DYNAMIC MODEL OF THE INDIVIDUAL CONSUMER

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Abstract

This paper presents an alternate formulation of consumer theory that allows the consumer to be modeled as acquiring his/her goods dynamically, i.e. through a series of incremental decisions based on the outcomes of their predecessors. The model begins with the assumption that the consumer knows his /her Marginal Rates of Substitution (MRS), and defines a utility-like quantity in terms of their integral. The model is developed using the mathematics of vector analysis, which clarifies the intuition of what such integrals mean and provides a simple and useful means of expressing the convexity of indifference surfaces. A concept of marginal demand is introduced to capture the difference in the mix of goods a consumer would procure, were he/she to acquire them incrementally rather than through a single, utility maximizing decision.

Key Words

Dynamic Consumer Theory, Integrability, Convex Indifference Surface, Engle's Law, Antonelli Conditions, Marginal Demand, Willingness to Pay, Contingent Valuation, Vector Analysis

JEL Classification Codes

B21, B41, C50, C60, D01, D11, D50

1) Introduction and Overview

Significant drawbacks of the utility maximization paradigm have remained with us since it was first introduced by William Stanley Jevons in 1862[1]. While recognition of the marginal relationships between economic quantities was his great insight, Jevons' method of predicting observable phenomena from unobservable, hypothetical causes such as utility (or preference) has been met with reservations from the very beginning. Problems that the paradigm induces range from the inability to aggregate individual's preferences[2] to confusion over exactly what should or should not be considered as contributing to economic value[3]. A particular concern is that that the utility maximization paradigm renders the consumer choice problem completely static. While it allows one to calculate equilibrium prices and quantities, it is unable to describe the means by which the market arrives at such. As result, we can say very little with regard to how fast markets adjust to changing circumstances, or what may happen while the market is in transition.

The dynamic model which this paper presents, takes the consumer's Marginal Rates of Substitution (MRS) as given, without attempting to explain them in terms of antecedent causes. From them it derives a utility-like function by vector line integration. This allows the consumer to be modeled as making sequential, differentially small transactions, each based on the outcomes of the transactions preceding it. The consumer "income" is understood literally as a flow of increments ΔW to his wealth. With each incremental transaction, the consumer spends his most recent wealth increment on durable goods, causing him to progress along his income expansion path as shown in Figure 1-1.

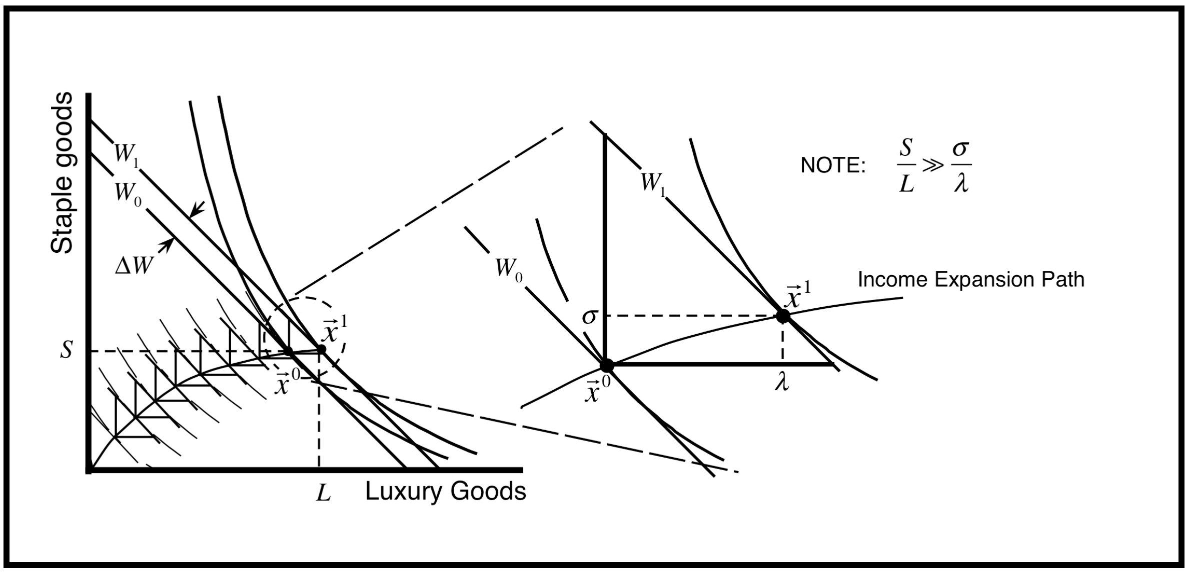

Figure 1-1

[1] Jevons Ideas in the matter were first communicated to the Statistical or Economic Section of the British Association at the Cambridge meeting in 1862 See Jevons (1865). See Black (1972) These were later expanded into his Theory of Political Economy
[2] Robbins (1935) pp.139-40
[3] Black (1972) see also Pattanaik (2009) p.328-334

Unless we are able to assume (as is current common practice) that the consumer's preferences are homoethic[4], the ratio of goods that the consumer would purchase in any given transaction would be different if he purchased them incrementally than if he were to purchase them all at once. From Figure 1-1 it is readily apparent that the ratio of goods S/L that the consumer would acquire, were he to purchase his optimal bundle $\vec{x}^1$ in a single transaction, is larger that the ratio σ/λ he would acquire in an incremental transaction, were he to already hold the bundle $\vec{x}^0$. We know however from Engle's law that a consumer's preferences cannot generally be assumed to be homoethic. Thus if the goods supplied and demanded in the market at any moment result from incremental decisions, modeling the agents as making non-incremental decisions would likely produce biased results.

The dynamic model of the consumer presented here will be an alternate *formulation* of neoclassical consumer theory, much as Lagrangian mechanics is alternative to the Newtonian formulation of Classical Mechanics[5]. From a formal standpoint, the dynamic model is a complete theory, beginning from its own set of assumptions and deriving its conclusions using the mathematical technique of *vector analysis*. From a scientific standpoint however the dynamic model is not a different *theory* of consumer behavior. When applied to the same problems, the dynamic model produces the same results as the utility maximization paradigm, though often with greater detail. Its primary advantage of course is that it can address problems that the utility maximization paradigm cannot. As is the case with the alternate formulations in physics, the analyst is free to use the one that best suits the problem at hand. He or she may also move freely between the two formulations as the need arises.

To summarize the dynamic model, we begin by assuming that: For every bundle of goods $\vec{x}$ the consumer might hold, he or she knows his or her MRS, and that such meet minimal norms of logical consistency. By assuming the existence of money, a numeraire good that is used primarily for that purpose,[6] we define the consumer's *marginal price* for any good x_i as the consumer's MRS for that good in terms of money. The consumer's marginal price $r_i(\vec{x})$ for good x_i is the maximum she would be willing to pay for a unit of it given her holding of the entire bundle $\vec{x}$. The consumer's set of MRS for all goods in $\vec{x}$ are defined by a vector function $\vec{r}(\vec{x})$ indicating the maximum she would pay for any good x_i, given her holding of all other goods in $\vec{x}$. The function $\vec{r}(\vec{x})$, contains all the information knowable about the consumer's choice behavior[7] and (at least in principle) is empirically measurable.

The *Marginal Value* dV the consumer would place on a differential increase $d\vec{x}$ to her bundle is given by the incremental quantities of each good added, priced according to her willingness to pay for them, i.e.: $\vec{r}(\vec{x})\bullet d\vec{x}$. The *Use Value* $V(\vec{x}'-\vec{x}_0)$ she places on some finite bundle of goods $\vec{x}'$, measured with respect to the value she places on some reference bundle $\vec{x}^0$ is the *line integral* taken over a consumption path in commodity space from $\vec{x}^0$ to $\vec{x}'$ as shown in Figure 1-2.

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[4] Or that all goods purchased are a flow of non durables.

[5] In fact The Utility Maximization approach is highly analogous to Legrangian Machanics (which relies on energy minimization), while the Dynamic model presented here has greater resemblance to the Newtonian approach.

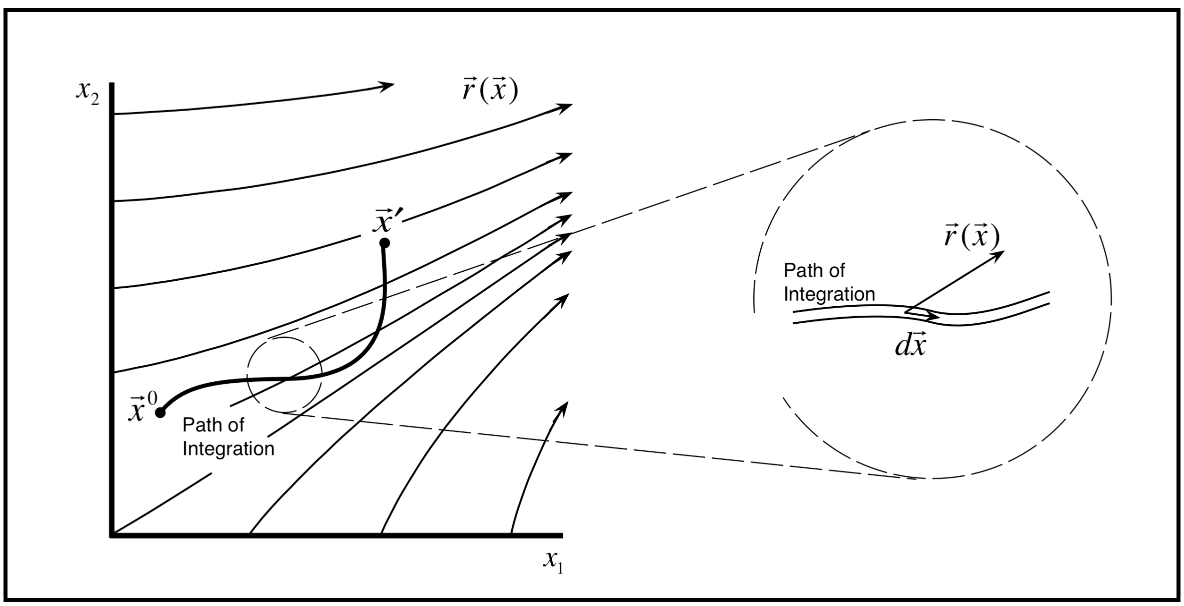
[6] The problem of their being n relationships between n+1 goods is solved by not considering money to be a good that is traded for its own sake.

[7] The substitute and compliment information is contained in the derivatives of the marginal price vector with respect to the goods in question.

Use Value defined in this manner is functionally equivalent to Jevons' utility. Like energy in Physics however, its existence is owed entirely to its mathematical definition. Because of the way it is defined, The use value the consumer places on a bundle can be measured in terms of the numeraire commodity.

The dynamic model rests on five basic assumptions: The first two, which contain a weak notion of rationality, insure that the marginal price function exists. A third assumption guarantees the existence of a suitable numeraire commodity. A fourth assumption is needed to insure that the integral of $\vec{r}(\vec{x})$ exists so that $V(\vec{x}' - \vec{x}_0)$ may be defined in terms of it. A discussion of this assumption will be provided that follows the integrability literature as it proceeded from Pareto[8] through Samuelson[9] and Howthakker[10]. As will be shown, integrability of $\vec{r}(\vec{x})$ requires only that that the complementary and substitutionary effects between goods are mutual (as A compliments B, so must B compliment A).

To insure that the consumer problem has a unique solution, a fifth assumption is needed to guarantee that the consumer's indifference curves be convex to the origin. As will be shown, this requires only that we assume a diminishing marginal price for all *linear combinations of goods*. As will become evident, vector analysis provides a straight forward and easily applied means of stating the convexity assumption, one from which the familiar quadratic form can be derived.



<u>Figure 1-2.</u> A Path of integration in the $x_1 - x_2$ plane. The marginal price function is depicted as curved streamlines.

[8] See Pareto (1971 A) and Pareto (1971 B)
[9] Samuelson (1950)
[10] Houthakker, (1950)

2) Some Notes on Vector Analysis

While I will not pretend to provide a tutorial on vector analysis here, I will point out some of the difference between the way vectors will be used in this paper, and the way they often appear in the economics literature. Vectors are *not* to be understood as linear arrays of unrelated objects. A vector is a *type of number* having properties interpretable as *magnitude* and *direction*, that can be expressed equivalently with respect to any coordinate system. While some physical quantities such as mass and temperature can be expressed as *scalars* (numbers having only magnitude) quantities such as velocity and momentum must be expressed as vectors. If two vehicles collide, the force of impact will depend on their relative directions of travel as well as the magnitude of their speeds.

It is with respect to *a given coordinate system*, that a vector can be expressed as an array of elements. Such an array is shorthand for a vector sum of components, where each component is the scalar product of the vector in question and a basis vector of unit length that defines a coordinate axis. For the familiar three-dimensional Cartesian system with axes labeled x-y-z, the basis vectors are be denoted: $\hat{\varphi}_x, \hat{\varphi}_y, \hat{\varphi}_z$ respectively as shown in Figure 2-1. For a given vector $\vec{A}$ its three respective components are:

$$\begin{matrix} a_x\hat{\varphi}_x & & a_x = \vec{A} \bullet \hat{\varphi}_x \\ a_y\hat{\varphi}_y & \text{where} & a_y = \vec{A} \bullet \hat{\varphi}_y \\ a_z\hat{\varphi}_z & & a_z = \vec{A} \bullet \hat{\varphi}_z \end{matrix} \tag{2-1}$$

We can write the vector as the sum of its components i.e.:

$$\vec{A} = \quad \vec{A}_x + \vec{A}_y + \vec{A}_z = \quad a_x\hat{\varphi}_x + a_y\hat{\varphi}_y + a_z\hat{\varphi}_z \tag{2-2}$$

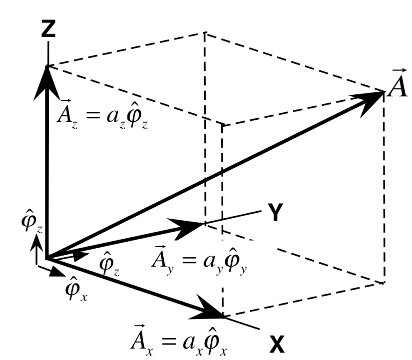


Figure 2-1. A Vector as a sum of components

Vector *functions* "map" vectors to points in space, and are commonly used to describe motion of extended, elastic bodies such as fluids. The motion of particles suspended within a fluid vary with their position within the fluid. The velocity of a particle suspended in a stream of water will be a function of its position relative to the riverbank. Particles closer to the shore will move more slowly and with a trajectory that follows curves in the riverbank, while particles near the center will move faster and in more of a straight line. Figure 2-2 illustrates of a vector field showing the velocity of exhaust gas as it escapes from an automotive tailpipe.

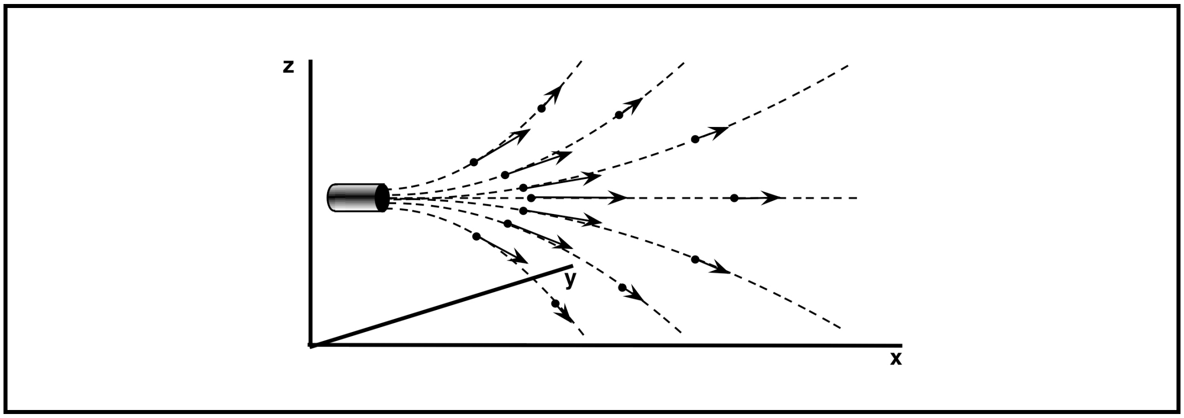


Figure 2-2. Vector field depicting the velocity of gas escaping from a pipe.

With respect to a given coordinate system, vector functions are expressed in component form, where the coefficients are all scalar functions of the same argument. A velocity function $\vec{v}(x,y,z)$ used to describe the velocity of a particle suspended at a point x,y,z can be expressed in component form as $\vec{v}(x,y,z) = v_x(x,y,z)\hat{\varphi}_x + v_y(x,y,z)\hat{\varphi}_y + v_z(x,y,z)\hat{\varphi}_z$.

3) The Marginal Price Function

I n this section I will define the marginal price function once I discuss the assumptions upon which that definition rests. The first two assumptions entail consumer "rationality", understood to consist of little more that logical consistency. The first assumption is merely that: Given any bundle of goods the consumer might posses, he / she knows how much of any one good he / she would exchange for one more unit of any other.

> **Assumption 1: (Existence of the MRS)**
> Given an economy with *n+1* commodities, x_i where $i \in \{1,2,\ldots n+1\}$. For any bundle of commodities $(x_1, x_2, \ldots x_{n+1})$, and any pair of commodities x_j and x_k within that bundle, the consumer's MRS_{j-k} (defined by Equation 3-1 below) exists with non-negative real values.

$$MRS_{i-k}(x_1, x_2, \ldots x_{n+1}) \triangleq \frac{dx_i}{dx_k} \tag{3-1}$$

Even though this assumption is quite obvious, it eliminates study of a myriad of pathological preference orderings that appear in the literature, for which the MRS do not exist[11]. I argue that

[11] See Scarf (1960) and Ingrao and Israel (1990) pp.138-40. Such pathological orderings include lexiocographic or Leontief orderings, perfect compliments and the like. Scarf's examples of general equalibria which were not globally stable was based on agents for whom goods were perfect compliments. I argue that such orderings cannot represent a

such orderings would represent a customer that is "confused" as to the rates at which he would exchange certain goods, and would thus be unable to participate in market transactions.

Assumption (2), which states that the exchanges the consumer is willing to make are logically consistent, is given formally as follows:

Assumption (2) Transitivity of the MRS

Given an economy with *n+1* commodities, for each set of commodities x_i, x_k, x_l where $i,k,l \in (1,2,\ldots n+1)$

$$MRS_{k-i} = MRS_{k-l} \bullet MRS_{l-i} \tag{3-2}$$

The following third assumption establishes a numeraire commodity as the standard by which economic value is measured. While numeraire commodities are already familiar, the utility maximization paradigm tacitly assumes economic value is measured in terms of some notion of satisfaction or "happiness". We hence can talk about a "diminishing marginal utility of money". In the dynamic framework introduced here, such notions of satisfaction are done away with completely. Recall however that for an economy with n goods, there will be only $n-1$ unique marginal rates of substitution. If we can assume that there is at least one good whose primary use is as money, we can eliminate it from the analysis by declaring it the standard by which the value placed on other goods is measured. This eliminates the troublesome extra degree of freedom. This appears to mirror what societies have done in practice. If one considers economies before the invention of currency, there were still goods such as precious gems and metals that were used for money and little else. This provides one more reason why pretty pieces of rock and metal with no practical use might be prized so highly.

Assumption (3) Existence of a standard commodity (Money)

Given an economy of $n+1$ commodities, there exists one numeraire commodity M that consumers use solely as a medium of exchange, a store of value, and/or as a unit of account.

Using Assumptions (1) through (3), we can define the consumer's marginal price for a single good. This is the maximum money price, that the consumer would be willing to pay for one additional unit of that good, given his holdings of all goods.

Definition: Marginal Price (of the i^th good)

For an economy with n goods $(x_1, x_2, \ldots x_n)$ and numeraire M, the consumer's *marginal price* for good x_i is a scalar function of the goods and money the consumer holds, given by:

"rational" consumer since one would be foolish to purchase one of such goods without considering purchase of the others. Such goods are therefore sold in sets (a pair of shoes), We therefore define a set of perfect compliments as a single commodity. Elements of sets of perfect compliments may be sold separately as replacement parts. In such case though the consumer is deciding between purchasing the replacement part (and "fixing" the set that he has), or replacing the entire set. In this case the consumer does have an MRS since the "replacement part" and the "new set" are not perfect compliments.

$$r_i(x_1, x_2, \ldots x_n) = \frac{dM}{dx_i}$$

This defines a set of n functions, which contain all the information that can be observed with regard the consumer's choice behavior. Before defining the Marginal Price (vector) function I will provide a formal definition of the commodity vector space, if for no other reason than to clarify the notation.

> **<u>Definition: Commodity Vector Space</u>**
> For an economy with n goods $(x_1, x_2, \ldots x_n)$ and numeraire M, The Commodity Vector Space is the real Cartesian space $\Re^n$ spanned by the mutual orthogonal basis vectors $\hat{\varphi}_i$, (where $i = 1,2,\ldots n$). where $\hat{\varphi}_i$ represents the i[th] commodity with all its defining attributes.

Notice that the commodity space has no coordinate axis corresponding to M. The numeraire appears only as the unit of measure for the other axes.

> **<u>Definition: Marginal Price</u>**
> For a consumer possessing a bundle $x_1, x_2 \ldots x_n = \vec{x}$, and marginal prices $r_i(x_1, x_2, \ldots x_n) = r_i(\vec{x})$ for each commodity x_i, the consumer's marginal price function $\vec{r}(\vec{x})$ is defined by:
>
> $$\vec{r}(\vec{x}) = r_1(\vec{x})\hat{\varphi}_1 + r_2(\vec{x})\hat{\varphi}_2 + \cdots + r_n(\vec{x})\hat{\varphi}_n \quad (4.3\text{-}1)$$

4) Use Value and Integrability

As mentioned earlier, the use value the consumer places on a quantity of goods will be defined as the integral of his marginal prices, taken over them as they are acquired incrementally. This of course raises the rather thorny historical issue of i*integrability*, which has appeared in the literature from the time it was first raised by Pareto until it was laid to rest by Samuelson and Howthakker a half century later. Before delving into the conditions that must be met for the integral of marginal prices to "exist", let us first consider what the integral of marginal prices might mean intuitively.

Consider an economy containing two goods x_1 and x_2, and a consumer who acquires them through a series of incremental transactions. The consumer begins at time t^0 with some bundle $\vec{x}[t^0] = \vec{x}^0 = x_1^0\hat{\phi}_1 + x_2^0\hat{\phi}_2$ as shown in Figure 4-1. The consumer receives increments of wealth w in the form of a stream of income $I[t]dt = dw[t]$ with which she purchases a series of incremental bundles $d\vec{x} = dx_1\hat{\varphi}_1 + dx_2\hat{\varphi}_2$. The ratio of goods $x_1[t]/x_2[t]$ purchased in each transaction will depend in part on their current relative market prices $\vec{p}[t] = p_1[t]\hat{\varphi}_1 + p_2[t]\hat{\varphi}_2$ which we presume vary with time t. The price variations lead the consumer along a *consumption path* shown as *Path A* in Figure 4-1.

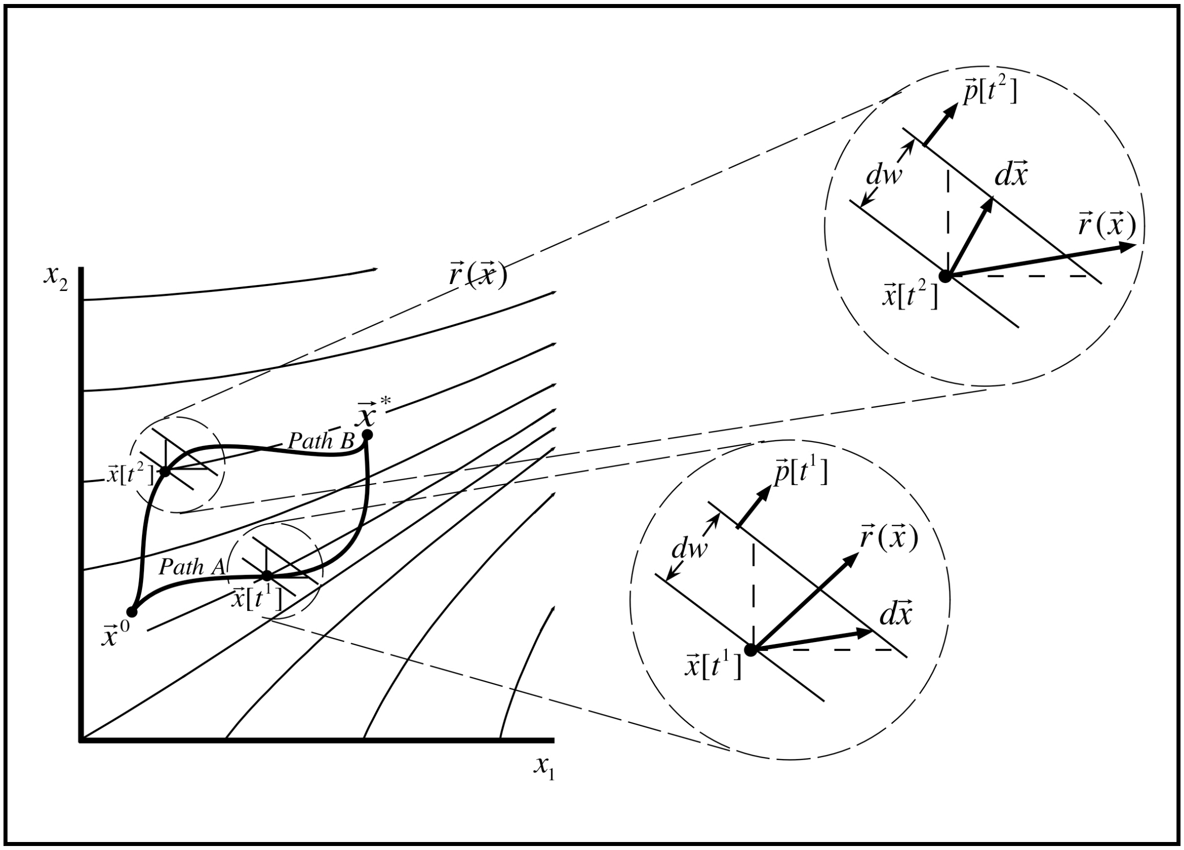


Figure 4-1

Consider he incremental purchase $d\vec{x}$ she makes at time t^1 just after the sum of her prior acquisitions total some intermediate bundle $\vec{x}[t^1]$. The use value she places on each good within $d\vec{x}$ is based on her willingness to pay for it, given that she holds $\vec{x}[t^1]$. The incremental use value $dV[t^1]$ she places on $d\vec{x}$ is just $r_1(\vec{x}[t^1])dx_1 + r_2(\vec{x}[t^1])dx_2 = \vec{r}(\vec{x}[t^1]) \bullet d\vec{x}$. If The consumer continues acquiring goods until time t^* when she holds bundle $\vec{x}^*$, the value she will place on all goods acquired along $Path\ A$ from $\vec{x}^0$ to $\vec{x}^*$ is the integral:

$$\int_{\vec{x}^0\ \ Path\ A}^{\vec{x}^*} \vec{r}(\vec{x}) \bullet d\vec{x}$$

Now let the consumer repeat the process, this time acquiring her goods facing a different scenario of market prices. These prices will lead her along $Path\ B$ in Figure 4-1. As before, consider the incremental purchase she makes at time t^2. The bundle $\vec{x}[t^2]$ she holds at that time will generally be different than was $\vec{x}[t^1]$ in the previous example. This will also be true of prices $\vec{p}[t^2]$. As result, the mix of goods she acquires at time t^2 will generally be different and so will the value $dV[t^2]$ she places on the incremental bundle[12]. If at time $t*$ the

[12] There is no reason to believe that at any time t, dv[t]=dw[t]. Prices p[t] may change fast enough to produce "corner solutions" for some incremental transactions. Hence the consumer may realize a surplus from any given transaction.

consumer ends up with the same bundle $\vec{x}*$ the value she places on it measured with respect to $\vec{x}^0$ will be given by:

$$\int_{\vec{x}^0\ \ Path\ B}^{\vec{x}*} \vec{r}(\vec{x}) \bullet d\vec{x}$$

From the information we have so far, there is no reason to believe that the value of this integral equals that of the integral taken over Path A.

Intuitively however, the only difference between these two exercises is that the consumer has acquired her goods in a different order[13]. It was Pareto's speculation into the possible significance of such order of consumption that initially raised the so called problem of integrability[14]. Using Samuelson's Strong Axiom of Revealed Preferences (SARP) however, Howthakker demonstrated that the order in which one acquires goods cannot effect his or her preferences over them. He did so by showing that if such order did matter, it would be possible for a consumer to prefer a given bundle of goods to itself[15]. Similar reasoning is provided in the following argument.

We run the consumption process along Path B backwards. The consumer begins with $\vec{x}*$ and *un*-acquires goods along Path B until she ends with $\vec{x}^0$. Essentially, she incrementally sells her goods back to the market at the prices she initially bought them along the path. Unless her tastes (represented by her marginal price function) have changed, we would intuitively expect that the use value she places on $\vec{x}^0$ would be the same as it had been at time t^0 when she started out with it. We would therefore expect the mathematics to show no net gain or loss in use value, were the consumer to acquire her goods along Path A, then un-acquire them along Path B, i.e. [16]:

$$\int_{\vec{x}^0\ \ Path\ A}^{\vec{x}^*} \vec{r}(\vec{x}) \bullet d\vec{x} + \int_{\vec{x}^*\ \ Path\ B}^{\vec{x}^0} \vec{r}(\vec{x}) \bullet d\vec{x} \quad = \oint \vec{r}(\vec{x}) \bullet d\vec{x} = 0 \qquad (4\text{-}1)$$

If Equation (4-1) holds, then it is apparent that the use value $V(\vec{x}^* - \vec{x}^0)$ is truly a function of the goods themselves and not of the order by which they were acquired. It is intuitively reasonable to assume such from the outset, and thus conclude that the integral's value independent of the path taken. There is however, more compelling reasoning available. According to *Stokes' Theorem*, a central result of vector analysis, the following three statements are equivalent:

$$\oint \vec{r}(\vec{x}) \bullet d\vec{x} = 0 \qquad \Leftrightarrow \qquad \vec{r}(\vec{x}) = \nabla V(\vec{x}) \qquad \Leftrightarrow \qquad \frac{\partial r_i(\vec{x})}{\partial x_k} = \frac{\partial r_k(\vec{x})}{\partial x_i} \forall_{i,k} \qquad (4\text{-}2)$$

[13] Pareto (1971 B)
[14] Samuelson (1950)
[15] Houthakker, (1950)
[16] The circle symbol on the last integral on the right side of Equation 4-1 represents integration around a closed path.

The left and middle statements confirm that the order in which the goods are acquired is irrelevant as long as $r(\vec{x})$ is a complete differential of some scalar function $V(\vec{x})$. The right hand equation is equivalent to the so-called *Antonelli conditions* for the integrability of demand functions[17]. If any one of the statements given in Equations (4-4) may be assumed, the remaining two follow as conclusions. It is the third statement that is intuitively the most compelling. The term $\partial r_i(\vec{x})/\partial x_k$ reflects the degree to which a consumer's holding of some good x_k impacts his willingness to acquire an additional quantity of some other good x_i. If $\partial r_i(\vec{x})/\partial x_k$ is positive, x_k complements x_i, if $\partial r_i(\vec{x})/\partial x_k$ is negative, x_k is a substitute for x_i. Since this notion is slightly different from cross price elasticity, It is instructive at this point to define a term *complementarity*:

Definition: Complementarily[18]
For a given consumer with marginal price function $\vec{r}(\vec{x})$ holding bundle $\vec{x} = \left(x_1, x_2, \ldots x_n\right)$, the complementary effect of her possession of good x_k on the marginal price $r_i(\vec{x})$ she would pay for another good x_i is defined to be: $\partial r_i(\vec{x})/\partial x_k$.

Given this definition of complementarity, it is apparent that the Antonelli conditions state that complementary (or substitutionary) effect between goods must be mutual. Intuitively we would expect that if x_k is a substitute for x_i then the reverse must be true as well. We can now state the right hand equation of Equations 4-2 as a formal assumption.

Assumption (4) [Mutual Complementarily][19]
For a given consumer with marginal price function $\vec{r}(\vec{x})$ holding bundle $\vec{x} = \left(x_1, x_2, \ldots x_n\right)$,the complementary effect of his possession of any good x_i on the marginal value $r_k(\vec{x})$ he would place on another good x_k is equal to the complementary effect of his possession of good x_k on the marginal value $r_i(\vec{x})$ he places on good x_i. Thus:

$$\frac{\partial r_i(\vec{x})}{\partial x_k} = \frac{\partial r_k(\vec{x})}{\partial x_i} \forall_{i,k} \tag{4-3}$$

If we are able to assume Equation 4-3, we know that $\vec{r}(\vec{x})$ is a complete differential of a scalar function of $V(\vec{x})$ which we can now define:

Definition: Use-Value
Given a consumer with marginal prices given by $\vec{r}(\vec{x})$, for which Assumption (4) is satisfied. The Use-Value a consumer places on a bundle of goods $\vec{x}'$, measured with respect to the value she places on some other bundle $\vec{x}^0$ is defined to be:

$$V\left(\vec{x}' - \vec{x}^0\right) = \int_{\vec{x}^0}^{\vec{x}'} \vec{r}(\vec{x}) \bullet d\vec{x} \tag{4-4}$$

[17] Antonelli (1971)
[18] This refers to net-complementarity.
[19] Eugen Slutsky recognized this as a testable hypothesis that must be true if demand functions were integrable See Samuelson (1950) p.357

where integral is evaluated over any path between $\vec{x}^0$ and $\vec{x}'$.

Defining the value the consumer places on a bundle $\vec{x}'$ with respect to a reference bundle $\vec{x}^0$ has empirical advantages, as that in practice, identifying a consumer who has no goods at all would be difficult to do. Measurements can thus be made with respect to a minimum, or subsistence reference bundle of the analyst's choosing.

The locus of points for which $V(\vec{x}' - \vec{x}^0)$ equals some constant is an *iso-value* (i.e. an indifference) curve or surface. Depending on the analysis, it may be convenient to represent the consumer's characteristics with a network diagram showing both his marginal prices and his indifference curves as given in Figure 4-3

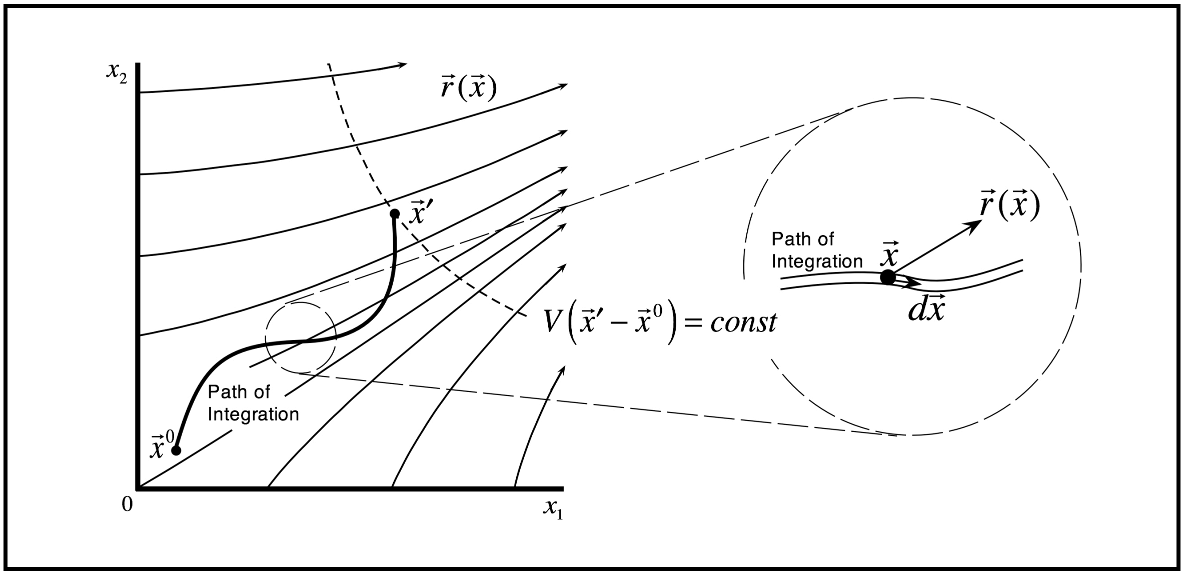


Figure 4-2

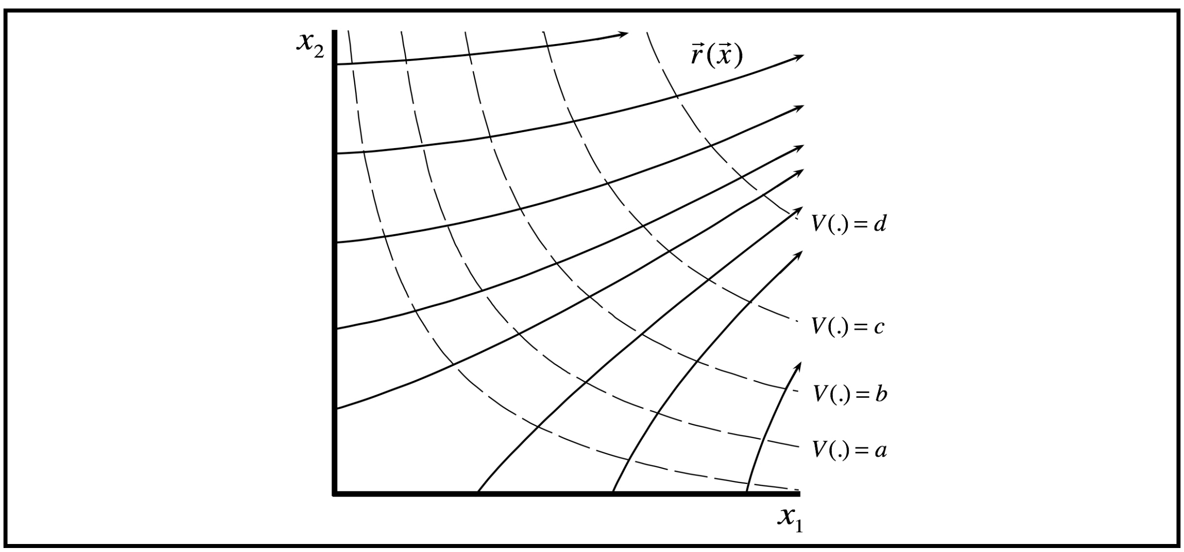


Figure 4-3

5) Convexity and the Assumption of Non Addiction

The final assumption will guarantee that the indifference curves of the use value function be convex to the origin. This assumption will be expressed in terms of the consumer's marginal prices, and its interpretation explored from a psychological and social perspective to see it if it is justified. As we will see, such an assumption is merely an extension of Jevons' law of diminishing marginal utility to include linear combinations of goods. The role the assumption plays in the workings of markets though may explain why most cultures restrict the presence of goods for which consumers may become addicted, i.e. the more they consume of these goods, the greater effort they will expend to acquire more.

To insure a unique solution to the consumer problem we require that for every possible set of positive prices, the budget plane must make contact with the one of the consumer's indifference surfaces at exactly one point[20]. This condition will be met if the indifference surfaces of the consumer's use value function are *convex*. Geometrically, convexity is assured if, for every pair of points A and B on an indifference surface, the chord joining them lies entirely interior to the surface. Such surfaces of course appear as rounded, thought not necessarily symmetric bowls, with their bottoms oriented towards the origin.

Figure 5-1 illustrates how this description of convexity can be translated into the language of vectors. Consider a convex surface, a slice of which is represented as the curve joining points A and B. For the sake of generality, we allow the surfaces' radii of curvature to be different in different directions as well as non-constant as one moves from point to point on the surface. Additionally we allow the surface to twist slightly as one proceeds from A to B. Vectors $\vec{N}_A$ and $\vec{N}_B$ are of arbitrary positive magnitude, and are normal to the surface at points A and B respectively. (To illustrate the twist, $\vec{N}_A$ is shown pointing out of the plane of the drawing and towards the viewer, while $\vec{N}_B$ points out of the plane and away from the viewer.) From Figure 5-1 it is apparent that if we project $\vec{N}_A$ and $\vec{N}_B$ onto the chord $\overrightarrow{AB}$, the projections would point towards each other. A projection of their vector difference $\vec{N}_B - \vec{N}_A$ onto the chord would thus point in the opposite direction as the vector $\overrightarrow{AB}$. Using this result, a convex surface can thus be defined as follows:

> **<u>Definition: Convex Surface</u>**
> A surface is said to be convex if for every pair of points A and B it contains, the corresponding normal vectors $\vec{N}_A$ and $\vec{N}_B$ satisfy:
>
> $$\left(\vec{N}_B - \vec{N}_A\right) \bullet \overrightarrow{AB} < 0$$
>
> where $\overrightarrow{AB}$ is a vector from A to B.

[20] If we allow the surfaces to be "quasi" convex (to have flat spots), the plane will touch the surface over the entire flat region, assuming the plane is oriented parallel to the flat region.

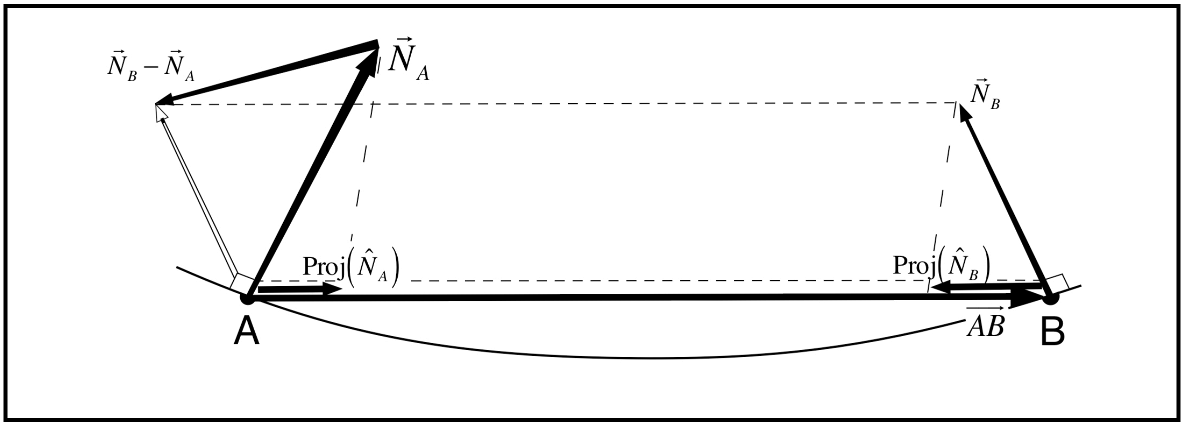


Figure 5-1

For our purposes, the properties of an indifference surface are of less interest than the properties of the marginal price function that generates them. Since $\vec{r}(\vec{x})$ is the gradient of $V(\vec{x})$ we know that for every point $\vec{x}$, $\vec{r}(\vec{x})$ is normal to the indifference surface of $V(\vec{x})$ passing through it. I define the vector $\overrightarrow{AB}$ as a displacement in commodity space $\Delta\vec{x}$. By replacing points A and B with $\vec{x}$ and $\vec{x}+\Delta\vec{x}$, the normal vectors at these points become $\vec{r}(\vec{x})$ and $\vec{r}(\vec{x}+\Delta\vec{x})$ respectively. Equation 5-1 becomes:

$$\left(\vec{r}(\vec{x}+\Delta\vec{x})-\vec{r}(\vec{x})\right)\bullet\Delta\vec{x}<0 \qquad (5\text{-}2)$$

If we consider $\Delta\vec{x}$ to be a small increment of a single good x_i Equation 5-2 reduces two the familiar law of diminishing marginal utility, stated in marginal price form[21].

$$\frac{r_i(\vec{x}+\Delta x_i)-r_i(\vec{x})}{\Delta x_i}\left(\Delta x_i\right)^2<0 \quad \Leftrightarrow \quad \frac{\partial r_i(\vec{x})}{\partial x_i}=\frac{\partial^2 V(\vec{x})}{\partial x_i^2}<0 \qquad (5\text{-}3)$$

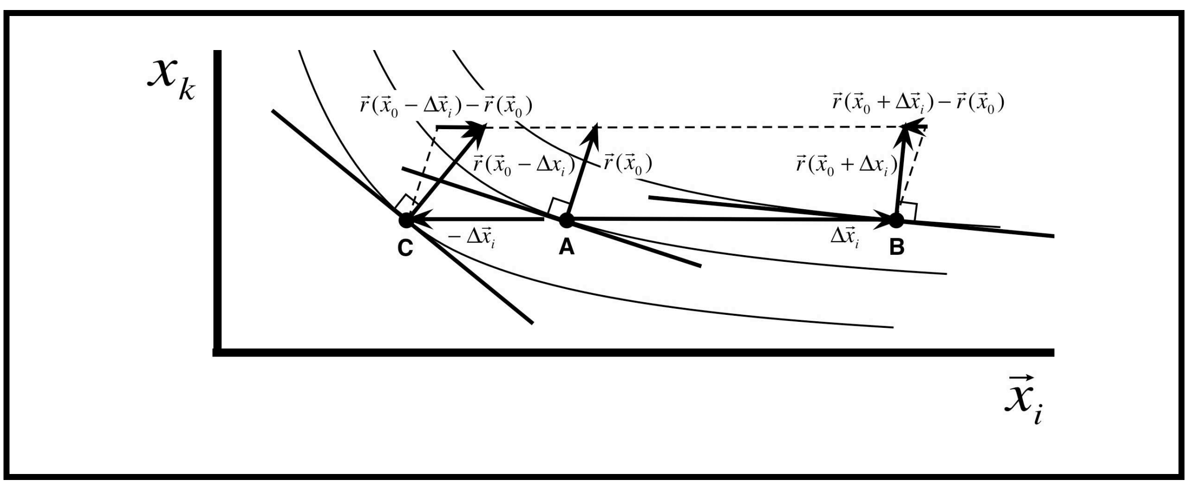


Figure 5-2

[21] Rather than saying that the benefit derived reduces with consumption, we say that it is the consumer's willingness to pay that reduces with consumption.

If $\Delta\vec{x}$ represents a linear combination of goods, the quadratic form used in the economics literature to characterize a convex utility function can be derived from Equation (5-2). We begin by expanding the dot product of Equation (5-2) according to its definition.

$$\sum_i \left[r_i\left(\vec{x}+\Delta\vec{x}\right) - r_i\left(\vec{x}\right) \right] \Delta x_i < 0 \tag{5-4}$$

Since the argument of each r_i is a function of all goods x_i, and each displacement Δx_i is small, we can apply the mean value theorem[22] to each term in the square brackets obtaining:

$$r_i\left(\vec{x}+\Delta\vec{x}\right) - r_i\left(\vec{x}\right) = \sum_k \frac{\partial r_i\left(\vec{x}+\theta\Delta\vec{x}'\right)}{\partial x_k}\Delta x_k \qquad 0<\theta<1 \tag{5-5}$$

Since $\theta\Delta\vec{x}'$ represents a very small displacement from $\vec{x}$, we can ignore it and substitute Equation (5-5) into Equation (5-4):

$$\sum_i \sum_k \frac{\partial r_i\left(\vec{x}\right)}{\partial x_k}\Delta x_k \Delta x_i < 0 \tag{5-6}$$

When written in matrix form the partial derivatives in Equation (5-6) form a matrix that it will be convenient to define formally.

Definition (Complementarity Tensor $\mathbf{C}(\vec{x})$)
For a consumer possessing a bundle $\vec{x}$ $x_1, x_2 \ldots x_n = \vec{x}$, and with marginal prices $\vec{r}(\vec{x})$, the complementarity tensor $\mathbf{C}(\vec{x})$ is the *n*x*n* matrix defining the complementary effect of each good upon all other goods, evaluated at $\vec{x}$.

$$\mathbf{C}(\vec{x}) = \begin{pmatrix} \frac{\partial r_1(\vec{x})}{\partial x_1} & \frac{\partial r_1(\vec{x})}{\partial x_2} & \cdots & \frac{\partial r_1(\vec{x})}{\partial x_n} \\ \frac{\partial r_2(\vec{x})}{\partial x_1} & \frac{\partial r_2(\vec{x})}{\partial x_2} & \cdots & \frac{\partial r_2(\vec{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r_n(\vec{x})}{\partial x_1} & \frac{\partial r_n(\vec{x})}{\partial x_2} & \cdots & \frac{\partial r_n(\vec{x})}{\partial x_2} \end{pmatrix}$$

Equation (5-6) can be expressed in tensor form as: $\left(\Delta\vec{\mathbf{x}}\right)\mathbf{C}(\vec{x})\left(\Delta\vec{\mathbf{x}}\right)^{\mathbf{T}}$ where $\left(\Delta\vec{\mathbf{x}}\right)^{\mathbf{T}}$ is the vector $\Delta\vec{x}$ expressed as a column. From Assumption (4) we know that $\mathbf{C}(\vec{x})$ is symmetric, i.e.

[22] See Taylor and Mann (1983) p.204

that $\mathbf{C}(\vec{x}) = \mathbf{C}(\vec{x})^{\mathbf{T}}$. From inequality (5-6) we know that $\mathbf{C}(\vec{x})$ is negative definite[23]. Substituting $\partial V(\vec{x}) / \partial x_i$ for each $r_i(\vec{x})$ in $\mathbf{C}(\vec{x})$ transforms it into the familiar Jacobean matrix commonly used to describe the convexity of utility functions.

We now take an intuitive second look at what Equation (5-2) means. We consider what would happen if there were goods present for which Equation (5-2) was violated. Presence of goods for which the consumer's willingness to pay for them did not diminish with their consumption would ultimately lead the consumer to expend all of his or her resources to acquire more. Such behavior of course is that of addiction. While such behavior may provide the consumer with short term pleasure, it usually results in long term damage to his well being. Our intuitive belief that addictive behavior destroys the addict as well as those with whom he interacts, is reflected in the mathematics that show that non-convexities in consumer's use value functions wreak havoc on markets. This may explain why societies have recognized that addictive behavior represents a kind of "rational-irrationality" that requires social intervention.

Whether it be narcotics, alcohol, sex, or other "vices", societies have evolved institutions that restrict consumption of goods to which some individuals may become addicted. Whether such restrictions be primitive taboo's or formal statutes, societies have acted to remove addicted behaviors from their market places. It is thus socially reasonable to assume the absence of such, which we now do.

<u>Definition (Addiction)</u>

For a consumer possessing a bundle $\vec{x} \quad x_1, x_2 \ldots x_n = \vec{x}$, and with marginal prices $\vec{r}(\vec{x})$, the consumer is said to be $addicted$ to some good x_i, or to a set of goods, $(x_i x_k \ldots)$ if her marginal price for that good or set of goods does not diminish with her consumption of them.[24] That is to say, for a positive increment of this good or set of goods $\Delta x'$ we have:

$$\left[\vec{r}(\vec{x} + \Delta\vec{x}') - \vec{r}(\vec{x})\right] \bullet \Delta\vec{x}' \geq 0 \tag{5-8}$$

<u>Assumption (5): Non Addiction</u>

For a consumer possessing a bundle $\vec{x} \quad x_1, x_2 \ldots x_n = \vec{x}$, and with marginal prices $\vec{r}(\vec{x})$, there is no good or set of goods to which the consumer is addicted. In other words there are no possible incremental bundles $\Delta\vec{x}'$ for which the following inequality does not hold:

$$\left[\vec{r}\left(\vec{x} + \Delta\vec{x}\right) - \vec{r}(\vec{x})\right] \bullet \Delta\vec{x} < 0 \tag{5-9}$$

[23] If we were to allow for quasi-convexity, Inequality (5-6) would become a weak inequality, and both C and the Jacobian would be negative semi-deffinite.

[24] To be completely rigorous, strong addiction and weak addiction should be defined in terms of whether or not the inequality in Equation 5-6 is strict. That detail is omitted here, as it does not contribute significantly to the argument.

6) The Consumer Choice Problem

At this point we could return to the utility maximization paradigm and maximize the consumer's use value function by the Lagrange method. Because of the way use value has been defined, such a problem is almost trivial. All we seek to do is:

$$Max\left\{\int \vec{r}(\vec{x})\bullet d\vec{x}\right\} \qquad \text{subject to:} \qquad \vec{p}\bullet\vec{x}=w \tag{6-1}$$

By inspection, we see that the first order conditions are simply:

$$r_i\left(\vec{x}^*\right)=p_i\ \forall_i \quad \Rightarrow \quad \vec{r}\left(\vec{x}^*\right)=\vec{p} \qquad \text{and} \qquad \vec{p}\bullet\vec{x}^*=w \tag{6-2}$$

In Equations (6-2) $\vec{x}^*$ represents the optimal bundle that solves the problem implied by Equations (6-1). If the solution is unique, we can represent $\vec{x}^*$ as a parametric function $\vec{x}^*[\vec{p},w]$ of market prices and wealth. This of course is just the system of demand functions written in vector form. The set of points, $\vec{x}^*[\vec{p}^0,w]$ where $\vec{p}^0$ is a given price vector, forms a parametric curve which is the familiar *wealth expansion path*.

To obtain this result in a dynamic problem is a bit more involved. Finding $\vec{x}^*$ will require multiple incremental steps that must be shown to converge to an equilibrium that specifies a single bundle. It must then be shown that the bundle to which the process converges is in fact the one that provides the consumer the greatest use value. Finally, it must be shown that the equilibrium is "stable", i.e. that if the consumer's equilibrium bundle were changed slightly, he would undergo transactions so as to restore his bundle to its equilibrium value. That discussion will be differed to a separate work that addresses general equilibrium[25]. For the moment, let it be said simply that the achievement of equilibrium in a dynamic model, which maximizes the consumer's use value can be proven from the assumptions given previously, and a corollary to Assumption (1) which will be developed here.

Rather than assume outright that the consumer can identify his most preferred bundle, we simply say that he will take advantage of a "good deal", or will try to get the most benefit per unit of numeraire spent. We begin by defining the "benefit" or "deal" that the consumer seeks to obtain by making a marginal exchange. This is simply his or her consumer's marginal surplus. Such surplus is the difference between what a consumer would be *willing* to pay for a marginal amount of a good, and what he is *required* to pay in the market.

<u>Definition: Consumer's Marginal Surplus (for a single good)</u>

For a consumer described by a marginal price function $\vec{r}(\vec{x})$, who is holding a bundle $\vec{x}$, the marginal surplus the consumer would enjoy from purchasing (or selling) a differential quantity dx_i of some good x_i is given by:

$$\left[r_i(\vec{x})-p_i\right]dx_i \tag{6-1}$$

[25] See McLaren (2015)

Notice that if the consumer would be willing to pay more for the good than its market price, the consumer would gain surplus by acquiring the good. In this case, both $[r_i(\vec{x}) - p_i]$ and dx_i are positive, and so is the surplus. If the consumer values a good less than does the market, he gains surplus by selling some of it. In this case both $[r_i(\vec{x}) - p_i]$ and dx_i are negative, and the surplus is again positive.

We will assume that the consumer will try to maximize the surplus obtained for each transaction. This requires that he adjust the relative quantities of the goods dx_i bought and sold, which will be reflected in the direction of the vector $d\vec{x}$.

> **Assumption 1: Part B (Maximization of the Consumer's Marginal Surplus)**
> Given, a consumer described by a marginal price function $\vec{r}(\vec{x})$, who is holding a bundle $\vec{x}$. For all goods x_i for which the consumer's marginal price $r_i(\vec{x})$ differs from the price p_i he or she is offered, the consumer will buy quantities dx_i, or sell quantities $-dx_i$ as necessary to gain the maximum total marginal surplus, subject to the budget constraint.
> $p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n = \vec{p} \bullet d\vec{x} = 0$

7) Marginal Demand

As mentioned in Section 1, a quantity of interest from the dynamic perspective is the mix of (durable) commodities a consumer would acquire in incremental transactions (i.e. how the consumer would spend an increment of wealth on additions to his stock of goods). This would be the marginal demand as is defined below[26]:

> **Definition (Marginal Demand)**
> Given a consumer holding the optimal bundle $\vec{x}^*(\vec{p},w)$ corresponding to his or her wealth w, and market prices $\vec{p}$. The consumer's marginal demand $\Delta\vec{x}^*(\vec{p},w)$ is defined to be:

$$\Delta\vec{x}^*(\vec{p},w) \triangleq \frac{\partial}{\partial w}\left[\vec{x}^*(\vec{p},w)\right] \tag{7-1}$$

If the consumer's use value function is homoethic as is commonly assumed, $\Delta\vec{x}^*$ is constant with respect to w and everywhere proportional to $\vec{p}$ (i.e. the wealth expansion paths are always straight lines through the origin. If the consumer's preferences are not homoethic, as Engle's law would imply, $\Delta\vec{x}$ is generally oblique to both $\vec{r}(\vec{x}^*)$ and $\vec{p}$ as is shown in Figure 7-1.

[26] In the static case, or if all the goods are presumed to be non durable, the notion of marginal demand would be meaningless.

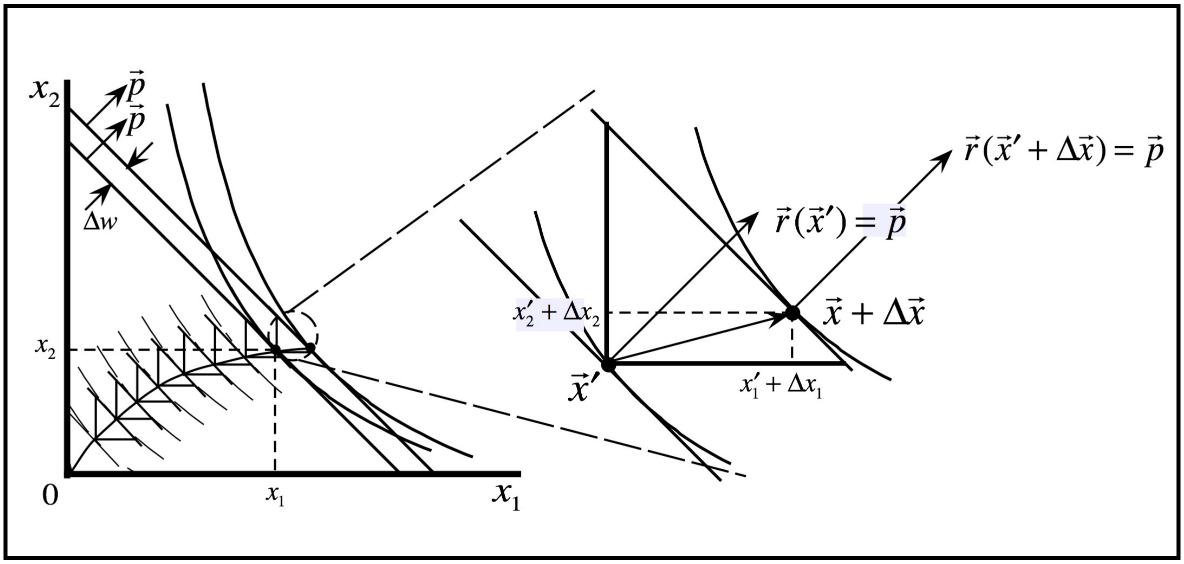


Figure 7-1

We can find $\Delta\vec{x}^*$ using the Mean Value Theorem as in Section 5 above. Since $\Delta\vec{x}^*(p,w')$ represents the slope of the income expansion path at $x' = \vec{x}^*(\vec{p},w')$, we can consider $\Delta\vec{x}^*$ to be the arbitrarily small displacement between two points on the expansion path: $\vec{x}'$ and $\vec{x}' + \Delta\vec{x}^*$. We therefore know that $\vec{r}(\vec{x}') = \vec{r}(\vec{x}' + \Delta\vec{x}^*) = \vec{p}$, hence $\vec{r}(\vec{x}' + \Delta\vec{x}^*) - \vec{r}(\vec{x}') = 0$. For this vector to be zero, each of its components must be zero, thus:

$$r_i(\vec{x}' + \Delta\vec{x}^*) - r_i(\vec{x}') = \sum_k \frac{\partial r_i(\vec{x}')}{\partial x_k} \Delta x_k^* = 0 \quad \forall i \tag{7-2}$$

Equations (7-2) define a system of n homogeneous linear equations in n unknowns that can be written as:

$$\mathbf{C}(\vec{x})(\Delta\vec{x}^*)^{\mathbf{T}} = 0 \tag{7-3}$$

Since by Assumption (5) we know that $\mathbf{C}(\vec{x})$ is negative definite, its determinant in non-zero and Equations (7-3) have a non-trivial solution.

4.6 Conclusion

The task of this paper has been to develop an alternative to the utility maximization or Lagrangian formulation of consumer theory, that allows the consumer to be modeled as exchanging goods dynamically through many incremental sequential transactions. The dynamic formulation has been developed completely independently from the utility maximization framework, in that it proceeds from its own self contained set of assumptions. This paper has however only gone so far as to show that the results produced by the dynamic framework are compatible with the utility maximization framework in that it provides the same answers to the same questions. What remains to be shown is that from the dynamic framework, a model of general equilibrium can be built that accounts for achievement of equilibrium as a dynamic process occurring among interacting agents.

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