation under the Student-t distribution

In this subsection, we report the simulation results on the finite sample properties of the pre-test and averaging GMM estimators, when the residual term u in the structure equation (7.4) of CLS is a student-t random variable with degree of freedom 2. We draw the latent random variables $(Z_1, \ldots, Z_{18}, \varepsilon_1, \ldots, \varepsilon_6, u^*)$ from the student-t distribution with degree of freedom 2 and correlation matrix $\text{diag}(I_{18 \times 18}, \Sigma_{7 \times 7})$ (where $\Sigma_{7 \times 7}$ is defined in (7.6) of CLS), and then replace (7.7) in CLS by $u = u^*$.[23] In this simulation study, we consider r_0 to be a scalar that takes values on the grid points between 0 and 20 with the grid length 0.5. The rest of the simulation design is the same as what is used in Section 7 of CLS.

The finite sample MSEs, biases and variances of the GMM estimators are reported in Figures

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[23]We use the matlab to perform all the simulation studies in this paper. Let $C_\Omega \equiv \text{diag}(I_{18 \times 18}, \Sigma_{7 \times 7})$, the matlab function mvtrnd(C_Ω, 2, n) will draw n such student-t random vectors independently. Each vector is constructed in the following way. First, a random vector, say $(\mathcal{N}_1, \ldots, \mathcal{N}_{25})$, will be simulated from the joint normal distribution with mean zero and variance-covariance matrix C_Ω. Then a vector of Chi-square random variables with degree of freedom 2, say $(\chi_1^2(2), \ldots, \chi_{25}^2(2))$, will be simulated independently. The Chi-square random variables are independent with each other. The student-t random vector is defined as

$$(t_1(2), \ldots, t_{25}(2)) = \left(\mathcal{N}_1/(\chi_1^2(2)/2)^{1/2}, \ldots, \mathcal{N}_{25}/(\chi_1^2(25)/2)^{1/2} \right).$$

E.6 - E.8. For different values of r_0, the maximum and minimum MSE of the restricted JS estimator in this simulation study are very close to each other. Therefore, we only present its minimum MSE in Figure E.6. For different values of r_0, the maximum and minimum bias (or variance) of the conservative GMM estimator in this simulation study are different from each other. Therefore, we present both its minimum and maximum bias (and minimum and maximum variance) in Figure E.7 (and Figure E.8 respectively).

It is interesting to see that in this simulation design, both the pre-test GMM estimator and the averaging GMM estimator has smaller finite sample MSEs than the conservative GMM estimator. The main reason for this phenomenon is that the residual term u in the structural equation is Student-t distributed with degree of freedom 2, which implies that u has infinite variance and hence the conservative GMM estimator has large variance in finite samples. This can be clearly seen from Figure E.8. When the extra IVs Z_j^* $(j = 1, \ldots, 6)$ are used in the GMM estimation, the finite sample variances of the GMM estimator is greatly reduced. Therefore, the finite samples biases of the pre-testing estimator and the averaging GMM estimator introduced by the extra IVs Z_j^* $(j = 1, \ldots, 6)$ are more than offset by the reduced finite sample variances, which enables both estimator have smaller finite sample MSEs.

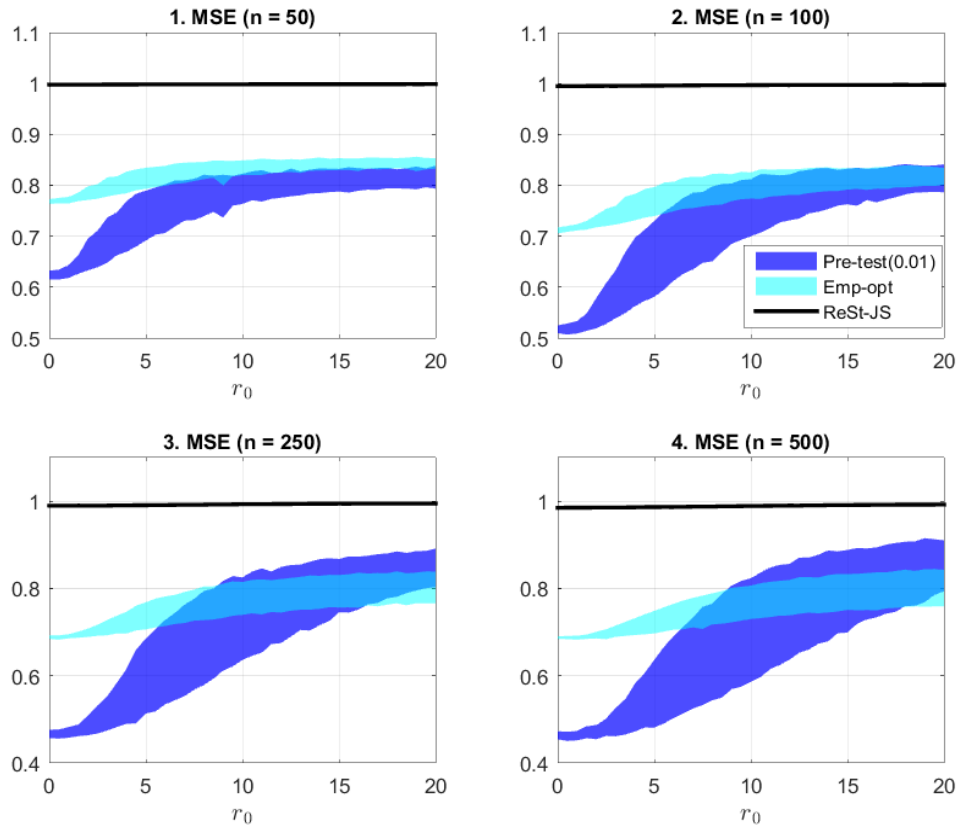Figure E.6. Finite Sample MSEs of the Pre-test and Averaging GMM Estimators

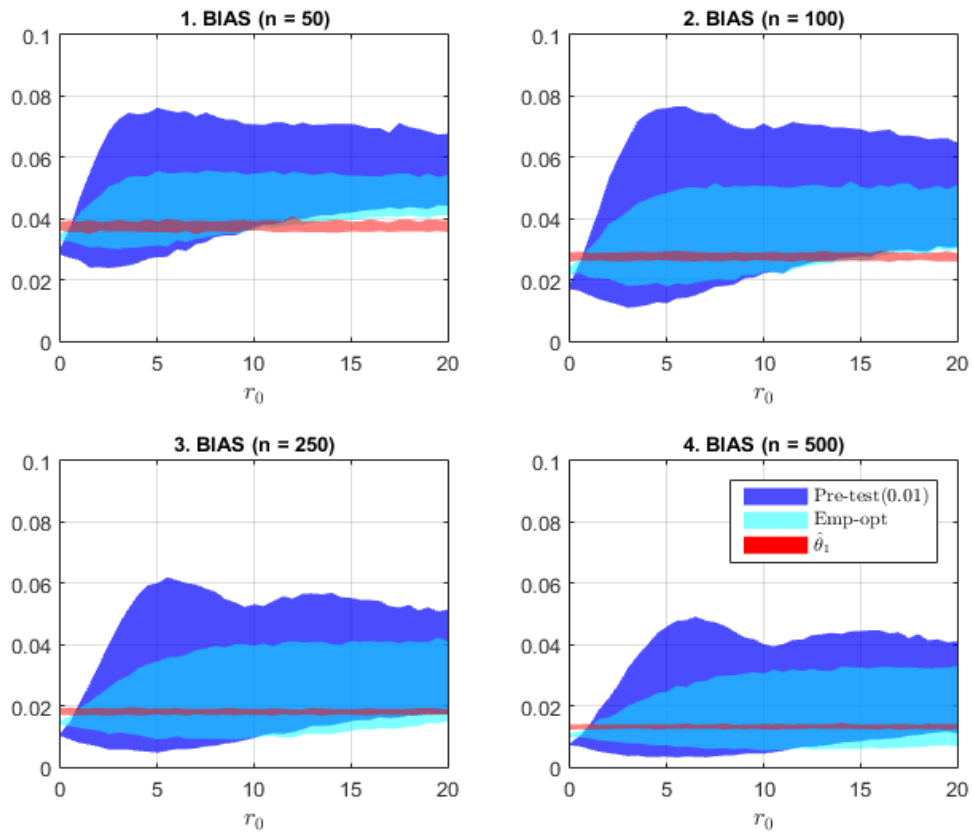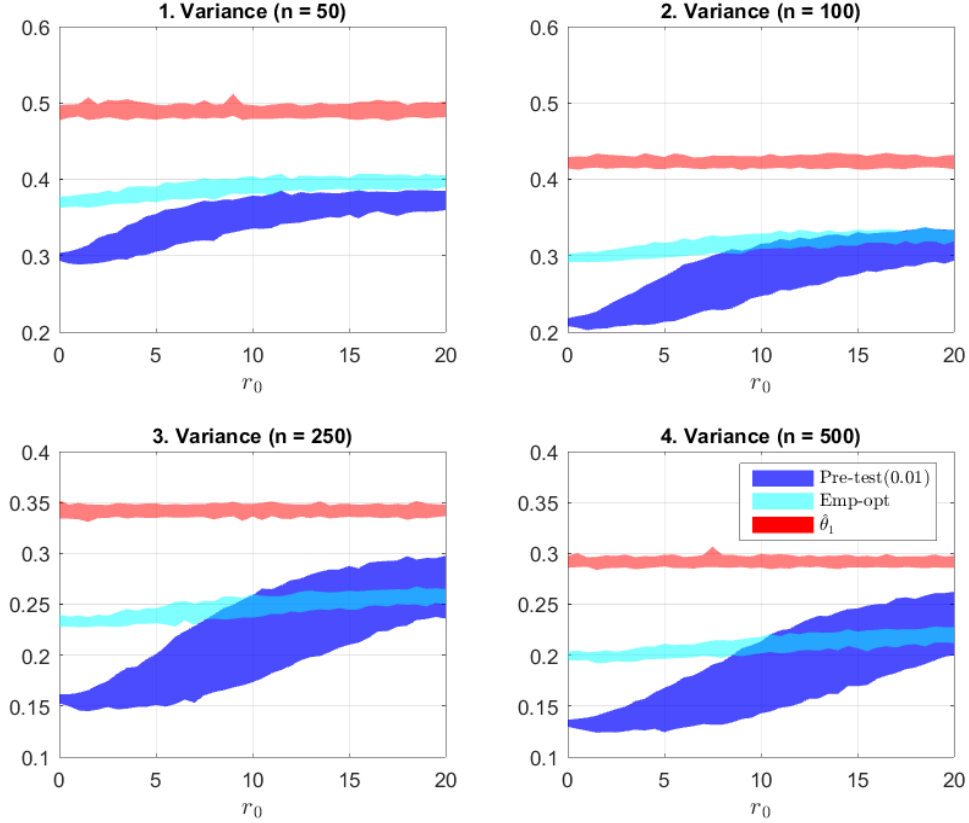Figure E.7. Finite Sample Biases of the Pre-test and Averaging GMM Estimators

Figure E.8. Finite Sample Variances of the Pre-test and Averaging GMM Estimators

F   Illustration in Gaussian Location Model

This section shows that in a Gaussian location model, the averaging GMM estimator dominates the conservative GMM estimator in finite samples, i.e., it exhibits JS phenomenon.

Suppose that we have one observation $(X', Y')'$ from the normal distribution

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(\begin{pmatrix} \theta \\ \theta + d \end{pmatrix}, I_{2k} \right) \tag{F.1}$$

where θ and d are $k \times 1$ vectors and I_{2k} is a $2k \times 2k$ identity matrix. We are interested in estimating θ.

Let Υ be the $k \times k$ identity matrix. The conservative GMM estimator $\widehat{\theta}_1 = X$ has risk $\text{tr}(\Upsilon I_k) = k$. On the other hand, the aggressive GMM estimator is $\widehat{\theta}_2 = (X+Y)/2$, which has risk $k/2 + \|d\|^2/4$. The empirical optimal weight defined in (4.7) becomes

$$\widetilde{\omega}_{eo} = \frac{2k}{2k + (Y-X)'(Y-X)}, \tag{F.2}$$

which together with the conservative and aggressive GMM estimators leads to the averaging estimator

$$\widehat{\theta}_{eo} = X + \frac{k}{2k + (Y-X)'(Y-X)}(Y-X). \tag{F.3}$$

Lemma F.1 *The averaging estimator $\widehat{\theta}_{eo}$ defined in (F.3) satisfies*

$$\mathbb{E}\left[||\widehat{\theta}_{eo} - \theta||^2 - ||\widehat{\theta}_1 - \theta||^2\right] < 0 \tag{F.4}$$

for any $k \geq 4$.

The inequality (F.4) shows that the risk of the averaging estimator is strictly smaller than the conservative estimator if $k \geq 4$, for any θ and any d.

Proof of Lemma F.1. By definition

$$\mathbb{E}\left[||\widehat{\theta}_{eo} - \theta||^2\right] - \mathbb{E}\left[||\widehat{\theta}_1 - \theta||^2\right]$$
$$= \mathbb{E}\left[\frac{k^2(Y-X)'(Y-X)}{(2k + (Y-X)'(Y-X))^2}\right]$$
$$+ 2\mathbb{E}\left[\frac{k(X-\theta)'(Y-X)}{2k + (Y-X)'(Y-X)}\right]. \tag{F.5}$$

Let $J_1 \equiv \mathbb{E}\left[\frac{(X-\theta)'(Y-X)}{2k+(Y-X)'(Y-X)}\right]$ and $J_2 \equiv \mathbb{E}\left[\frac{(Y-X)'(Y-X)}{(2k+(Y-X)'(Y-X))^2}\right]$. Let $X^* = X - \theta$, $Y^* = Y - \theta$ and $Z^* = (X^{*\prime}, Y^{*\prime})'$. Then we can write

$$J_1 = \mathbb{E}\left[\frac{(X-\theta)'(Y-X)}{2k + (Y-X)'(Y-X)}\right] = \mathbb{E}\left[\frac{X^{*\prime}(Y^* - X^*)}{2k + (Y^* - X^*)'(Y^* - X^*)}\right] = \mathbb{E}\left[\frac{Z^{*\prime}D_1 Z^*}{2k + Z^{*\prime}D_2 Z^*}\right] \tag{F.6}$$

where

$$D_1 = \begin{pmatrix} -I_k & 0_k \\ I_k & 0_k \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} I_k & -I_k \\ -I_k & I_k \end{pmatrix}. \tag{F.7}$$

Note that

$$\mathbb{E}\left[D_1 Z^* Z^{*\prime} D_1'\right] = D_2 \tag{F.8}$$

by definition and the Gaussian assumption. Let $\eta(x) = x/(x'D_2 x + 2k)$. Then its derivative is

$$\frac{\partial \eta(x)'}{\partial x} = \frac{1}{x'D_2 x + 2k}I_k - \frac{2}{(x'D_2 x + 2k)^2}D_2 x x'. \tag{F.9}$$

By Lemma 1 of Hansen (2016), which is a matrix version of the Stein's Lemma (Stein, 1981),

$$
\begin{aligned}
J_1 &= \mathbb{E}\left(\eta(Z^*)'D_1 Z^*\right) = \mathbb{E}\left[\operatorname{tr}\left(\frac{\partial \eta(Z^*)'}{\partial x}D_1\right)\right] \\
&= \mathbb{E}\left[\frac{\operatorname{tr}(D_1)}{2k+Z^{*\prime}D_2 Z^*}\right] - 2\mathbb{E}\left[\frac{\operatorname{tr}(D_2 Z^* Z^{*\prime}D_1)}{(2k+Z^{*\prime}D_2 Z^*)^2}\right] \\
&= \mathbb{E}\left[\frac{-k}{2k+Z^{*\prime}D_2 Z^*}\right] - 2\mathbb{E}\left[\frac{Z^{*\prime}D_1 D_2 Z^*}{(2k+Z^{*\prime}D_2 Z^*)^2}\right] \\
&= \mathbb{E}\left[\frac{-k}{2k+Z^{*\prime}D_2 Z^*}\right] + 2\mathbb{E}\left[\frac{Z^{*\prime}D_2 Z^*}{(2k+Z^{*\prime}D_2 Z^*)^2}\right] \\
&= \mathbb{E}\left[\frac{2-k}{2k+Z^{*\prime}D_2 Z^*}\right] + \mathbb{E}\left[\frac{-4k}{(2k+Z^{*\prime}D_2 Z^*)^2}\right] \qquad \text{(F.10)}
\end{aligned}
$$

where fourth equality is by the following result

$$
D_1 D_2 = \begin{pmatrix} -I_k & I_k \\ I_k & -I_k \end{pmatrix} = -D_2.
$$

Moreover,

$$
k^2 J_2 = \mathbb{E}\left[\frac{k^2(Y-X)'(Y-X)}{(2k+(Y-X)'(Y-X))^2}\right] = \mathbb{E}\left[\frac{k^2}{2k+Z^{*\prime}D_2 Z^*}\right] - \mathbb{E}\left[\frac{2k^3}{(2k+Z^{*\prime}D_2 Z^*)^2}\right] \qquad \text{(F.11)}
$$

which together with (F.10) implies that

$$
\begin{aligned}
&\mathbb{E}\left[||\widehat{\theta}_{eo}-\theta||^2\right] - \mathbb{E}\left[||\widehat{\theta}_1-\theta||^2\right] \\
&= \mathbb{E}\left[\frac{2k(2-k)+k^2}{2k+Z^{*\prime}D_2 Z^*}\right] - \mathbb{E}\left[\frac{2k^3+8k^2}{(2k+Z^{*\prime}D_2 Z^*)^2}\right] \\
&= \mathbb{E}\left[\frac{k(4-k)}{2k+Z^{*\prime}D_2 Z^*}\right] - \mathbb{E}\left[\frac{2k^2(k+4)}{(2k+Z^{*\prime}D_2 Z^*)^2}\right]. \qquad \text{(F.12)}
\end{aligned}
$$

The asserted result follows by the fact that D_2 is positive semi-definite and the second term in the right-hand side of the second equality of (F.12) is always negative. ∎