Existence and Stability of Dynamic Exchange Equilibra

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ABSTRACT

Economic general equilibrium results when the forces of supply and demand bring market participants into agreement as to the prices at which good should sell, and the quantities that should be sold. While it has been shown mathematically, that such exchange prices and quantities can be simultaneously determined for all goods in an economy, a mechanism that describes how such equilibria are achieved in practice has never been proposed. This paper demonstrates that such equilibria result naturally from the dynamic interaction of market participants. The dynamic interaction modeled in this paper can be used to study the interaction of demographic groups, which drives the evolution of a community. This model should also be useful in studying the effects of public policy and other external factors on markets.

INTRODUCTION

Since the time of Leon Walras, the tatonnement process by which exchange equilibrium is achieved has been modeled as occurring in an imaginary market, whose participants exchanged goods at prices called out by a virtual auctioneer [2, pp.620-26]. The first set of prices called out would usually result in unsold surpluses of some goods, and shortages of others. The auctioneer would then adjust the prices so as to at reduce the surpluses and shortages, before initiating a new round of exchanges. Through repetition, this tatonnement, (groping) process allows the auctioneer to eventually arrive at a single set of prices that would "clear" the market, with no goods unsold or in short supply.

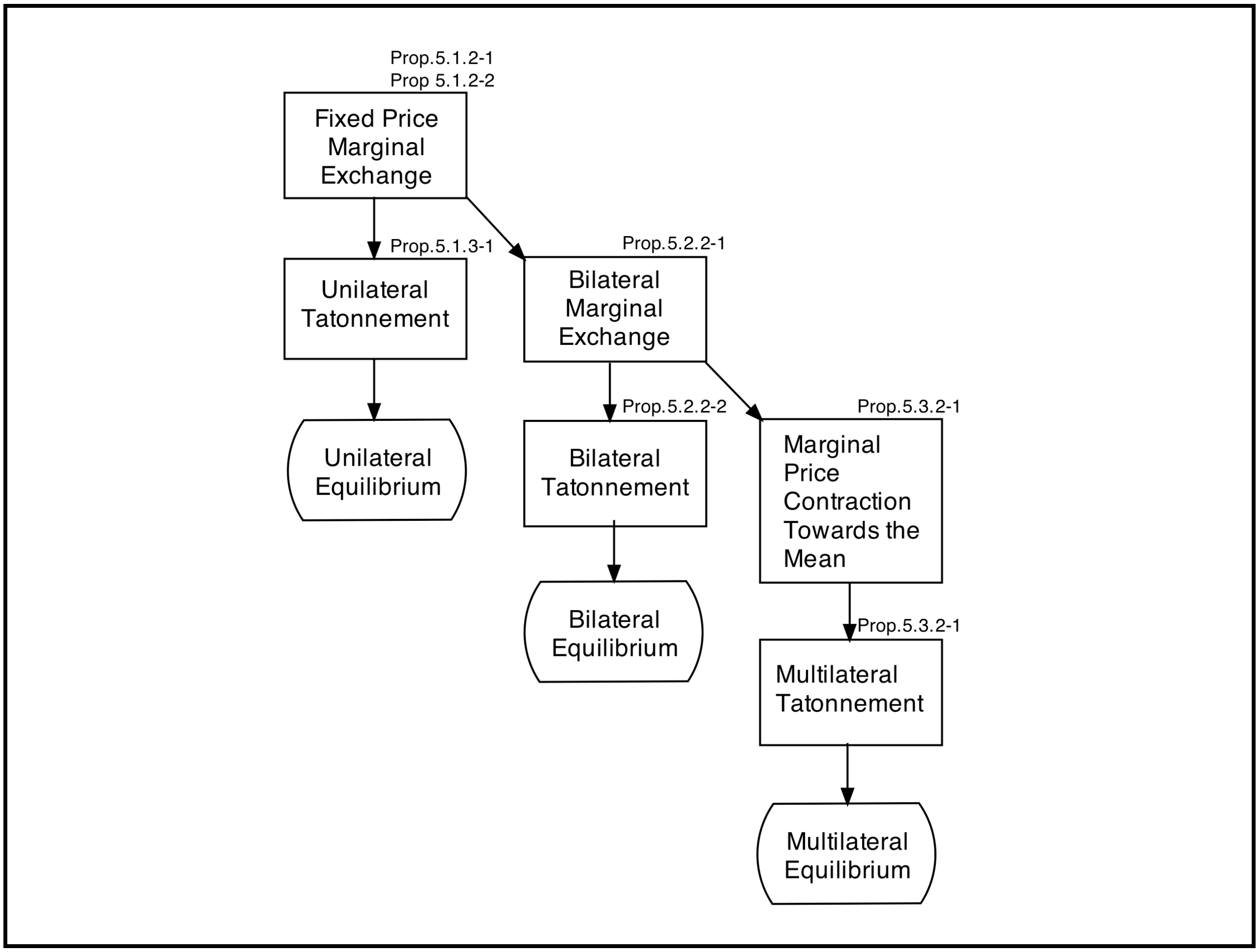
Such a hypothetical process may be sufficient to show that equilibria exist, but provides no real insight as to how they might be achieved in an actual market. As result, general equilibrium theory can say nothing about how markets might behave when out of equilibrium, or even guarantee that equilibria would be stable once received.

The dynamic model presented here solves these problems by breaking the achievement of equilibria into many small exchanges that over time equalize the consumers' marginal prices. Marginal Price theory is discussed in detail elsewhere [3]. Briefly, the consumer's *marginal price* for a good is defined as the maximum he would be willing to pay an additional unit of it, or the minimum price for which he would sell an additional unit of it. The consumer's marginal price for any given good is generally a function of the quantities of *all* goods currently in the consumer's possession, and changes predictably with the transactions the consumer makes.

In each step, buyers and sellers seek each other out and exchange a marginal bundle of goods. Following their transactions, they each move on to different trading partners with whom they transact additional business. The process of exchange redistributes goods so as to cause the consumer's marginal prices to converge into a common set of "market" prices. This is analogous to the way random collisions of gas molecules redistribute kinetic energy throughout a container, until equilibrium temperature and pressure is reached [4, pp. 153-60].

A distinct advantage of the dynamic process is that equilibria unconditionally stable. Exchanges stop as soon as an equilibrium is reached and do not begin again unless the equilibrium is exogenously disturbed. Should such disturbance occur, the process begins again and continues until equilibrium is reestablished.

FIGURE 0-1



The structure of the propositions (or theorems) describing dynamic equilibria is as shown in Figure 0-1. The case of Multilateral Equilibrium (exchange of many goods among many consumers) is built up from the bilateral case, which in turn is built on the unilateral or "fixed price" case. The propositions describing the marginal exchanges show that each occurrence of such will increase the use value which we shall define momentarily) enjoyed by the consumers involved, as well as appropriately adjusting their marginal prices. The propositions describing the tatonnements simply show that the repeated adjustments of the consumers' marginal prices will cause them to converge.

In the unilateral case, a single consumer exchanges marginal quantities of goods with a "market" at pre-determined fixed prices. The consumer buys goods for which his or her marginal price is higher than the fixed price, and sells goods for which her marginal price is lower. As result of the exchange, the consumer's marginal prices for goods she has purchased falls, while her prices for goods she has sold rises. In both cases, the consumer's marginal prices contract towards the fixed price. In the tatonnement process, such exchanges continue until the consumer's marginal prices match the market prices.

In the Bilateral case, two consumers meet and agree to exchange goods at a set of mutually beneficial prices. Using the reasoning from the fixed price case, we show that such exchange draws the consumers' marginal prices closer to each other. In the tatonnement process, constant exchange of marginal bundles, at constantly renegotiated prices, causes the consumers' marginal prices to ultimately merge.

In the multilateral case, consumers meet in pairs that engage in bilateral marginal exchanges. In addition to being drawn together, each exchange partner's marginal prices are drawn closer to the mean set of marginal prices for all consumers. In the tatonnement process, exchanges among different members of the community cause the marginal prices of all consumers to contract to the (continuously adjusting) mean, which becomes the set of "market" prices.

The theory of Dynamic General Equilibrium is based on a dynamic theory of consumer behavior developed elsewhere [3], and presented here only briefly. That theory (as well as this one) is built using vector analysis, a technique described in [1] and many texts on multivariate calculus. The theory is built on five assumptions, only three of which will be needed here. The first (part of) the first assumption (Assumption 1a) states that For any bundle of goods a consumer might hold, he or she knows how much of any one good she would be willing to trade for a single unit of any other. For an economy in which *n* goods $(x_1, x_2 \ldots x_n)$ are available, the quantities of these making up the consumer's bundle are represented by the vector $(x_1, x_2 \ldots x_n) \triangleq \vec{x}$. The consumer's marginal price for some good x_i is represented by $r_i(x_1, x_2, \ldots x_n) = r_i(\vec{x})$. The consumer's set of marginal prices for *all* goods is the *vector function* $\vec{r}(\vec{x})$. In terms of mathematics, Assumption 1a merely states that $\vec{r}(\vec{x})$ exists for all possible values of $\vec{x}$ (i.e. $x_i \geq 0$ for all x_i). The second part of the first assumption (Assumption 1b), which states what the consumer does with these prices, will be given in the next section.

With each marginal exchange the consumer makes, we define the use value he places on his marginal bundle $d\vec{x}$ as the quantity or each good in the bundle times the consumer's marginal price for it, or:

$$dV(\vec{x}) \triangleq r_1(x)dx_1 + r_2(x)dx_2 + \cdots + r_1(x)dx_1 = \vec{r}(\vec{x}) \bullet d\vec{x} \tag{1-1}$$

The signs of each dx_i can be plus or minus, depending on whether the good is bought (dx_i) or sold ($-dx_i$). Since with many marginal transactions, the goods in the consumer's bundle may have been acquired in any order. One may ask if this matters. By Assumption (4), The answer, in no.

If the consumer begins with some bundle $\vec{x}'$ and accumulates marginal bundles until he arrives at bundle $\vec{x}''$, the value he places on the total goods acquired is:

$$V(x'' - x') \triangleq \int_{x'}^{x''} \vec{r}(\vec{x}) \bullet d\vec{x} \tag{1-2}$$

Since with many marginal transactions, the goods in the consumer's bundle may have been acquired in any order. One may ask if this matters. The answer, by Assumption (4) in no (see [3]).

While Assumption (1b) will be functionally similar to the standard assumption of utility maximization, Assumption (5) is analogous the *Law of Diminishing Marginal Utility*, and is slightly stronger than the standard assumption of *Convexity*. Assumption (5) states that there are no goods for which a consumer's marginal prices do not fall as he acquires more of them. Mathematically, this is stated as:

$$\left[r_i(\vec{x}+d\vec{x})-r_i(\vec{x})\right]\left(dx_i\right)<0 \qquad \text{for all goods } x_i \tag{1-3}$$

The standard assumption of convexity is slightly looser but easier to use. It is stated in vector form as:

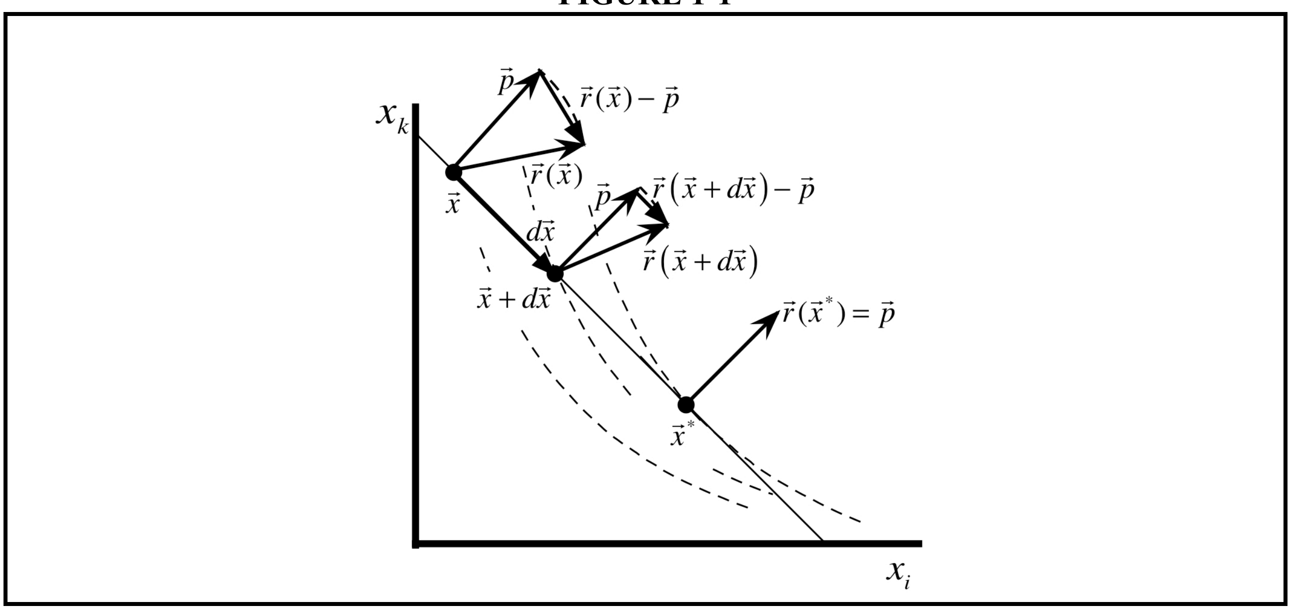
$$\sum_{i=1}^{n}\left[r_i(\vec{x}+d\vec{x})-r(\vec{x})\right]\left(dx_i\right)=\left[\vec{r}(\vec{x}+d\vec{x})-\vec{r}(\vec{x})\right]\bullet d\vec{x}<0 \tag{1-4}$$

1) UNILATERAL EQUILIBRIUM

We begin our discussion with the case of a single consumer exchanging goods with a "market" that allows him or her to exchange as much of any good as he or she desires at fixed "market" prices.

After making the necessary definitions, we begin by showing that whenever the consumer's marginal prices $\vec{r}(\vec{x})$ do not equal $\vec{p}$, the consumer will benefit by exchanging a differentially small bundle of goods $d\vec{x}$. As result of the exchange, the consumer's marginal prices will be brought "closer" to $\vec{p}$ as shown in Figure 5.1-1. We then show that the consumer will continue to make these marginal exchanges until $\vec{r}(\vec{x})=\vec{p}$. Finally, we show that the bundle $\vec{x}^*$, for which $\vec{r}(\vec{x}^*)=\vec{p}$ is the one that offers the consumer the greatest benefit. This is done by showing that it the consumer were to continue making marginal exchanges once $\vec{x}^*$ has been acquired, she would begin to loose the benefit she had previously gained.

FIGURE 1-1



1.1) Definitions and Assumptions

In chapter 4 we formally stated only the first part of Assumption 1; that the consumer knows what his marginal prices are. We now need to state the second part, which indicated how he would respond to an opportunity for exchange. Intuitively, we want to say that the consumer will take advantage of a "good deal", or will try to get the most benefit per unit of numeraire spent.

We begin by defining the "benefit" or "deal" that the consumer seeks to obtain by making a marginal exchange. This is simply his or her consumer's marginal surplus, exactly as Dupuit envisioned it (see Section 3.2). It is the difference between what a consumer would be willing to pay for a marginal amount of a good, and what he is required to pay by an exchange partner.

Definition: Consumer's Marginal Surplus (for a single good)

For a consumer described by a marginal price function $\vec{r}(\vec{x})$, and holding a bundle $\vec{x}$, the marginal surplus the consumer would enjoy from purchasing (or selling) a differential quantity dx_i of some good x_i is given by:

$$\left[r_i(\vec{x}) - p_i\right]dx_i \tag{1.1-1}$$

Notice that if the consumer would be willing to pay more for the good than its market price, the consumer would gain surplus by acquiring the good. In this case, both $[r_i(\vec{x}) - p_i]$ and dx_i are positive, and so is the surplus. If the consumer values a good less than does the market, he gains surplus by selling some of it. In this case both $[r_i(\vec{x}) - p_i]$ and dx_i are negative, and the surplus is again positive.

We will assume that the consumer will try to maximize the surplus obtained for each transaction. This requires that he adjust the relative quantities of the goods dx_i bought and sold, which will be reflected in the direction of the vector $d\vec{x}$.

Assumption 1b

Given, a consumer described by a marginal price function $\vec{r}(\vec{x})$, and holding a bundle $\vec{x}$. For all goods x_i (and only for such goods) for which the consumer's marginal price differs from the price $\vec{p}$ he or she is offered, the consumer will buy quantities dx_i, or sell quantities $-dx_i$ as necessary to gain the maximum total marginal surplus, subject to the budget constraint. $p_1dx_1 + p_2dx_2 + \cdots + p_ndx_n = \vec{p} \bullet d\vec{x} = 0$

By assumption 1b, the consumer solves the following problem:

$$\underset{\Delta x}{Max}\left\{\left(\vec{r}(\vec{x}) - \vec{p}\right) \bullet d\vec{x}\right\} = \underset{\Delta x}{Max}\left\{\vec{r}(\vec{x}) \bullet d\vec{x} - \vec{p} \bullet d\vec{x}\right\} \text{ s.t. } \vec{p} \bullet d\vec{x} = 0 \tag{1.1-2}$$

We can substitute the budget constraint into the objective function, re writing the problem as:

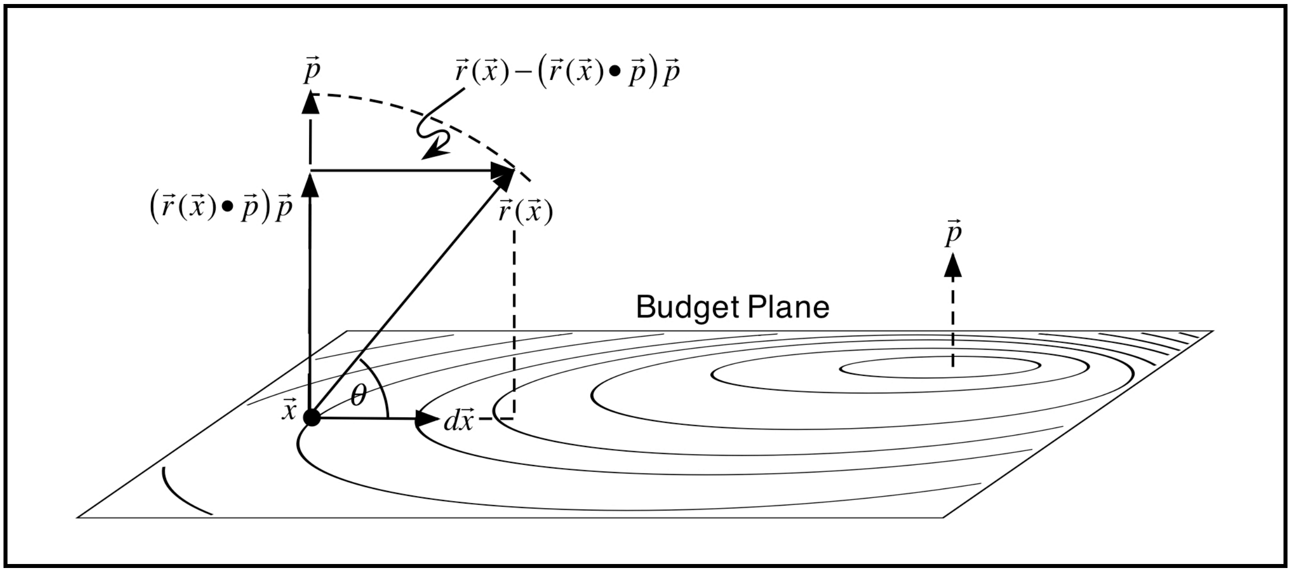
$$\underset{\theta}{Max}\left\{\left|\vec{r}(\vec{x})\right|\left|d\vec{x}\right|\cos\theta\right\} \tag{1.1-3}$$

As can be seen from Figure 5.1.1-1, the objective function is maximized when the angle θ between $\vec{r}(\vec{x})$ and $d\vec{x}$ is minimized. This occurs when $d\vec{x}$ lies in the intersection of the budget plane, determined by $\vec{p} \bullet d\vec{x} = 0$ and the plane determined by $\vec{r}(\vec{x})$ and $\vec{p}$ as shown in Figure 1.1-1. The exchange bundle $d\vec{x}$ has differential magnitude $|d\vec{x}|$, and has direction parallel to $\vec{r}(\vec{x}) - (\vec{r}(\vec{x}) \bullet \vec{p})\vec{p}$. The marginal exchange bundle can therefore be written as:

$$d\vec{x} = \frac{\vec{r}(\vec{x}) - (\vec{r}(\vec{x}) \bullet \vec{p})\vec{p}}{\left|\vec{r}(\vec{x}) - (\vec{r}(\vec{x}) \bullet \vec{p})\vec{p}\right|}|d\vec{x}| \qquad (1.1\text{-}4)$$

This of course is simply the projection of the price difference $\vec{r}(\vec{x}) - \vec{p}$ into the budget plane as shown in Figure 1.1-1.

FIGURE 1.1-1



1.2) Propositions and Proofs

The following two propositions describe the exchange of a marginal bundle between a consumer and anyone else at predetermined prices. The first of these propositions states in essence that whenever a consumer's marginal prices differs from those he is offered, benefit from exchange is possible and the consumer will engage in a marginal exchange. When benefit from exchange is not possible, the consumer will refrain from exchange. This will be useful later in proving that equilibria are stable.

The second proposition indicates that, the consumer's marginal price will "contract" towards $\vec{p}$ with each marginal exchange. The collective difference between the consumer's marginal prices and $\vec{p}$ is measured by the magnitude of the difference $\vec{r}(\vec{x}) - \vec{p}$. Before the consumer exchanges $d\vec{x}$ the difference between the prices is $|\vec{r}(\vec{x}) - \vec{p}|$ while after the exchange it is $|\vec{r}(\vec{x} + d\vec{x}) - \vec{p}|$ as shown in Figure 5.1.2-1

Proposition 1.2-1 (Benefit from Marginal Exchange)

Given a consumer who is described by marginal price function $\vec{r}(\vec{x})$ and possesses a bundle $\vec{x}$. If (and only if) the consumer is given the opportunity to exchange goods at a price $\vec{p}$ for which $\vec{r}(\vec{x}) \neq \vec{p}$, the following will result:

a) The consumer will exchange a small bundle $d\vec{x}$ constructed such that:

$$\left[r_i(\vec{x}) - p_i\right]dx_i > 0 \quad \forall i \tag{1.2-1}$$

and

$$p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n = \vec{p} \bullet d\vec{x} = 0\,. \tag{1.2-2}$$

b) Such exchange will increase the use value of the consumer's holdings, i.e.

$$V(\vec{x} + d\vec{x}) > V(\vec{x})\,. \tag{1.2-3}$$

PROOF:

By definition, $\vec{r}(\vec{x})$ and $\vec{p}$ are of unit magnitude, hence $\sum_{i=1}^{n} r_i(\vec{x}) \equiv \sum_{i=1}^{n} p_i \equiv 1$.

If (and only if) there exists some good x_i for which $r_i(x) > p_i$ then there must be at least one good x_k for which $r_k(x) < p_k$. Assuming that goods are divisible, and the consumer already possesses some of the good (or goods) x_k, the consumer is able to devise an exchange bundle $d\vec{x}$ containing only goods, the exchange of which will grant the consumer a positive surplus while satisfying the budget condition $\vec{p} \bullet d\vec{x} = 0$. By Assumption 1b, the consumer will exchange this bundle. Therefore, the consumer will make a marginal exchange whenever his or her $\vec{r}(\vec{x}) \neq \vec{p}$, and will refrain from making an exchange when $\vec{r}(\vec{x}) = \vec{p}$. This completes the proof of part (a).

The increase in use value that a customer holding a bundle $\vec{x}$ would gain by exchanging a bundle $d\vec{x}$ is by definition Equation (1-2):

$$V(\vec{x} + d\vec{x}) - V(\vec{x}) = \int_{0}^{\vec{x}+d\vec{x}} \vec{r}\left(\vec{x}\right) \bullet d\vec{x} - \int_{0}^{\vec{x}} \vec{r}\left(\vec{x}\right) \bullet d\vec{x} = \vec{r}\left(\vec{x}\right) \bullet d\vec{x} \tag{1.2-4}$$

By Assumption 1b, the marginal surplus the consumer gains from the exchange of all goods in $d\vec{x}$ is positive, hence:

$$0 < \sum_{i=1}^{n} \left[r_i\left(\vec{x}\right) - p_i\right] dx_i = \left[\vec{r}\left(\vec{x}\right) - \vec{p}\right] \bullet d\vec{x} = \vec{r}\left(\vec{x}\right) \bullet d\vec{x} - \vec{p} \bullet d\vec{x} \tag{1.2-5}$$

Since the last term on the right is zero we have:

$$0 < \vec{r}\left(\vec{x}\right) \bullet d\vec{x} \tag{1.2-6}$$

Hence from Equation (1.2-4) we have $V(\vec{x}+d\vec{x}) > V(\vec{x})$ for every exchange. This completes the proof of Part B.
QED.

Proposition 1.2-2 (Price Contraction from Marginal Exchange)

Given a consumer described by marginal price function $\vec{r}(\vec{x})$ and possessing a bundle $\vec{x}$. If such consumer, who is given the opportunity to exchange goods at prices $\vec{p}$, exchanges a marginal bundle $d\vec{x}$ as defined by Assumption 1b, the differences between the consumer's $\vec{r}(\vec{x})$ and $\vec{p}$ will contract, i.e.:

$$\left|\vec{r}(\vec{x})-\vec{p}\right| > \left|\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right| > 0 \qquad (1.2\text{-}7)$$

PROOF:
From Assumption 5 (Equation 1-4) we have:

$$\left[\vec{r}(\vec{x}+d\vec{x})-\vec{r}(\vec{x})\right]\bullet d\vec{x} < 0 \qquad (1.2\text{-}8)$$

Since $d\vec{x}$ is very small, we can assume from Equation (1-4) that $\vec{r}(\vec{x}+d\vec{x})\bullet d\vec{x}$ is positive whenever $\vec{r}(\vec{x})\bullet d\vec{x}$ is positive, thus:

$$\vec{r}(\vec{x})\bullet d\vec{x} > \vec{r}(\vec{x}+d\vec{x})\bullet d\vec{x} > 0 \qquad (1.2\text{-}9)$$

Since $\vec{p}\bullet d\vec{x}=0$ we can subtract it from all terms in Equation (1.2-9) without altering the inequality, leaving:

$$\left(\vec{r}(\vec{x})-\vec{p}\right)\bullet d\vec{x} > \left(\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right)\bullet d\vec{x} > 0 \qquad (1.2\text{-}10)$$

Substituting $d\vec{x}$ from its value given in Equation (1.1-4) and cancelling the denominator, we have:

$$\begin{aligned}&\left(\vec{r}(\vec{x})-\vec{p}\right)\bullet\left(\vec{r}(\vec{x})-\left(\vec{r}(\vec{x})\bullet\vec{p}\right)\vec{p}\right) > \\ &\qquad \left(\vec{r}(\vec{x}+d\vec{x})-\vec{p}\right)\bullet\left(\vec{r}(\vec{x}+d\vec{x})-\left(\vec{r}(\vec{x}+d\vec{x})\bullet\vec{p}\right)\vec{p}\right) > 0\end{aligned} \qquad (1.2\text{-}11)$$

Multiplying out the dot product and collecting terms leaves:

$$\left|\vec{r}(\vec{x})\right|^2-\left(\vec{r}(\vec{x})\bullet\vec{p}\right)^2 > \left|\vec{r}(\vec{x}+d\vec{x})\right|^2-\left(\vec{r}(\vec{x}+d\vec{x})\bullet\vec{p}\right)^2 > 0 \qquad (1.2\text{-}12)$$

From the Pythagorean theorem we know that:

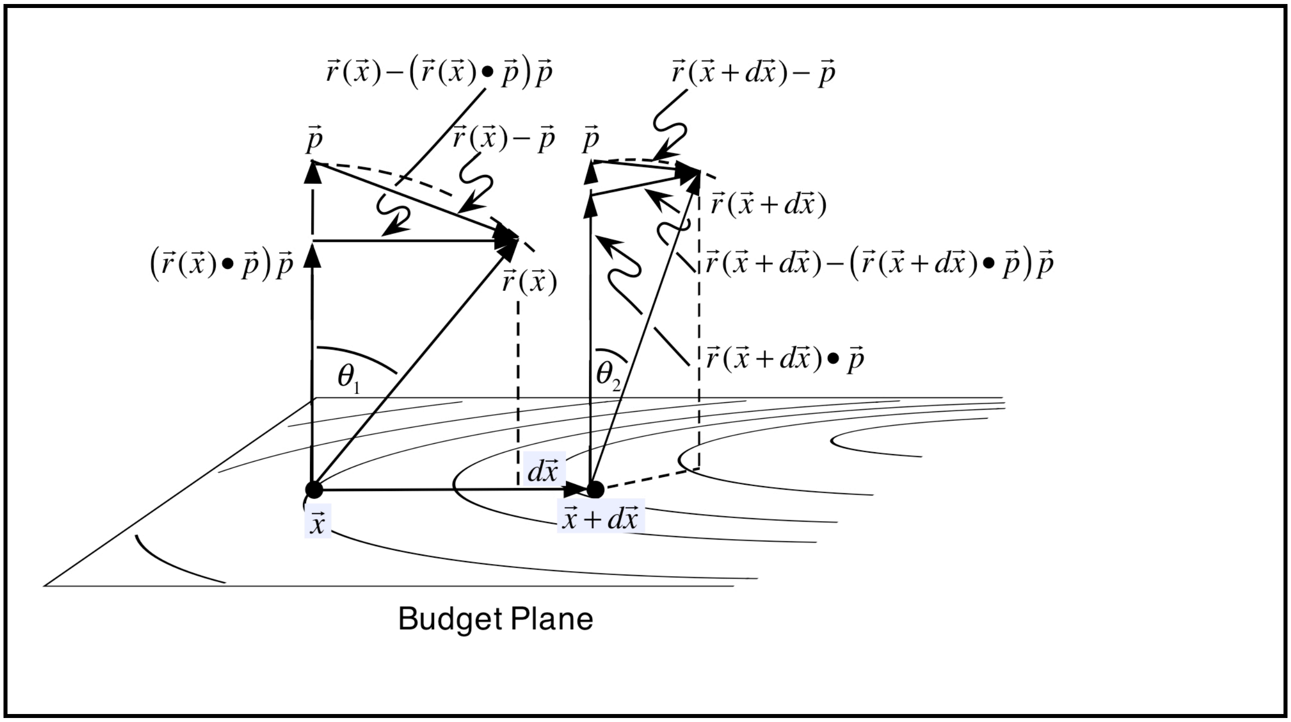
$$\left|\vec{r}(\vec{x})\right|^2-\left|\vec{r}(\vec{x})\bullet p\right|^2=\left|\vec{r}(\vec{x})-\vec{r}(\vec{x})\bullet p\right|^2=\left|\vec{r}(\vec{x})\right|^2\left(\sin\theta_1\right)^2$$
$$\left|\vec{r}(\vec{x}+dx)\right|^2-\left|\vec{r}(\vec{x}+dx)\bullet p\right|^2=\left|\vec{r}(\vec{x}+dx)-\vec{r}(\vec{x}+dx)\bullet p\right|^2=\left|r(x+dx)\right|^2\left(\sin\theta_2\right)^2$$

Since $\left|\vec{r}(\vec{x})\right|\equiv\left|\vec{r}(\vec{x}+dx)\right|\equiv 1$, from Equation 1.2-12 we have:

$$\sin\theta_2 < \sin\theta_1 \quad\Rightarrow\quad \theta_2 < \theta_1 \qquad (1.2\text{-}13)$$

Since $\vec{p}$ also has unit magnitude, the decrease in angle indicates that $\vec{r}(\vec{x}+d\vec{x})$ is "closer" to $\vec{p}$ than is $\vec{r}(\vec{x})$, thus $|\vec{r}(\vec{x})-\vec{p}|>|\vec{r}(\vec{x}+d\vec{x})-\vec{p}|>0$ as claimed. This completes the proof.
QED.

FIGURE 1.2-1



The final proposition models the tatonnement process as a sequence of marginal exchanges made over time. Since the difference between the consumer's marginal prices and $\vec{p}$ reduce with each exchange, they muse eventually reach zero.

> **Proposition 1.2-3: Unilateral Tatonnement**
> Given a consumer described by marginal price function $\vec{r}(\vec{x})$, and at time t_0 possesses an initial bundle $\vec{x}[t_0]$. Given also that the consumer is given the opportunity to exchange any number of marginal bundles at a fixed prices $\vec{p}$. The consumer will, at t_0, and in future time periods t_0+n, exchange marginal bundles $d\vec{x}[t_0+n]$, until he attains a bundle $\vec{x}[t_0+z]$ for which $\vec{r}(\vec{x}[t_0+z])=\vec{p}$. Furthermore, the total use-value $V(x[t_0+z]-x[t_0])$ gained by the consumer will be the maximum available to him at prices $\vec{p}$ given his wealth $w=\vec{p}\bullet\vec{x}[t_0]$.

PROOF:
For every time period t_0+n for which the consumer's marginal prices $\vec{r}(\vec{x}[t_0+n])$ do not equal $\vec{p}$, Proposition 5.1.2-1 implies that the consumer will exchange a marginal bundle $d\vec{x}[t_0+n]$. As result, the use value the consumer enjoys will have increased, i.e.: $V(x[t_0+n]+dx[t_0+n])>V(x[t_0+n])$. Per Proposition 5.1.2-2 we know that for every time period we have $|\vec{r}(\vec{x}[t_0+n])-\vec{p}|>|\vec{r}(\vec{x}[t_0+n]+d\vec{x}[t_0+n])-\vec{p}|>0$.

At the beginning of every time period $t_0 + n + 1$, the consumer's bundle is simply the one he held previously, adjusted by the bundle exchanged, $\vec{x}[t_0 + n + 1] \triangleq \vec{x}[t_0 + n] + d\vec{x}[t_0 + n]$. Per Propositions 5.1.2-1 and 5.1.2-2 we thus have:

$$V(x[t_0 + n + 1]) > V(x[t_0 + n]) \tag{5.1.2-14}$$

$$\left|\vec{r}(\vec{x}[t_0 + n]) - \vec{p}\right| > \left|\vec{r}(\vec{x}[t_0 + n + 1] - \vec{p}\right| > 0 \tag{5.1.2-15}$$

From Equation 5.1.2-15 it is apparent that:

$$\underset{n\to\infty}{\text{limit}}\left|\vec{r}(\vec{x}[t_0 + n]) - \vec{p}\right| = 0 \tag{5.1.2-16}$$

For practical purposes, we will choose some number ε that is negligibly close to zero. Since Equation 5.1.2-16 approaches zero monotonically, there must be some number $0 < z < \infty$ such that:

$$\left|\vec{r}(\vec{x}[t_0 + z]) - \vec{p}\right| < \varepsilon \tag{5.1.2-17}$$

Therefore, at least for practical purposes, $\vec{x}[t_0 + z]$ is the bundle for which the consumer's marginal prices equal $\vec{p}$.

According to Proposition 5.1.1-1 exchange will stop at this point, and will not restart as long as $\vec{p}$ or $\vec{x}[t_0 + z]$ remain unchanged.

To show that $V(\vec{x}[t_0 + z])$ provides the maximum use value available at prices $\vec{p}$, we assume for a moment that it does not. By Assumption 5 (Equation 1-4), the indifference curves of $V(\vec{x})$ are convex. Thus, if $V(\vec{x}[t_0 + z])$ is not the maximum, there is some marginal bundle $d\vec{x}'$ the consumer could exchange, for which $V(\vec{x}[t_0 + z] + d\vec{x}') > V(\vec{x}[t_0 + z])$. If the consumer were to make such exchange, his marginal price vector $\vec{r}(\vec{x}[t_0 + z] + d\vec{x}')$ must satisfy Equation 4.4-5 (Assumption 5), thus:

$$\left[\vec{r}(\vec{x}[t_0 + z] + d\vec{x}') - \vec{r}(\vec{x}[t_0 + z])\right] \bullet d\vec{x}' < 0 \tag{5.1.2-18}$$

Since at equilibrium $\vec{r}(\vec{x}[t_0 + z]) = \vec{p}$, and $\vec{p} \bullet d\vec{x}$ is always zero, Equation (5.1-18) becomes:

$$\vec{r}(\vec{x}[t_0 + z] + d\vec{x}') \bullet d\vec{x}' < 0 \tag{5.1-19}$$

If the consumer, who now holds $x[t_0 + z] + d\vec{x}'$ were to reverse his exchange of $d\vec{x}'$, he would gain a positive surplus since: $\vec{r}(\vec{x}[t_0 + z] + d\vec{x}') \bullet (-d\vec{x}') > 0$. We thus have $V(x[t_0 + z] + d\vec{x}) < V(x[t_0 + z])$ which contradicts our temporary assumption. We have thus shown that $V(\vec{x}[t_0 + z])$ is the maximum value available to the consumer. This completes the proof.
QED.

2) BILATERAL EQUILIBRIUM

Bilateral exchange, or the exchange of goods between two individuals, is modeled as an extension of the of the fixed price exchange. Two individuals with marginal different prices $\vec{r}^{\,1}(\vec{x}^1)$ and $\vec{r}^{\,2}(\vec{x}^2)$ which are functions of the bundles $\vec{x}^1$ and $\vec{x}^2$ they respectively hold, engage in a sequence of bilateral marginal exchanges. In each round, the individuals agree to a price $\vec{p}$ that lies "between" their marginal prices, at which the bundle $d\vec{x} = d\vec{x}^1 = -d\vec{x}^2$ is to be exchanged. Once the price has been agreed upon, the remainder of the exchange is, to each consumer, no different from a fixed price exchange. We know therefore that each marginal exchange benefits each consumer, and causes his marginal prices to contract towards $\vec{p}$. Since $\vec{p}$ is "between" $\vec{r}^{\,1}(\vec{x}^1)$ and $\vec{r}^{\,2}(\vec{x}^2)$, we show that these marginal prices have contracted towards each other. In the tatonnement process, we show that the contraction continues until the consumers' marginal prices merge into the equilibrium price. Finally we show that the use value enjoyed by each consumer is the maximum available to them given the bundles they started out with, and the equilibrium price.

2.1) More Definitions

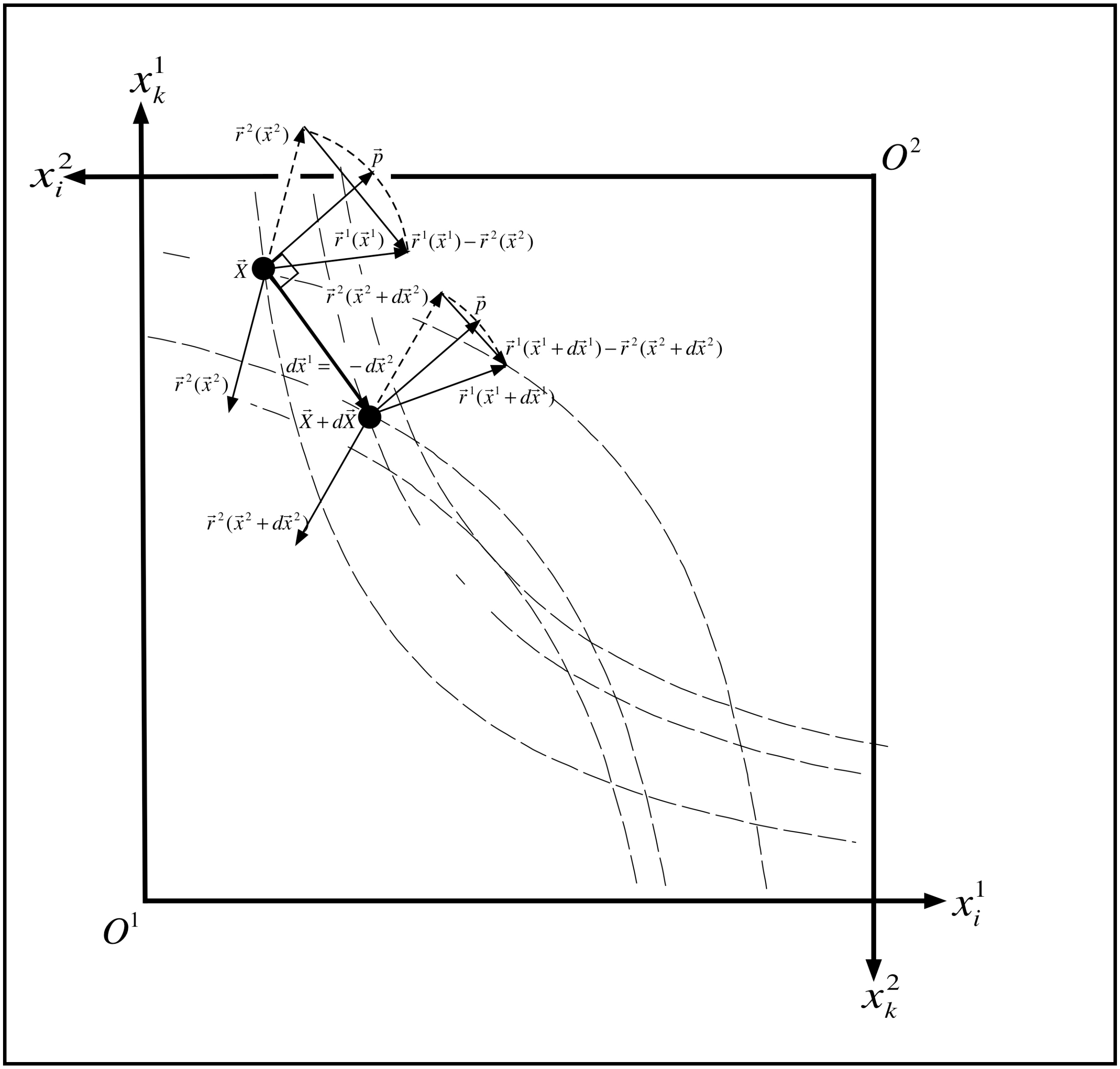
We begin by defining what it means for a vector to lay "between" another pair of vectors. There are two different notions of "between-ness" that we will have occasion to use. The first applies to a vector that lies in the same plane as the vectors it is "between". Such a vector can be described algebraically in terms of the other vectors. The second notion of "between-ness" applies to a vector whose components lie between the components of the bounding vectors. Such a vector lies in the hyper-rectangular region of space defined by the vectors it is said to be between, as shown in Figure 5.2.1-1.

Definition [A vector that lies "Between" a pair of vectors]
Given Three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, each of n components: Vector $\vec{B}$ lies "between" $\vec{A}$ and $\vec{C}$, if and only if $\vec{B}$ can be expressed in the form: $j\vec{B} = \vec{C} + k(\vec{A} - \vec{C})$ where. $0 < k < 1$ and $j > 0$.

Definition [Box Defined by Two Vectors]
Given pair of vectors $\vec{A} = A_1\hat{\varphi}_1 + A_2\hat{\varphi}_2 + \cdots + A_n\hat{\varphi}_n$ and $\vec{C} = C_1\hat{\varphi}_1 + C_2\hat{\varphi}_2 + \cdots + C_n\hat{\varphi}_n$ The Box defined by these vectors consists of the set $\aleph$ of all vectors $\vec{\chi}$ such that for all components χ_i:

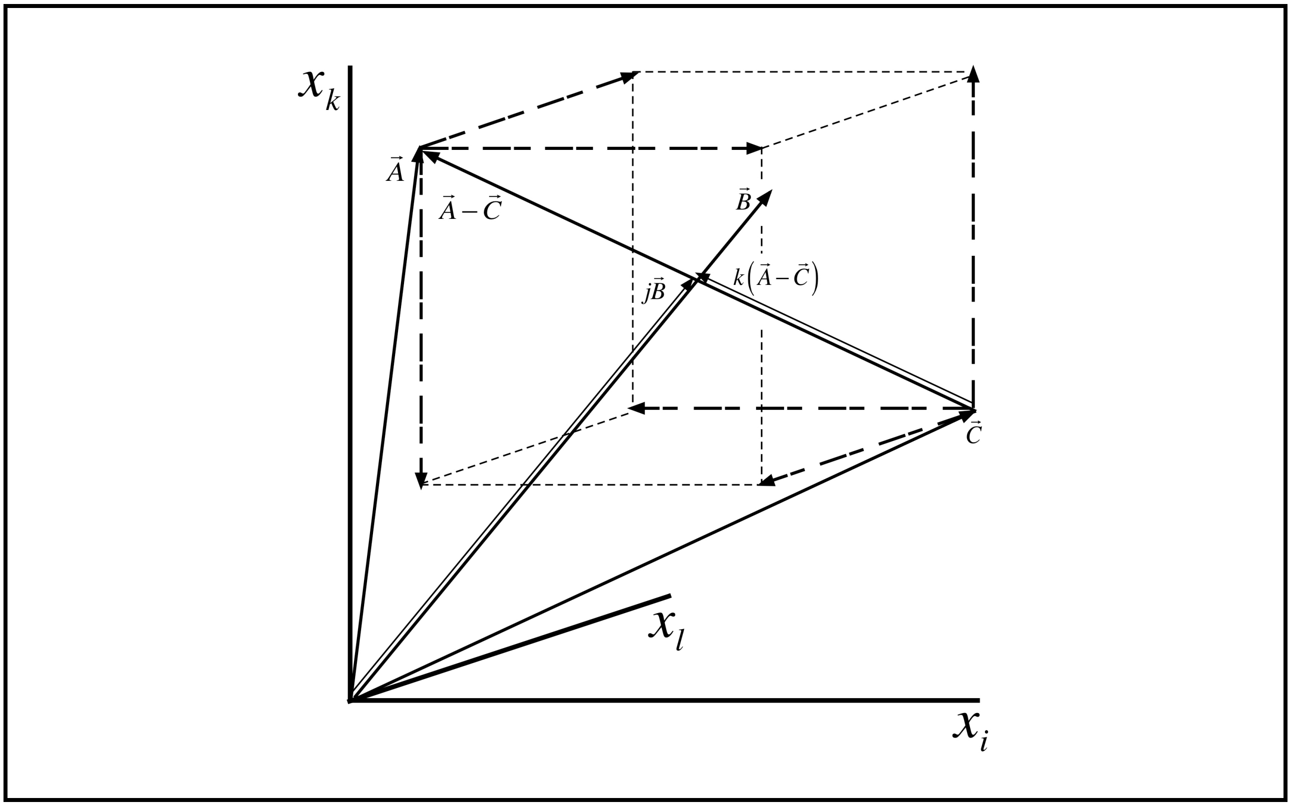
$$A_i > C_i \Rightarrow A_i \geq \chi_i \geq C_i \qquad or \qquad C_i > A_i \Rightarrow C_i \geq \chi_i \geq A_i \tag{2.1-1}$$

FIGURE 2.1-1



x_k^1
$\vec{r}^{\,2}(\vec{x}^{\,2})$
$\vec{p}$
x_i^2
O^2
$\vec{r}^{\,1}(\vec{x}^{\,1})$
$\vec{r}^{\,1}(\vec{x}^{\,1})-\vec{r}^{\,2}(\vec{x}^{\,2})$
$\vec{X}$
$\vec{r}^{\,2}(\vec{x}^{\,2}+d\vec{x}^{\,2})$
$\vec{p}$
$\vec{r}^{\,1}(\vec{x}^{\,1}+d\vec{x}^{\,1})-\vec{r}^{\,2}(\vec{x}^{\,2}+d\vec{x}^{\,2})$
$d\vec{x}^{\,1}=-d\vec{x}^{\,2}$
$\vec{r}^{\,2}(\vec{x}^{\,2})$
$\vec{r}^{\,1}(\vec{x}^{\,1}+d\vec{x}^{\,1})$
$\vec{X}+d\vec{X}$
$\vec{r}^{\,2}(\vec{x}^{\,2}+d\vec{x}^{\,2})$
x_i^1
O^1
x_k^2

FIGURE 2.1-2



2.2) More Propositions and Proofs

<u>Proposition 2.2-1 Bilateral Marginal Exchange</u>

Given two consumers who are described by marginal price functions $\vec{r}^{\,1}(\vec{x}^{\,1})$ and $\vec{r}^{\,2}(\vec{x}^{\,2})$, and who possess bundles $\vec{x}^{\,1}$ and $\vec{x}^{\,2}$ respectively.

If and only if $\vec{r}^{\,1}(\vec{x}^{\,1}) \neq \vec{r}^{\,2}(\vec{x}^{\,2})$, the consumers will agree to exchange a marginal bundle $d\vec{x} = dx^1 = -d\vec{x}^2$ of goods at a price $\vec{p}$ that lies between $\vec{r}^{\,1}(\vec{x}^{\,1})$ and $\vec{r}^{\,2}(\vec{x}^{\,2})$. As result of the exchange:

a) The use value of both consumers will have increased $V^1(\vec{x}^{\,1} + d\vec{x}^{\,1}) > V^1(\vec{x}^{\,1})$ and $V^2(\vec{x}^{\,2} + d\vec{x}^{\,2}) > V^2(\vec{x}^{\,2})$.

b) The marginal prices of the consumers will have contracted together:
$\left|\vec{r}^{\,1}(\vec{x}^{\,1}) - \vec{r}^{\,2}(\vec{x}^{\,2})\right| > \left|\vec{r}^{\,1}(\vec{x}^{\,1} + d\vec{x}^{\,1}) - \vec{r}^{\,2}(\vec{x}^{\,2} + d\vec{x}^{\,2})\right| > 0$

Proof:
Since $\vec{r}^{\,1}(\vec{x}^{\,1})$ and $\vec{r}^{\,2}(\vec{x}^{\,2})$ are of unit magnitude and $\vec{r}^{\,1}(\vec{x}^{\,1}) \neq \vec{r}^{\,2}(\vec{x}^{\,2})$, there exists at least one good x_i for which $\vec{r}_i^{\,1}(\vec{x}^{\,1}) > \vec{r}_i^{\,2}(\vec{x}^{\,2})$ and at least one other good x_k for which $\vec{r}_k^{\,2}(\vec{x}^{\,2}) > \vec{r}_k^{\,1}(\vec{x}^{\,1})$. Therefore there exists at least one price $\vec{p}$ that lies between $\vec{r}^{\,1}(\vec{x}^{\,1})$ and $\vec{r}^{\,2}(\vec{x}^{\,2})$. Therefore, by Proposition 1.2-1 there is an opportunity for both consumers to benefit from an exchange.

Per Assumption 1b the consumers will attempt to negotiate the price $\vec{p}$ and the contents of marginal bundle $d\vec{x}$ so as to satisfy:

$$\begin{aligned} consumer\,\#1: \quad & d\vec{x}^{\,1} = \vec{r}^{\,1}(\vec{x}^{\,1}) - \left(\vec{r}^{\,1}(\vec{x}^{\,1}) \bullet \vec{p}\right)\vec{p} \\ consumer\,\#2: \quad & d\vec{x}^{\,2} = \vec{r}^{\,2}(\vec{x}^{\,2}) - \left(\vec{r}^{\,2}(\vec{x}^{\,2}) \bullet \vec{p}\right)\vec{p} \\ & d\vec{x}^{\,1} = -d\vec{x}^{\,2} = d\vec{x} \end{aligned} \tag{2.2-1}$$

This will cause the consumers to choose a price that lays exactly half way between $\vec{r}^{\,1}(\vec{x}^{\,1})$ and $\vec{r}^{\,2}(\vec{x}^{\,2})$, i.e. with $k = 1/2$. To show this, we combine Equations 2.2-1 giving:

$$\begin{aligned} d\vec{x} &= \vec{r}^{\,1}(\vec{x}^{\,1}) - \left(\vec{r}^{\,1}(\vec{x}^{\,1}) \bullet \vec{p}\right)\vec{p} = \left(\vec{r}^{\,2}(\vec{x}^{\,2}) \bullet \vec{p}\right)\vec{p} - \vec{r}^{\,2}(\vec{x}^{\,2}) \\ &\Rightarrow \vec{r}^{\,1}(\vec{x}^{\,1}) + \vec{r}^{\,2}(\vec{x}^{\,2}) = \left(\left[\vec{r}^{\,1}(\vec{x}^{\,1}) + \vec{r}^{\,2}(\vec{x}^{\,2})\right] \bullet \vec{p}\right)\vec{p} \\ &\Rightarrow \vec{p} = \frac{\vec{r}^{\,1}(\vec{x}^{\,1}) + \vec{r}^{\,2}(\vec{x}^{\,2})}{\left[\vec{r}^{\,1}(\vec{x}^{\,1}) + \vec{r}^{\,2}(\vec{x}^{\,2})\right] \bullet \vec{p}} = \frac{\vec{r}^{\,1}(\vec{x}^{\,1}) + \vec{r}^{\,2}(\vec{x}^{\,2})}{\left|\vec{r}^{\,1}(\vec{x}^{\,1}) + \vec{r}^{\,2}(\vec{x}^{\,2})\right|} \end{aligned} \tag{2.2-2}$$

In the last step of Equation 2.2-2 we have recognized that the denominator is a scalar, hence the direction of $\vec{p}$ is given by the numerator. Since $\vec{p}$ is by definition a unit vector, its dot product with any vector parallel to it is simply the magnitude of the parallel vector.

By Proposition 1.2-2, the marginal prices of both consumers will contract towards $\vec{p}$, i.e.

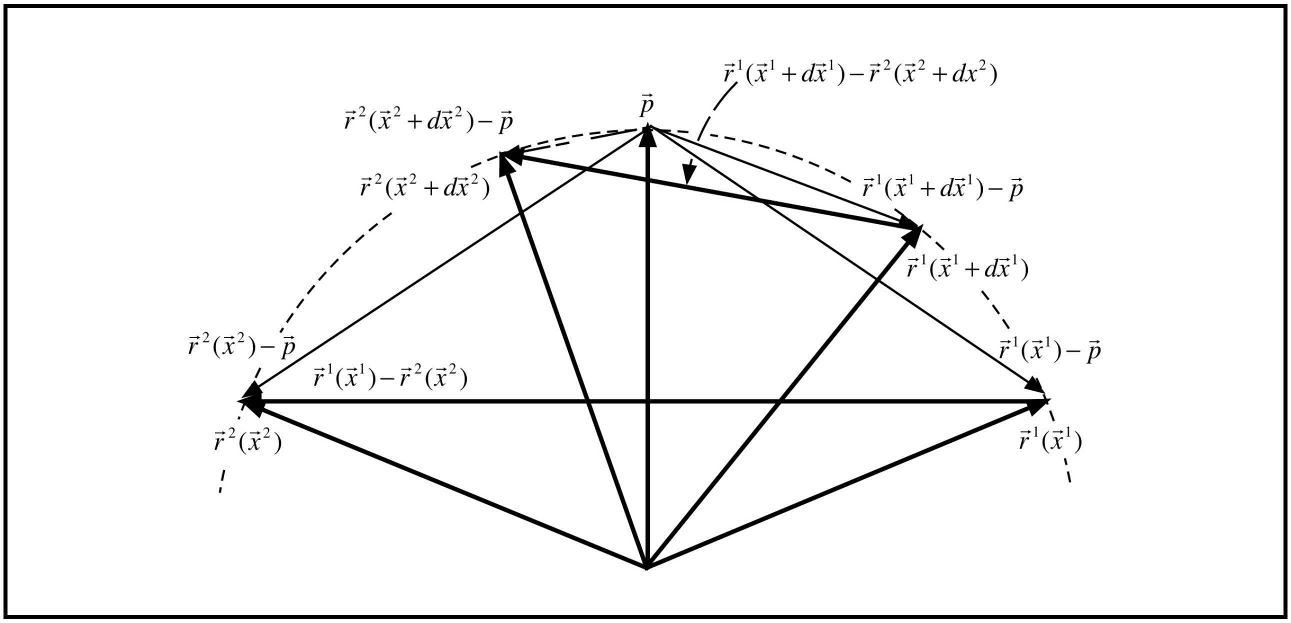
$$\begin{aligned} \left|\vec{r}^{\,1}(\vec{x}^{\,1}) - \vec{p}\right| &> \left|\vec{r}^{\,1}(\vec{x}^{\,1} + d\vec{x}^{\,1}) - \vec{p}\right| > 0 \\ \left|\vec{r}^{\,2}(\vec{x}^{\,2}) - \vec{p}\right| &> \left|\vec{r}^{\,2}(\vec{x}^{\,2} + d\vec{x}^{\,2}) - \vec{p}\right| > 0 \end{aligned} \tag{2.2-3}$$

Since $\vec{p}$ is between $\vec{r}^{\,1}(\vec{x}^{\,1})$ and $\vec{r}^{\,1}(\vec{x}^{\,1})$ as shown in Figure 5.2-2 we must have:

$$\left|\vec{r}^{\,1}(\vec{x}^{\,1}) - \vec{r}^{\,2}(\vec{x}^{\,2})\right| > \left|\vec{r}^{\,1}(\vec{x}^{\,1} + d\vec{x}^{\,1}) - \vec{r}^{\,2}(\vec{x}^{\,2} + d\vec{x}^{\,2})\right| > 0 \tag{2.2-4}$$

This completes the proof:
QED

FIGURE 2.2-1



Proposition 5.2.2-2 Bilateral Tatonnement

Given two consumers described by marginal price functions $\vec{r}^1(\vec{x})$, and $\vec{r}^2(\vec{x})$ respectively. At time t_0 they possess respective initial bundles $\vec{x}^1[t_0]$ and $\vec{x}^2[t_0]$. If, in any time period $t_0 + n$ the consumers are allowed to exchange marginal bundles $d\vec{x}[t_0 + n]$, they will do so until a time period $t_0 + z$ in which:

a) The consumers arrive at a common "market" set of prices $\vec{p}[t_0 + z]$ where: $\vec{r}^1(\vec{x}^1[t_0 + z]) = \vec{r}^2(\vec{x}^2[t_0 + z]) = \vec{p}[t_0 + z]$

b) The consumers will have obtained the maximum use value available to them at price $\vec{p}[t_0 + z]$, given their initial bundles $\vec{x}^1[t_0]$ and $\vec{x}^2[t_0]$.

Proof:

At any time $t_0 + n$, where $n = 0,1,2,\ldots$ unless the marginal prices of the two consumers are already equal, Proposition 5.2-1 indicates that they will exchange a marginal bundle $d\vec{x}[t_0 + n]$ at a mutually agreed price $\vec{p}[t_0 + n]$. After the exchange is completed, the consumers' marginal prices will have contracted towards each other, i.e.:

$$\left|\vec{r}^1(\vec{x}^1[t_0 + n]) - \vec{r}^2(\vec{x}^2[t_0 + n])\right| > \left|\vec{r}^1(\vec{x}^1[t_0 + n + 1]) - \vec{r}^2(\vec{x}^2[t_0 + n + 1])\right| > 0$$

Where:

$$\vec{x}^1[t_0 + n + 1] = \vec{x}^1[t_0 + n] + d\vec{x}$$
$$\vec{x}^2[t_0 + n + 1] = \vec{x}^1[t_0 + n] - d\vec{x}$$

Since this applies to every time period, we must have:

$$\lim_{n\to\infty}\left|\vec{r}^{\,1}(\vec{x}^{\,1}[t_0+n])-\vec{r}^{\,2}(\vec{x}^{\,2}[t_0+n])\right|=0 \qquad (2.2\text{-}16)$$

As before, we choose some number $\varepsilon > 0$ that is negligibly close to zero. Since Equation 1.2-16 approaches zero monotonically, there must be some number $0 < z < \infty$ such that:

$$\left|\vec{r}^{\,1}(\vec{x}^{\,1}[t_0+n])-\vec{r}^{\,2}(\vec{x}^{\,2}[t_0+n])\right|<\varepsilon \qquad (2.2\text{-}17)$$

Thus equilibrium is achieved (and exchange stops) at time $t_0 + z$ when: $\left|\vec{r}^{\,1}(\vec{x}^{\,1}[t_0+z])=\vec{r}^{\,2}(\vec{x}^{\,2}[t_0+z])\right|$ as claimed. This completes the proof of (a)

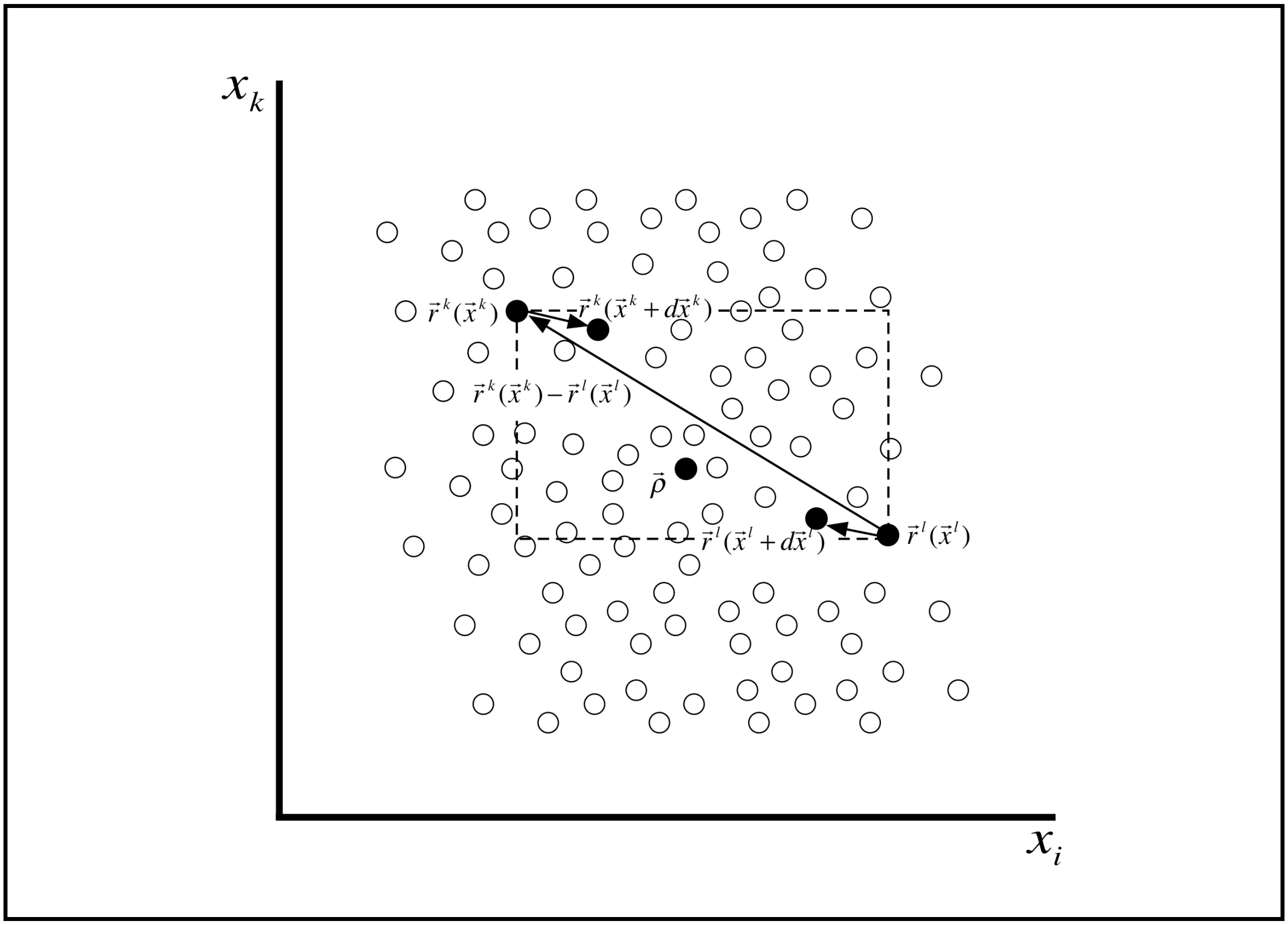
To prove part (b) we know from the definition of use value and from Assumption (4) that the use value each consumer gains through the course of their marginal exchanges does not depend on the exchanges themselves. For Consumer 1 $V^1(\vec{x}^{\,1}[t_0+z])-V^1(\vec{x}^{\,1}[t_0])$ would be the same whether he acquired $\vec{x}^{\,1}[t_0+z]$ through bilateral marginal exchanges or through fixed price exchanges made at $\vec{p}[t_0+z]$. The maximization result follows from the unilateral tatonnement (Proposition 1.2-3). This completes the proof of part (b)
QED.

3) MULTILATERAL EQUILIBRIUM

The multilateral case, where goods are exchanged among many consumers, is broken into many bilateral marginal exchanges. Each exchange brings the marginal prices of the trading partners closer to the averages for the whole community. The partners to any given marginal exchange do not necessarily continue making exchanges with each other. They may meet and exchange only a single marginal bundle before moving on to find other partners. If we were to plot the marginal price vectors for each member of the community as points in any coordinate plane, they would appear as a random cluster that is collapsing onto its center, as shown in Figure 3-1. With each marginal exchange, the mean shifts to compensate. The mean to which the points collapse is therefore constantly readjusting so as to maintain its central position in the cluster.

Consumers would be expected to “shop around” for partners with whom trade will provide the greatest benefit. These are of course those individuals whose marginal prices differ the most from their prospective partners. We model this by only considering exchanges between consumers whose marginal price vectors define a “box” that contains the current mean. This indicates that while one partner’s marginal price for a given good is at or above the mean, the other’s marginal price is at or below the mean. Thus individuals will choose partners whose marginal prices are somewhat “across the cluster” in Figure 5.3-1, as opposed to nearby neighbors.

FIGURE 5.3-1



3.1 Definitions

<u>DEFFINITION: Mean Marginal Price</u>

Given a community of *m* consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of *n* goods $i \in (1,2,\ldots n)$. Given that each consumer μ holds a specific bundle $\vec{x}^{\mu}$ and is described by his or her individual marginal price function:

$$\begin{aligned}\vec{r}^{\,\mu}(\vec{x}^{\,\mu}) &= r_1^{\mu}(x_1^{\mu},x_2^{\mu}\ldots x_n^{\mu})\hat{\varphi}_1 + r_2^{\mu}(x_1^{\mu},x_2^{\mu}\ldots x_n^{\mu})\hat{\varphi}_2 + \cdots + r_n^{\mu}(x_1^{\mu},x_2^{\mu}\ldots x_n^{\mu})\hat{\varphi}_n \\ &= \sum_{i=1}^{n} r_i^{\mu}(\vec{x}^{\,\mu})\hat{\varphi}_i\end{aligned} \qquad (3.1\text{-}1)$$

The Mean Marginal Price is a vector $\vec{\rho} = \rho_1\hat{\varphi}_1 + \rho_2\hat{\varphi}_2 + \cdots + \rho_n\hat{\varphi}_n$, each component ρ_i of which is the mean of the marginal prices $r_i^{\mu}(\vec{x}^{\,\mu})$ of the consumers μ for the good x_i given by:

$$\rho_i \triangleq \frac{1}{m}\sum_{\mu=1}^{m} r_i^{u}(\vec{x}^{\mu}) \tag{3.1-2}$$

<u>DEFFINITION: Deviation from the Mean Marginal Price</u>

Given a community of *m* consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of *n* goods $i \in (1,2,\ldots n)$. Given that each consumer μ holds a specific bundle $\vec{x}^{\mu}$ and is described by his or her individual marginal price function $\vec{r}^{\,\mu}(\vec{x}^{\mu})$. The Deviation from the Mean Marginal Price by the marginal price of the μ^{th} consumer is the vector $\vec{s}^{\,\mu} = \vec{r}^{\,\mu}(\vec{x}^{\mu}) - \vec{\rho}$

<u>DEFFINITION: Average Deviation from the Mean Marginal Price</u>

Given a community of *m* consumers $\mu \in (1,2,\ldots m)$, able to choose among the same set of *n* goods $i \in (1,2,\ldots n)$. Given that each consumer μ holds a specific bundle $\vec{x}^{\mu}$ and is described by his or her individual marginal price function $\vec{r}^{\,\mu}(\vec{x}^{\mu})$. The Average Deviation from the Mean Marginal Price σ is the average of the magnitudes of the deviations from the from the mean marginal price given by:

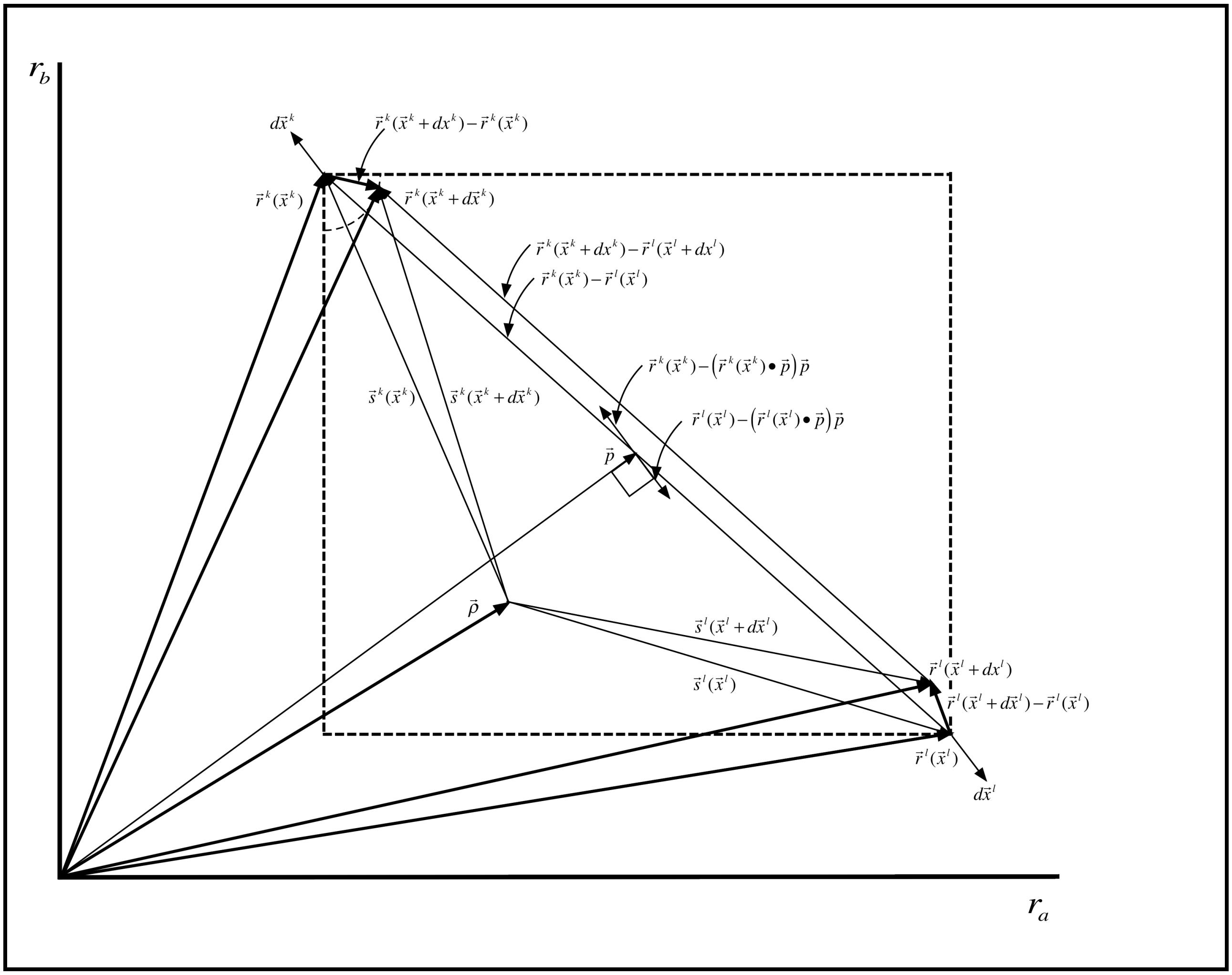
$$\sigma \triangleq \frac{1}{m}\sum_{\mu=1}^{m}\left|\vec{s}^{\,\mu}\right| = \frac{1}{m}\sum_{\mu=1}^{m}\left(\sum_{i=1}^{n}\left(\vec{r}_i^{\,\mu}(\vec{x}^{\mu}) - \vec{\rho}_i\right)^2\right)^{\frac{1}{2}} \tag{3.1-3}$$

3.2) Propositions

For a pair of consumers whose marginal prices define a box that contains the mean, the following proposition indicates that a bilateral marginal exchange between them will result in both of their marginal prices drawing closer to the mean. We show this in two steps: First we show that the Assumption of Non-Addiction in its strongest form will cause the marginal prices for both consumers to shift into the interior of the box. We do this by considering the impact of Assumption (5) on each component of the marginal price vectors individually. As result of the exchange the shifted marginal prices $\vec{r}^{\,k}(\vec{x}^{k} + d\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l} + d\vec{x}^{l})$ will lie in the corner regions of the box near $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l})$ respectively.

The second step of the proof will be to show that $\vec{r}^{\,k}(\vec{x}^{k} + d\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l} + d\vec{x}^{l})$ must be closer to any point $\vec{\rho}$ that is interior to the box, than are $\vec{r}^{\,k}(\vec{x}^{k})$ and $\vec{r}^{\,l}(\vec{x}^{l})$. This is apparent from Figure 3.2-1.

FIGURE 3.2-1



<u>PROPOSITION 3.2-1 Marginal Price Contraction towards the Mean</u>

Given two consumers who are described by marginal price functions $\vec{r}^{\,k}(\vec{x})$ and $\vec{r}^{\,l}(\vec{x})$ respectively who are members of a community of *m* consumers, having mean marginal price $\vec{\rho}$. Given also that $\vec{\rho}$ lies in the box defined by $\vec{r}^{\,k}(\vec{x}^k)$ and $\vec{r}^{\,l}(\vec{x})$.

The consumers will exchange a marginal bundle $d\vec{x} = d\vec{x}^1 = -d\vec{x}^2$ after which, their marginal prices will have contracted towards the mean, i.e.: $\left|\vec{s}^{\,k}(\vec{x}^k + dx^k)\right| < \left|\vec{s}^{\,k}(\vec{x}^k)\right|$ and $\left|\vec{s}^{\,l}(\vec{x}^l + dx^l)\right| < \left|\vec{s}^{\,l}(\vec{x}^l)\right|$

Proof:

We begin by showing that $\vec{r}^{\,k}(\vec{x}^k + d\vec{x}^k)$ lies within the box defined by $\vec{r}^{\,k}(\vec{x}^k)$ and $\vec{r}^{\,l}(\vec{x}^l)$.

From the proof of Proposition 2.2-1 we know that the prices $\vec{p}$ at which the consumers will agree to trade lie between $\vec{r}^{\,k}(\vec{x}^k)$ and $\vec{r}^{\,l}(\vec{x}^l)$. Thus, for any good x_i for which $r_i^k(\vec{x}^k) > r_i^l(\vec{x}^l)$ it must also be true that $r_i^k(\vec{x}^k) > p_i$. Consumer *k* will therefore purchase a (positive) quantity dx_i^k. After the purchase is completed, Assumption 5 implies that Consumer k's marginal price for x_i will have shifted so that $\left(r_i^k(\vec{x}^k + d\vec{x}^k) - r_i^k(\vec{x}^k)\right)dx_i^k < 0$ since dx_i^k is positive the term in brackets must be negative and:

$$r_i^k(\vec{x}^k) > r_i^k(\vec{x}^k + d\vec{x}^k) > r_i^l(\vec{x}^l) \qquad (3.2\text{-}1)$$

Similarly, for any good x_i for which $r_i^k(\vec{x}^k) < r_i^l(\vec{x}^l)$ it must be true that $r_i^k(\vec{x}^k) < p_i$. In this case Consumer k will sell a (negative) quantity $-dx_i^k$. Again after the purchase is completed, Assumption 5 (Equation 1-3) implies that $\left(r_i^k(\vec{x}^k + d\vec{x}^k) - r_i^k(\vec{x}^k)\right)(-dx_i^k) < 0$ since the exchange quantity this time is negative we must have:

$$r_i^k(\vec{x}^k) < r_i^k(\vec{x}^k + d\vec{x}^k) < r_i^l(\vec{x}^l) \qquad (3.2\text{-}2)$$

From Equations (3.2-1) and (3.2-2) we know that by definition, $\vec{r}^{\,k}(\vec{x}^k + d\vec{x}^k)$ lies within the box defined by $\vec{r}^{\,k}(\vec{x}^k)$ and $\vec{r}^{\,l}(\vec{x}^l)$. By similar reasoning it can be shown that $\vec{r}^{\,l}(\vec{x}^l + d\vec{x}^l)$ lies within the box as well.

We now show that because these vectors lie within the box, they have contracted towards the mean. Notice from Figure 3.2-1 that such box is also defined by the vectors by $\vec{s}^{\,k}(\vec{x}^k)$ and $\vec{s}^{\,l}(\vec{x}^l)$. Since $\vec{r}^{\,k}(\vec{x}^k + d\vec{x}^k)$ and $\vec{r}^{\,l}(\vec{x}^l + d\vec{x}^l)$ lie differentially close to $\vec{r}^{\,k}(\vec{x}^k)$ and $\vec{r}^{\,l}(\vec{x}^l)$ respectively we must have, for all components $s_i^k(\vec{x}^k + d\vec{x})$:

$$\begin{aligned} s_i^k(\vec{x}^k) > s_i^l(\vec{x}^l) \quad &\Rightarrow \quad s_i^k(\vec{x}^k) > s_i^k(\vec{x}^k + d\vec{x}) > \rho_i > s_i^l(\vec{x}^l) \\ s_i^k(\vec{x}^k) < s_i^l(\vec{x}^l) \quad &\Rightarrow \quad s_i^k(\vec{x}^k) < s_i^k(\vec{x}^k + d\vec{x}) < \rho_i < s_i^l(\vec{x}^l) \end{aligned} \qquad (3.2\text{-}3)$$

Without loss of generality, assume components $s_a^k(x^k) < s_a^l(x^l)$ and $s_b^l(x^l) < s_b^k(x^k)$ are as shown in Figure 3.2-1. Assume that the only component of $\vec{s}^{\,k}(\vec{x}^k + d\vec{x}^k)$ that differs from $\vec{s}^{\,k}(\vec{x})$ is the ath component $\vec{s}_a^{\,k}(\vec{x}^k + d\vec{x}^k)$. For all points within the box for which $\vec{s}_a^{\,k}(\vec{x}^k + d\vec{x}^k) \neq \vec{s}_a^{\,k}(\vec{x}^k)$ we must have $\vec{s}_a^{\,k}(\vec{x}^k + d\vec{x}^k) < \vec{s}_a^{\,k}(\vec{x}^k)$ in this case:

$$\left|\vec{s}^{\,k}(\vec{x}^k + d\vec{x})\right| = \left(\left(s_a^k(\vec{x}^k + d\vec{x})\right)^2 + \sum_{i \neq a}\left(s_i^k(x^k)\right)^2\right)^{1/2} < \left(\sum_{i=1}^{n}\left(s_i^k(x^k)\right)^2\right)^{1/2} = \left|\vec{s}^{\,k}(\vec{x}^k)\right| \qquad (3.2\text{-}4)$$

Using the same reasoning, we can show that Equation 3.2-2 will hold for every component $s_i^k(\vec{x}^k + d\vec{x}^k)$ were it the only one allowed to deviate from its corresponding component $s_i^k(\vec{x}^k)$. Since all vectors $\vec{s}^{\,k}(\vec{x}^k + d\vec{x})$ contain at least one component that is smaller than their corresponding components of $s_i^k(\vec{x}^k)$ (and no components which are larger), it must generally be true that $\left|\vec{s}^{\,k}(\vec{x}^k + d\vec{x}^k)\right| < \left|\vec{s}^{\,k}(\vec{x}^k)\right|$. By similar reasoning, it can be shown that $\left|\vec{s}^{\,l}(\vec{x}^l + d\vec{x}^l)\right| < \left|\vec{s}^{\,l}(\vec{x}^l)\right|$. Thus, the marginal exchange has caused both consumers' marginal prices to contract towards the mean as claimed.

QED.

PROPOSITION 3.2-2 Multilateral Tatonnement

Given a community of m consumers $\mu \in (1,2,\ldots m)$, with each consumer μ described by a marginal price function $\vec{r}^{\mu}(\vec{x})$ and holding a bundle $\vec{x}^{\mu}[t_0+n]$ at time t_0+n. Given also that at any time period the community is described by a mean marginal price $\vec{\rho}[t_0+n]$ and average deviation of marginal prices $\sigma[t_0+n]$. Given that in any time period in which the consumers' marginal prices are not all equal, a pair of consumers k and l, whose marginal prices $\vec{r}^{k}(\vec{x}^{k}[t_0+n])$ and $\vec{r}^{l}(\vec{x}^{l}[t_0+n])$ enclose the mean, are allowed to exchange a marginal bundle $d\vec{x}[t_0+n]$. Therefore the following will occur:

a) Marginal exchanges will commence and continue until a time period t_0+z at which time all consumers arrive at a common set of "market" prices $\vec{p}[t_0+z]$ where: $\vec{r}^{\mu}(\vec{x}^{\mu}[t_0+z]) = \vec{p}[t_0+z] \quad \forall \mu$

b) The use values $V^{\mu}[t_0+z]$ of each consumer will be the maximum available to that consumer at prices $\vec{p}[t_0+z]$, given their initial bundles at time t_0.

Proof:

Exchange of the marginal bundle $d\vec{x}$ is implied by Proposition 2.2-1, as is the contraction of the consumers' marginal priced toward each other. Proposition 5.3.2-1 implies that the marginal prices of both consumers will contract towards the mean, i.e.:

$$\begin{aligned}&\left|\vec{s}^{k}(\vec{x}^{k}[t_0+n]+dx^{k}[t_0+n])\right| < \left|\vec{s}^{k}(\vec{x}^{k}[t_0+n])\right| \\ &\left|\vec{s}^{l}(\vec{x}^{l}[t_0+n]\ \ +dx^{l}[t_0+n])\right| < \left|\vec{s}^{l}(\vec{x}^{l}[t_0+n])\right|\end{aligned} \qquad (3.2\text{-}5)$$

Since none of the other consumers' marginal prices will have deviated we must have:

$$\begin{aligned}\sigma'[t_0+n] &\triangleq \frac{1}{m}\left(\left|\vec{s}^{k}\left(x^{k}[t_0+n]+dx^{k}[t_0+n]\right)\right| + \left|\vec{s}^{l}\left(x^{l}[t_0+n]+dx^{l}[t_0+n])\right)\right| + \sum_{\mu\neq k,l}^{m}\left|\vec{s}^{\mu}\left(x^{\mu}[t_0+n]\right)\right|\right) \\ &< \frac{1}{m}\left(\sum_{\mu\neq k,l}^{m}\left|\vec{s}^{\mu}\left(x^{\mu}[t_0+n]\right)\right|\right) = \sigma[t_0+n] \qquad (3.2\text{-}6)\end{aligned}$$

The shift in the marginal prices for the two consumers will in general shift the mean, causing all of the deviations to change slightly. At the beginning of the next time period the new mean will be $\vec{\rho}[t_0+n+1]$. For the consumers who were not engaged in the exchange, the new deviations will be:

$$\vec{s}^{\mu}(\vec{x}^{\mu}[t_0+n+1]) = \vec{r}^{\mu}(\vec{x}^{\mu}[t_0+n]) - \vec{\rho}[t_0+n+1] \qquad \mu \neq k,l \qquad (3.2\text{-}7)$$

For the consumers involved in the exchange:

$$\begin{aligned}\vec{s}^{k}\left(\vec{x}^{k}[t_0+n+1]\right) &= \vec{r}^{k}\left(\vec{x}^{k}[t_0+n]+d\vec{x}^{k}[t_0+n]\right) - \vec{\rho}[t_0+n+1] \\ \vec{s}^{l}\left(\vec{x}^{l}[t_0+n+1]\ \right) &= \vec{r}^{l}\ \left(\vec{x}^{l}[t_0+n]+d\vec{x}^{l}[t_0+n]\ \right) - \vec{\rho}[t_0+n+1]\end{aligned} \qquad (3.2\text{-}8)$$

Recalculation of the mean may decrease the average deviation but cannot increase it, therefore:

$$\sigma[t_0+n+1] \le \sigma'[t_0+n] < \sigma[t_0+n] \tag{3.2-9}$$

We know therefore, that in every time period the average deviation decreases due to the exchanges, hence:

$$\underset{n\to\infty}{\text{limit}}\,\sigma[t_0+n]=0 \tag{3.2-10}$$

Using the $\varepsilon - z$ reasoning as was done in Proposition 5.1.2-2 we know that there is some time period t_0+z at which $\sigma[t_0+z]$ differs from zero by a negligible amount. With $\sigma[t_0+z]=0$ we know that all consumers' marginal prices will have converged to the mean, which becomes the market price as claimed, i.e.:

$$\vec{r}^{\,\mu}(\vec{x}^{\,\mu}[t_0+z] = \vec{\rho}[t_0+z] \triangleq \vec{p} \quad \forall\mu \tag{3.211}$$

This completes the proof of (a)

To prove part (b) we know from the definition of use value and from Assumption (4) that the use value each consumer gains through the course of their marginal exchanges does not depend on the exchanges themselves. For Consumer μ $V^{\mu}(\vec{x}^{\,\mu}[t_0+z]) - V^{\mu}(\vec{x}^{\,\mu}[t_0])$ would be the same whether he acquired $\vec{x}^{\,\mu}[t_0+z]$ through multilateral marginal exchanges or through fixed price exchanges made at $\vec{p}[t_0+z]$. The maximization result follows from the unilateral tatonnement (Proposition 5.1.2-3). This completes the proof of part (b)
QED

4) CONCLUSION

The dynamic theory of exchange equilibrium presented here characterizes it as merely a social process that brings about agreement as to what prices should be. In all cases considered here, the tatonnement processes are highly robust, capable of bringing about equilibria for nearly every possible initial distribution of goods and for any set of consumers.

From the analysis presented, there are few ways in which any policies can impact the tatonnement processes: The most obvious of course would be artificially fixing prices. This is the only intervention that actually conflicts with the market mechanism. While some such actions are necessary, they are always costly. Attempts to prevent the sale of contraband goods, gives rise to black markets that are extremely difficult to curtail.

The important interventions to consider come in the form of goods redistribution, through taxes, transfers and government purchases; or in the form of regulations that impact the seller's marginal prices, which are driven by his marginal costs. The tatonnement processes are fully capable of coping with these interventions. Such interventions merely shift the equilibria. There is nothing in this analysis (or in current neoclassical theory) that establishes any one equilibrium as socially "better" than any another. The ability to determine which equilibria may be more desirable will require the additional analytic tools presented elsewhere [3].

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